Lesson #8 Handout (8/31/2009)

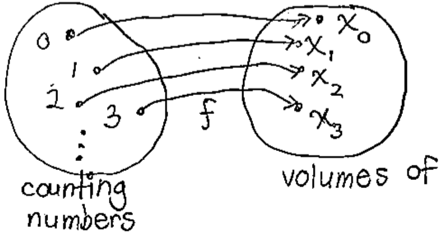
I. Sequences and Series
II. Counting
III. Inequalities and Equalities

Sequence: A function with domain consisting of the natural numbers (or integers)

(DSP computer algorithms)

$\{x_n : n \in \mathbb{N}\}$

$f(n) = x_n$, $n = 0, 1, 2, 3, \dots$



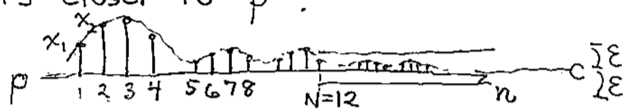
Convergence: The sequence $\{x_n\}$ in the real number system is said to converge if there exists a number $p \in \mathbb{R}$ such that:

- $\underset{\text{real \#}}{\forall \varepsilon} > 0$, $\underset{\text{integer}}{\exists N}$ such that , $|x_n - p| < \varepsilon$ whenever $n \geq N$
- we say $\{x_n\}$ converges to p

$$x_n \longrightarrow p$$

$$\lim_{n \to \infty} x_n = p$$

It's not enough to say that "as n gets bigger, x_n gets closer to p".



In the example of the picture, x_{n+1} is not necessarily nearer p than x_n, so "n is bigger" doesn't mean "x_n is closer to p". But $\{x_n\}$ does converge to p. I can have any requirement of "nearness". No matter how "near" the sequence should be near p (choose $\varepsilon > 0$), x_n will be within ε of p as long as $n \geq N$ (n is big enough).

Series: a sequence of partial sums (Fourier Series, Taylor Series)

$\{S_n\}$

$S_1 = x_1$

$S_2 = x_1 + x_2$

$S_3 = x_1 + x_2 + x_3$

$\boxed{S_n = \sum_{i=1}^{n} x_i}$ the n^{th} partial sum

"A series converges to S" means $\boxed{\sum_{i=1}^{\infty} x_i = \lim_{n \to \infty} S_n = S}$

Useful Series:

$$\boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}}$$

$$\begin{array}{r} 1 + 2 + \dots + n \\ n + n-1 + \dots + 1 \\ \hline (n+1)n = 2S \end{array} \Rightarrow S = \frac{n(n+1)}{2}$$

$$\boxed{\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}}$$

← Proof by collapsing sum

$$(i+1)^3 - i^3 = 3i^2 + 3i + 1$$

$$\underbrace{\sum_{i=1}^{n} [\quad]}_{(n+1)^3 - 1^3} = 3 \sum_{i=1}^{n} i^2 + 3\frac{(n)(n+1)}{2} + n$$

Geometric Series

$$\sum_{i=0}^{n} r^i = \sum_{j=1}^{n+1} r^{j-1} = 1 + r + r^2 + r^3 + \dots + r^n = \underset{\text{always OK}}{\frac{1 - r^{n+1}}{1 - r}}$$

$$\sum_{i=0}^{\infty} r^i = 1 + r + r^2 + \dots = \frac{1}{1-r} \text{ converges iff } |r| < 1$$

$$\begin{array}{rl} 1 + r + r^2 + \dots + r^n & = S_n \\ -(\quad r + r^2 + \dots + r^n + r^{n+1} & = rS_n) \\ \hline 1 - r^{n+1} & = (1-r)S_n \end{array}$$

$$\begin{array}{rl} 1 + r + r^2 + \dots & = S \\ -(\quad r + r^2 + \dots & = rS) \\ \hline 1 & = (1-r)S \end{array}$$

Change of index: Let $j - 1 = i$

$i: 0 \sim n$

$j = i+1: 1 \sim n+1$

Arithmetic Sequence: $\boxed{a_n - a_{n-1} = d}$ constant difference

$a_1,\ a_1+d,\ a_1+2d,\ \dots$

$\boxed{a_n = a_1 + (n-1)d}$

- N^{th} partial sum $\boxed{\begin{aligned} S_N &= \sum_{n=1}^{N} a_n = \sum_{n=1}^{N}\left[a_1+(n-1)d\right] \\ &= a_1 N + \frac{N(N+1)}{2}d - Nd \end{aligned}}$

* The number between two terms of an arithmetic sequence is the $\boxed{\text{arithmetic mean} \quad ? = \frac{x+y}{2}}$

$x,\ ?,\ y$

$a_n,\ a_n+d,\ a_n+2d$

$? = a_n + d = \dfrac{a_n + (a_n+2d)}{2}$

Geometric Sequence $\boxed{\frac{a_{n+1}}{a_n} = r}$ constant ratio

$a_1,\ ra_1,\ r^2a_1,\ \dots$

- The N^{th} partial sum is that of a geometric series
  $S_N = a_1\left(1+r+r^2+\dots+r^N\right) = \boxed{a_1 \frac{1-r^{N+1}}{1-r}}$
- The sum of the infinite series is
  $a_1(1+r+r^2+\dots) = \boxed{\frac{a_1}{1-r}}$ Only converges for $|r|<1$
- The number between two terms of a geometric sequence is the $\boxed{\text{geometric mean } \sqrt{xy}}$

$x,\ ?,\ y$

$a_n,\ ra_n,\ r^2a_n$

$? = ra_n = \sqrt{\underset{x}{a_n}\,(\underset{y}{r^2a_n})}$

(ex) Recursive Sequence

$\boxed{1}$ (A) $a_1 = 3$

$a_n = 2a_{n-1} + 5$

Find a_4

Soln:

a_1: 3 ↵

a_2: 2ans+5

a_3: entry

a_4: entry

$a_4 = 59$

Use calculator

(B) $a_1 = 1 \quad a_2 = 1$

$a_n = a_{n-1} + a_{n-2},\ n \geq 3$

find 1st 7 terms

Soln:

1, 1, 2, 3, 5, 8, 13

(ex) $\boxed{2}$ First term of geometric series is 64, common ratio $\frac{1}{4}$. For what n is $t_n = \frac{1}{4}$?

Soln: $t_1 = 64$

$t_n = 64\left(\frac{1}{4}\right)^{n-1} = \frac{1}{4}$

$4^3 \cdot 4 \cdot \frac{4}{4^n} = 1$

$4^5 = 4^n \Rightarrow n = 5$

(ex) $\boxed{3}$ Sequence 2, x, y, 9. 1st 3 terms: arithmetic
last 3 terms: geometric

x? y?

Soln:

method 1

$$\left[\begin{aligned} x &= \frac{2+y}{2} \\ y &= \sqrt{9x} \end{aligned}\right.$$

$\Rightarrow y^2 = 9x$

$= 9\left(\frac{2+y}{2}\right)$

$2y^2 - 9y - 18 = 0$

$y = \textcircled{6}$ or $-\frac{3}{2}$

$x = \textcircled{4}$ or $\frac{1}{4}$

Method 2: (longer way)

$$\begin{cases} x = 2+d \\ y = 2+2d \end{cases} \qquad \begin{cases} y = rx \\ 9 = r^2x \end{cases}$$

$\downarrow \qquad\qquad \downarrow$

$y = 2+2(x-2) \qquad 9+\left(\frac{y}{x}\right)^2 x$

$$\left[\begin{aligned} y &= 2x-2 \\ 9 &= \frac{y^2}{x} \end{aligned}\right.$$

Solve ...

ⓔⓧ In an arithmetic series, $S_n = 3n^2+2n$
$\boxed{4}$ Find the first three terms.

Soln: arithmetic sequence $a_1, a_2, a_3, \dots$ $\quad a_i = a_1+(i-1)d$
arithmetic series $a_1+a_2+a_3+\dots$

$$a_1 = S_1 = 3(1)^2+2(1) = \boxed{5}$$

$$= S_2 = 3(2)^2+2(2) = 16 \Rightarrow \overset{}{a_2} = \boxed{11} \Rightarrow d = 6$$

$$S_2 = a_1+a_2 \qquad a_2 = a_1+d$$

$$a_3 = a_2+d = 11+6 = \boxed{17}$$

ⓔⓧ 1, 2, 4, ... Geometric Sequence
$\boxed{5}$ Sum of first 7 terms?

Soln: $r = \frac{2}{1} = \frac{4}{2} = 2$

$$a_1 + ra_1 + r^2a_1 + \dots + r^6a_1 = a_1 \sum_{i=1}^{6} r =$$

$$= a_1 \frac{1-r^7}{1-r} = 1 \; \frac{1-2^7}{1-2} = \boxed{127}$$

ⓔⓧ Repeating decimal
$\boxed{6}$ 0.4545... as fraction?

Soln: $\frac{45}{100} + \frac{45}{(100)^2} + \frac{45}{(100)^3} + \dots = 45\left[\sum_{i=0}^{\infty}\left(\frac{1}{100}\right)^i - 1\right]$

$$45\left[\frac{1}{1-\frac{1}{100}} - 1\right]$$

or, $r = \frac{1}{100}$, $a_1 = \frac{45}{100}$

$$a_1 \sum_{i=0}^{\infty} r^i = \frac{a_1}{1-r} = \frac{45/100}{1-1/100} = \frac{45}{99} = \boxed{\frac{5}{11}}$$

Check with a calculator
$0.4545454545 \xrightarrow{\text{FRAC}} \frac{5}{11}$

$\boxed{7}$ $0.\underline{7178}\underline{7178}\dots = \frac{7178}{1000} + \frac{7178}{(1000)^2} + \dots = \frac{7178/1000}{1-\frac{1}{1000}}$

$$= \boxed{7178/9999}$$

Counting

- Factorials: $n!$ "n prime", "n factorial"

$$\boxed{n! = n(n-1)(n-2)\cdots 2\cdot 1} \quad (n \text{ an integer})$$

ⓔⓧ $\frac{(n+2)!-(n+1)!}{n!}$ Simplify
$\boxed{8}$

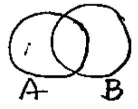
Soln: $(n+2)(n+1) - (n+1) = (n+1)(n+2-1) = \boxed{(n+1)^2}$

ⓔⓧ $2 + (-\frac{1}{2}) + \frac{1}{8} + (-\frac{1}{32}) + \dots = ?$
$\boxed{9}$

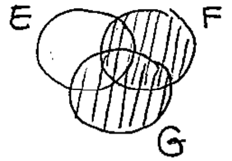
Soln: $\frac{-\frac{1}{2}}{2} = -\frac{1}{4}$, $\frac{\frac{1}{8}}{-\frac{1}{2}} = -\frac{1}{4}$ Geometric series

$$a_1 = 2, \; r = -\frac{1}{4}, \; a_1\sum_{i=0}^{\infty} r^i = a_1 \frac{1}{1-r} = \frac{2}{1-\frac{1}{4}} = \boxed{\frac{8}{3}}$$

- Venn Diagrams



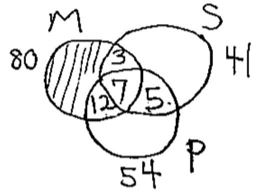
$$N(A\cup B) = N(A) + N(B) - N(A\cap B)$$



$$(E^c \cap G) \cup F$$

ⓔⓧ Among seniors, 80 take math, 41 take
$\boxed{10}$ Spanish, 54 take physics. ⑩ take math and Spanish; ⑲ take math & physics; and ⑫ take physics and Spanish. Seven take all three. How many seniors take math but not Spanish or physics?

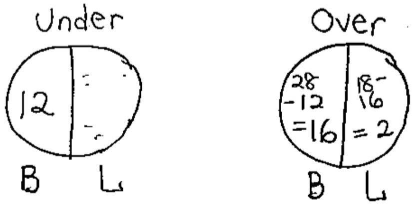
Soln:
80 total



$$80-(3+7+12) = \boxed{58}$$

ⓔⓧ $\boxed{11}$ There are 50 pets. 28 are birds, and 32 are under the age of 1 year. 12 are birds under the age of 1 year. How many lizards over the age of 1 year are in the room?

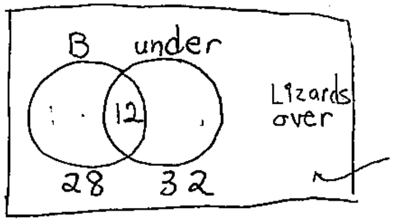
Soln: Method 1: 50 total



32 $\qquad$ 50-32=18

$\boxed{2}$ Lizards over 1 year old

Method 2:



$50-(28+32-12)$
$= 50-48 = \boxed{2}$

n different objects. Select r of them

Maxwell-Boltzmann Distribution $\boxed{17}$ n^r	Bose-Einstein (A) $\binom{n+r-1}{r}$	w/replacement (unlimited supply)
$\boxed{12}$ $_nP_r$	Fermi-Dirac $\boxed{13}$ $_nC_r$	w/o replacement: Take r of n items
ordered samples (Distinguishable AB ≠ BA)	Unordered (Indistinguishable Bag AB = BA)	

* Generalized Counting Principle $\boxed{16}$
$E_1, E_2, \ldots E_k$ are sets with
$n_1, n_2, \ldots n_k$ elements respectively
$\Rightarrow$ $n_1 \times n_2 \times \ldots \times n_k$ ways to choose one element from E_1, then one element from $E_2, \ldots$

ⓔⓧ There are n routes from Town A to B and m routes from Town B to C. For each way you get from A to B, there are n ways to get from B to C, so there are $n \times m$ ways to go from A to C via Town B. $A \xrightarrow[n \text{ routes}]{} B \xrightarrow[m \text{ routes}]{} C$

- Permutation: $_nP_r$. There are n different objects (books). Arrange r of them, where order matters (on a bookshelf)

$$\underbrace{n\,(n-1)\cdots(n-r+1)}_{r \text{ spots}} \qquad \boxed{{}_nP_r = \frac{n!}{(n-r)!}}$$

ⓔⓧ $\boxed{12}$ 20 members in team. Choose one president, one vice president, one treasurer. How many ways?

Soln: $\underset{P}{\underline{20}} \times \underset{V}{\underline{19}} \times \underset{T}{\underline{18}}$

For each way you choose a president, there are 19 ways you can choose a VP. And for each prez and VP, there are 18 ways to choose a treasurer

★ Power Set of A is the set of all subsets of A. If A has n elements, its power set has 2^n elements

- Each subset of A either has or does not have an element in A
$A=\{a_1, a_2, \dots, a_n\}$

$\leftarrow \underbrace{\underline{2} \times \underline{2} \times \dots \times \underline{2}}_{n}$

yes or no : the set has elt a_1

yes or no elt a_2

2^n possible yes/no patterns

- Combinations $\boxed{{}_nC_r = \binom{n}{r} = \frac{n!}{(n-r)!\,r!} = \binom{n}{n-r} \quad \binom{n}{0} = \binom{n}{n} = 1}$

There are n different objects. Take r of them, (without replacement) where order does not matter.

There are nC_r ways to do this.

Reasoning: $\underbrace{n(n-1)\dots(n-r+1)}_{r \text{ spots}} = {}_nP_r = \underbrace{{}_nC_r}_{\text{order does not matter}} \times \underbrace{r!}_{\#\text{ repeats}}$

ways to arrange r objects where order matters

ⓔⓧ 4 candies
C_1, C_2, C_3, C_4

Choose 3 of them, order does not matter

$$\underbrace{4\times 3\times 2}_{} = \underbrace{{}_4C_3}_{} \times \underbrace{3\times 2\times 1}_{}$$

ways to order 3 candies
$C_1C_2C_3 \neq C_2C_1C_3$

order doesn't matter
$C_1C_2C_3$
& $C_2C_1C_3$
$C_1C_3C_2$
$C_2C_3C_1$
$C_3C_1C_2$
$C_3C_2C_1$ same

For each combination of 3 candies, there are this many permutations (repetitions)

ⓔⓧ $\boxed{13}$ 20 members in team. Make a 3-person committee. How many possible committees?

Soln: Unlike (ex12), order doesn't matter.
If Mary, Alice, and Bill are on the committee, it doesn't matter if it's MAB, MBA, AMB, etc.

$$\binom{20}{3} = \frac{20!}{17!\,3!} = \frac{20\times 19\times 18}{3\times 2} = \boxed{1140}$$

★ Repetitions
$n = n_1+n_2+\dots+n_k$. There are n objects of k different types, where n_1 objects are type 1, n_2 are type 2, … n_k are type k.
The number of distinguishable permutations is $\dfrac{n!}{n_1!\,n_2!\dots n_k!}$

ⓔⓧ $\boxed{14}$ (A) How many permutations of the letters in STANFORD?

Soln: 8 different letters, $\boxed{8!}$ distinguishable perms.

(B) How many permutations of BERKELEY? Call it P

Soln: The 3 E's are indistinguishable

8! counts these as diferent

$$\left\{\begin{array}{ll} BYE_1RE_2LE_3K & BYE_2RE_3LE_1K \\ BYE_1RE_3LE_2K & BYE_3RE_1LE_2K \\ BYE_2RE_1LE_3K & BYE_3RE_2LE_1K \end{array}\right.$$

Wherever the 3 E's are, there are 3! indistinguishable permutations,

So $8! = P\cdot 3! \Rightarrow P = \frac{8!}{3!}$

ⓔⓧ $\boxed{15}$ How many distinguishable permutations of letters in BOOKSHOPS?
9 letters, 3 O's, 2 S's

$$\frac{9!}{3!\,2!} = \boxed{30,240}$$

ⓔⓧ There is an unlimited supply of 6 (n) colors of candy. Choose 3 candies and put them in a bag. How many 3-color combinations could be in the bag?

Soln: Imagine $\overset{r}{3}$ circles and $\overset{n-1}{5}$ sticks

○○○||||| means the bag has 3 candies of color 1 and no candy of any other color

○|○|||○| means the bag has 1 color-1 candy, 1 color-2 candy, and 1 color-5 candy

The 5 sticks separate the 3 circles into 6 compartments. If a ball is in compartment i, it means a candy of color i was chosen. There are how many permutations of the 3 circles and 5 sticks? $\boxed{\binom{8}{5}=\binom{8}{3}}$ $\boxed{=56}$

$$\underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_\ \_}$$

3+5 = 8 spots. Choose 5 of the spots for the location of the sticks. The other 3 spots will be the location of the circles,

$$\binom{r+n-1}{n-1}=\binom{n+r-1}{r}$$

ⓔⓧ $\boxed{16}$ 5 boys and 6 girls would like to be on the homecoming court, which will consist of 2 boys and 2 girls. How many different courts are possible?

Soln: For each way to pick 2 boys, there's a way to pick 2 girls $\binom{5}{2}\times\binom{6}{2}=\frac{5!}{2!3!}\ \frac{6!}{2!3!}=\boxed{150}$

ⓔⓧ $\boxed{17}$ A bit consists of a 0 or 1. How many n-bit codewords can you make?

Soln: $\underbrace{\overset{0\text{ or }1}{\underline{2}}\times\underline{2}\times\underline{\ }\ \underline{\ }\cdots\times\underline{2}}_{n}=2^n$ n-bit codewords

ⓔⓧ $\boxed{18}$ 10 points on a circle. How many heptagons can you make? How many triangles?

Soln: Choose 7 of the points (unordered) and connect the adjacent ones

$\binom{10}{7}=\binom{10}{3}=$ same number of heptagons as triangles

ⓔⓧ $\boxed{20}$ There are 2 red chairs and 4 blue chairs. How many ways can you arrange them in a row?

Soln: $\underbrace{\underline{R}\ \underline{B}\ \underline{R}\ \underline{B}\ \underline{B}\ \underline{B}}$

6 spots. Choose 4 of them for blue chairs. There are $\boxed{\binom{6}{4}=\binom{6}{2}}$ ways to fix the spots for blue chairs, (i.e. to arrange the chairs)