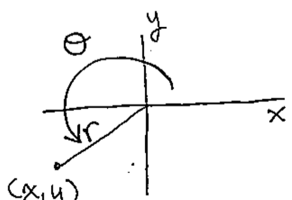
SAI Math Level 2 Prep

Lesson #7 Handout. Th. 8/27/2009

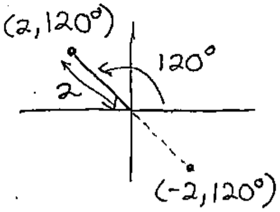
- I. Polar Coordinates, Unit Circle, Right Triangles
- II.
- II. Geometry

I. Polar Coordinates, Unit Circle, Right Triangles



- Rectangular/Cartesian coordinates
  $(x,y) = $ (horizontal position, vertical position)
- Polar Coordinates (r,θ)
  r = distance from origin, (length of terminal side)
  θ = rotation of terminal side from positive x-axis
  $\theta > 0$ ↺
  $\theta < 0$ ↻

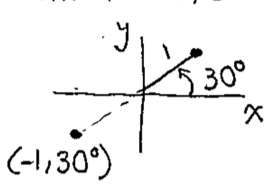
(ex) (2,120°)



$(2,120°) = (2,-240°) = (-2, 300°)$

(ex) [1] Plot the point $(-1, 390°)$

Soln: $(-1, 30°)$. Rotate terminal side of length 1, 30° counterclockwise from positive x-axis. Then go in the other direction.



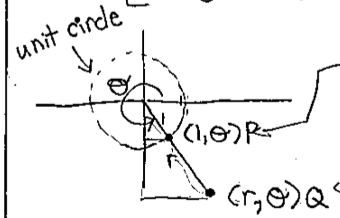
- Similarly for sin, tan, csc, sec, cot
- Sign by quadrant

- From polar to rectangular coordinates
  $(r,\theta) \longrightarrow (x,y)$ $\boxed{\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}}$ Always works

Why? Definitions

$\sin\theta \equiv$ y-coordinate on unit circle when rotated θ

$\cos\theta \equiv$ x-coordinate on unit circle

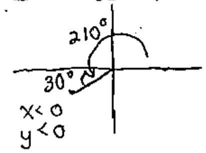


By the definitions, $(1,\theta)$ has rectangular coordinates $(\cos\theta, \sin\theta)$

Point Q in rectangular coordinates just multiplied the x-coordinate and the y-coordinate of point P by r (by similar triangles)

So Q has rectangular coordinates $(r\cos\theta, r\sin\theta)$

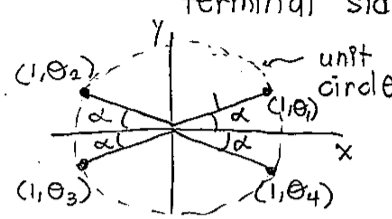
(ex) [2] Convert the polar coordinates $(3, 210°)$ to rectangular coordinates (x,y)



$x = 3\cos 210° = -3\cos 30° = -3\sqrt{3}/2$

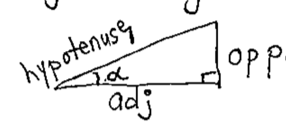
$y = 3\sin 210° = -3\sin 30° = -3(1/2)$

- Useful: Reference Angle = smallest angle between terminal side and x-axis



$0 \le \alpha \le 90°$

Right triangle



$\underbrace{|\cos\theta_1| = |\cos\theta_2| = |\cos\theta_3| = |\cos\theta_4|}$ $= \overbrace{\cos\alpha}$

- same absolute value of x-coordinate
- The sign depends on the quadrant of the terminal side; where is the x coodinate positive or negative?

- Be careful with the calculator: Inverse Trig Functions

$\sin^{-1} z \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\cos^{-1} z \in [0, \pi]$

$\tan^{-1} z \in (-\frac{\pi}{2}, \pi/2)$

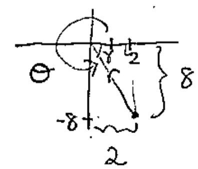
$\boxed{\text{Calculator may give angle in the wrong quadrant!}}$

ex 3 From Rectangular to Polar Coordinates

$(x,y) \longrightarrow (r,\theta)$

$\boxed{r = \sqrt{x^2+y^2}\ (>0) \quad \tan\theta = \frac{y}{x} \text{ Be careful of the quadrant!}}$

a) Convert $(2,-8)$ to polar coord.



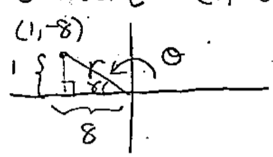
$r = \sqrt{2^2+8^2} = \boxed{\sqrt{68}}$

The calculator always gives the correct answer for a right triangle

$\tan\gamma = 8/2 \qquad \gamma = 75.96°$

$\theta = 360° - \gamma = \boxed{284°}$

b) Convert $(-1,+8)$ to polar coordinates

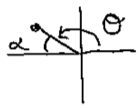


$r = \sqrt{1^2+8^2} = \sqrt{65}$

$\tan\gamma = \frac{1}{8} \Rightarrow \gamma = 71.25°$

$\theta = 180° - \gamma = 172.9°$

ex 4 $\frac{\pi}{2} \le \theta \le \pi$, $\sin\theta = \frac{1}{3}$. Find $\cos\theta$



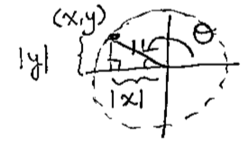
Method 1: $\sin^{-1}\frac{1}{3} = 19.47°$

$\cos 19.47° = 0.943$

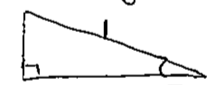
But θ is in Q2, $\cos\theta < 0$ $\quad -0.943$

Method 2:

Think of the unit circle. $\cos\theta = \frac{x\text{-coord on unit circle}}{1}$



$\sin\theta = y\text{coordinate} = \frac{1}{3}$



$\sqrt{1-(\frac{1}{3})^2} = \sqrt{\frac{8}{9}} = 2\sqrt{2}/3$

$\cos\theta = -\frac{2\sqrt{2}/3}{1}$

$\boxed{\text{Summary: Draw the right triangle with hypotenuse length 1 between the terminal side and the x-axis. Sign by quadrant}}$

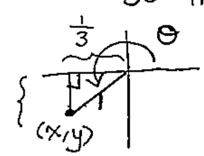
ex 5 $180° \le \theta \le 270°$ $\quad \cos\theta = -\frac{1}{3}$. Find $\tan\theta$

Method 1: $\tan(\cos^{-1}(\frac{-1}{3})) = 2.828$

θ is in quadrant 3, $\tan\theta > 0$

so the answer is $\boxed{+2.828}$

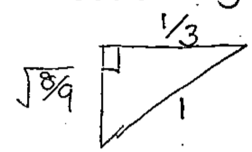
Method 2:



$\sqrt{1-(\frac{1}{3})^2} = \sqrt{8/9}$

$\tan\theta = \frac{y}{x}$ ← both are negative so $\tan\theta > 0$

$\cos\theta = -\frac{1}{3} = \underline{x\text{-coord. on unit circle}}$

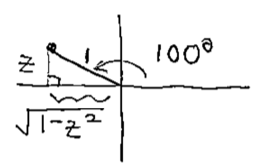


$\tan\theta = \frac{-\sqrt{8/9}}{-1/3} = \boxed{2\sqrt{2}}$

ex (HW6 #25) $\sin 100° = z$ $\quad \sin 200°$ in terms of z?

$\sin 200° = 2\underbrace{\sin 100°}_{} \cos 100°$

z = y-coordinate on unit circle in quadrant 2, so $z > 0$

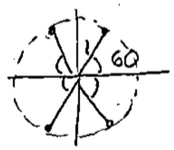


$\cos 100° = -\sqrt{1-z^2}$ = x-coord. on unit circle

$\sin 200° = -2z\sqrt{1-z^2}$

(ex) 6 Find t in the interval $[0, 360°)$ such that $|\cos t| = \frac{1}{2}$

Solution: $\cos 60° = \frac{1}{2}$.



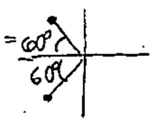
t = any angle whose reference angle is $60°$

$$\boxed{t = \underset{\theta_1}{60°},\ \underset{\theta_2}{180°-60°},\ \underset{\theta_3}{180°+60°},\ \underset{\theta_4}{360°-60°}}$$

since the absolute value of the x-coordinate on the unit circle is the same when the terminal side is rotated by $\theta_1, \theta_2, \theta_3, \theta_4$

(ex) 7 Find t in the interval $[0, 2\pi)$ such that $\cos t = -\frac{1}{2}$

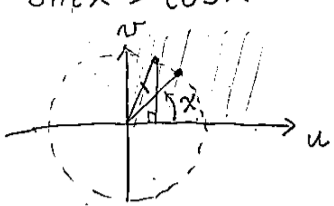
Solution: $\cos 60° = \frac{1}{2}$. Think of the points on the unit circle with reference angle $60°$ and negative x-coordinates



$$t = \boxed{\pi - \frac{\pi}{3},\ \pi + \frac{\pi}{3}}$$

(ex) 8 Find x in the interval $[0, \frac{\pi}{2}]$ such that

$\sin x > \cos x$



$\sin x = v$-coordinate on unit circle when terminal side rotated by x

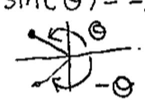
$\cos x = u$-coordinate "

$v = u$ when $\sin x = \cos x$, $x = \pi/4$

$v > u$ when the height is bigger than the base

$$\boxed{\frac{\pi}{4} < x \leq \frac{\pi}{2}}$$

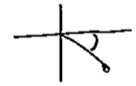
Note: $\sin(-\theta) = -\sin\theta$ odd



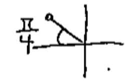
$\cos(-\theta) = \cos\theta$ even

$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = -\tan(\theta)$ odd

(ex) 9 $\tan(-60°) = \underset{\substack{\uparrow \\ y<0 \\ x>0}}{-} \tan 60° = -\sqrt{3}$

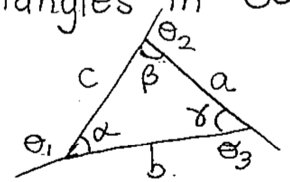


$\cos\frac{3\pi}{4} = \underset{\substack{\uparrow \\ x<0}}{-}\cos\frac{\pi}{4}$



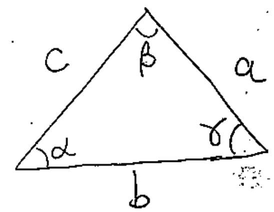
II. Geometry

① Triangles in General



$$\boxed{\begin{array}{ll} \alpha+\beta+\gamma = 180° & \text{interior} \\ \theta_1+\theta_2+\theta_3 = 360° & \text{exterior} \\ \theta_1 = \beta+\gamma & \\ \text{Area} = \frac{1}{2}\text{ base} \times \text{height} & \\ |b-c| < a < b+c & \end{array}}$$

Triangle Inequality: $|b-c| < a < b+c$



Law of Sines

$$\boxed{\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}}$$

↑ Useful when given

- 2 sides and angle opposite
- 2 angles and side opposite
- You have 2 pairs of opp. side & ∠

Law of Cosines

$$\boxed{\begin{aligned} c^2 &= a^2+b^2-2ab\cos\gamma \\ b^2 &= a^2+c^2-2ac\cos\beta \\ a^2 &= b^2+c^2-2bc\cos\alpha \end{aligned}}$$

← Useful when given

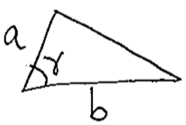
- 3 sides
- 2 sides and angle between

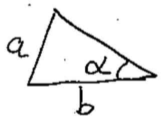
$$\boxed{\begin{aligned} \text{Area} &= \tfrac{1}{2}\text{ base} \times \text{height} \\ &= \tfrac{1}{2}ab\sin\gamma \\ &= \tfrac{1}{2}bc\sin\alpha \\ &= \tfrac{1}{2}ac\sin\beta \end{aligned}}$$

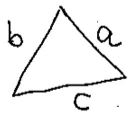
← Useful when given

- 2 sides and angle between

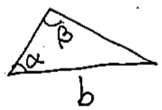
ex 10 What Law would you apply?

i) 

iv) 

ii) 

v) $a:b:c = 1:2:3$

iii) 

vi) 

Soln:

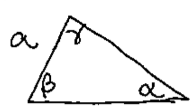
i) Law of Cosines or Area formula (2 sides & angle betw)

ii) Law of Cosines (3 sides)

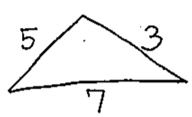
iii) Law of Sines (2 angles & side opposite β)

iv) Law of Sines (2 sides & angle opposite)

v) Law of Cosines (essentially 3 sides)

vi)  $\Rightarrow$  2 angles and side opposite α $\rightarrow$ Law of Sines

$\alpha = 180^\circ - \beta - \gamma$

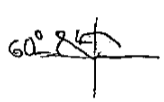
ex 11  Find the degree of the largest angle

Soln: Largest angle is opposite longest side

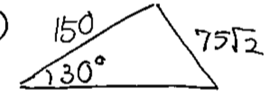
Given 3 sides $\rightarrow$ Use Law of Cosines

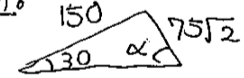
$7^2 = 5^2 + 3^2 - 2(5)(3)\cos\gamma$

$\cos\gamma = -\frac{1}{2}$ Find all γ in $[0,\pi]$



$\gamma = \boxed{120^\circ}$

ex 12  Find the other two angles

Soln: 

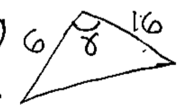
2 sides & angle opp. $\rightarrow$ Law of Sines

$$\frac{150}{\sin\alpha} = \frac{75\sqrt{2}}{\sin 30^\circ}$$

$\sin\alpha = \frac{\sqrt{2}}{2}$ Find all α in $[0,\pi]$

$\alpha = 135^\circ$ or 45°

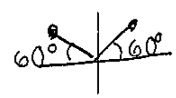
$\beta = 180^\circ - 135^\circ - 30^\circ = 15^\circ$ or $180^\circ - 45^\circ - 30^\circ = 105^\circ$

ex 13  area $= 24\sqrt{3}$ $\gamma = ?$

Soln: 2 sides & angle between $\rightarrow$ Area Formula

$\frac{1}{2}6(16)\sin\gamma = 24\sqrt{3}$

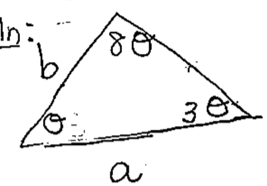
$\sin\gamma = 0.866$ Find all $\gamma \in [0,\pi]$



$\boxed{\gamma = 60^\circ \text{ or } 120^\circ}$

2 solutions!

ex 14 The angles of a triangle are in a ratio of 8:3:1. Ratio of longest side to next longest side?

Soln: 

$a:b$?

$$\frac{a}{\sin(8\theta)} = \frac{b}{\sin(3\theta)}$$

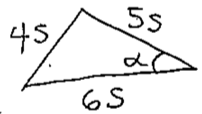
$\theta + 3\theta + 8\theta = 180^\circ \Rightarrow \theta = 15^\circ$

$$\Rightarrow \frac{a}{b} = \frac{\sin 120^\circ}{\sin 45^\circ} = \frac{\sqrt{3}/2}{\sqrt{2}/2}$$

$a:b = \boxed{\sqrt{3}:\sqrt{2}}$

Pn4

(ex) 15. The sides of a triangle are in ratio 4:5:6. The smallest angle is?

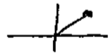
Soln: 

3 sides → law of cosines

let $s=1$

$4^2 = 5^2 + 6^2 - 2(5)(6)\cos\alpha$

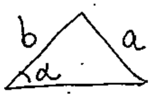
$\cos\alpha = 0.75$ Find all $\alpha \in [0, \pi]$



$\boxed{\alpha = 41.4^\circ}$

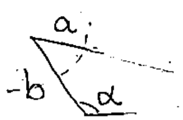
Triangles in General

Given 2 sides and an opposite angle how many triangles can you form?



Given α, a, b

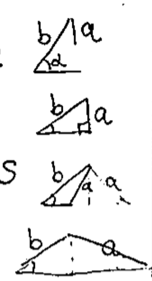
α obtuse $\alpha \geq 90^\circ$:



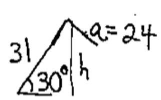
$\begin{cases} a \leq b & \text{no triangle} \\ a > b & \text{1 triangle} \end{cases}$

α acute $0 < \alpha < 90^\circ$:

$\begin{cases} a < b\sin\alpha & \text{no triangle} \\ a = b\sin\alpha & \text{1 triangle} \\ b\sin\alpha < a < b & \text{2 triangles} \\ a \geq b & \text{1 triangle} \end{cases}$



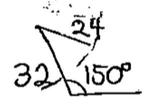
(ex) 16. How many triangles can be formed if $a=24$, $b=31$, $\angle A = 30^\circ$?

Soln: 

$h = 31\sin 30^\circ \approx 31 \cdot \frac{1}{2} \approx 15$

$h < a < b \Rightarrow$ 2 triangles

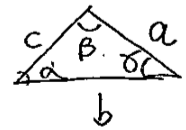
(ex) 17. $a=24$, $b=32$, $\angle A = 150^\circ$. How many triangles can be formed?



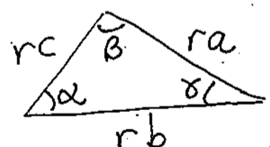
Soln: $a < b$ No triangle

Similar Triangles

Area ratio is square of side ratio



area $= A$

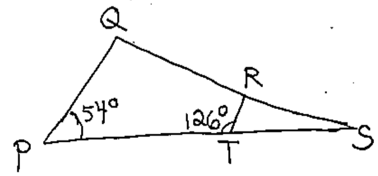


area $= \boxed{r^2 A}$

Why?

$\frac{1}{2}(ra)(rb)\sin\gamma$

$= (\frac{1}{2}ab\sin\gamma)\, r^2$

(ex) 18. 

$\overline{QR} = \overline{RS}$

area of ΔRST is $\frac{c}{2}$

Find area of ΔQSP

Soln: 

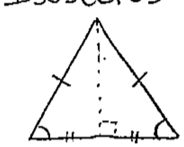
$180^\circ - 126^\circ = 54^\circ$

$\angle S$ is the same for both triangles

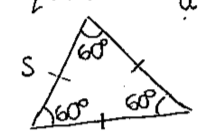
$\Rightarrow$ they are similar

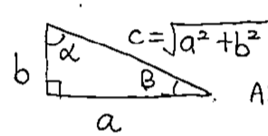
The sides of ΔQSP are 2 times the sides of ΔRST. So the area of ΔQSP is $2^2 \frac{c}{2} = \boxed{2c}$

Special Triangles

Isosceles 

Equilateral: area $= \frac{1}{2}s^2 \underbrace{\sin 60^\circ}_{\frac{\sqrt{3}}{2}} = \frac{s^2\sqrt{3}}{4}$

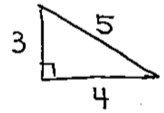


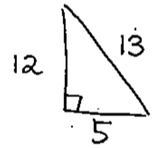
Right 

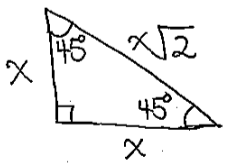
Area $= \frac{1}{2}ab$

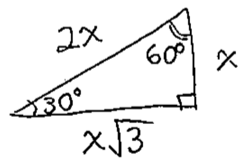
Special Right Triangles

Pythagorean Triplets:

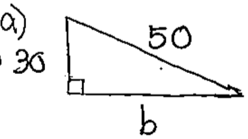








ex a) 15.2



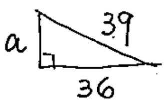
$b = ?$ Soln: 3-4-5 right triangle

30 - b - 50

$b = 40$

(Any 3-4-5 or 5-12-13 triangle must be a right triangle!)

b)

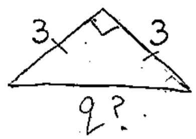


$a = ?$ Soln: 5-12-13 Δ

a - 36 - 39

$\boxed{3} \cdot 5 = \boxed{15}$

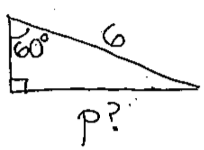
c)



Soln: 45°-45°-90° triangle

$q = 3\sqrt{2}$

d)

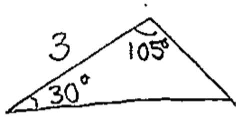


Soln: 30°-60°-90° Δ $6 = 2x$

$p = x\sqrt{3} = 3\sqrt{3}$

---

ex 16.2

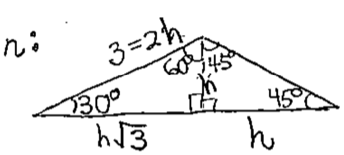


Area?

Method 2: $\frac{3}{\sin 45°} = \frac{a}{\sin 105°}$

$\frac{1}{2} a\, 3 \sin 30° = 3.07$

Soln:



$\text{Area} = \frac{1}{2} h\sqrt{3}\, h + \frac{1}{2} h\, h$

$h = \frac{3}{2} \Rightarrow \text{Area} = \frac{9}{8}(1+\sqrt{3})$

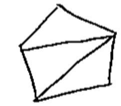
$= 3.07$

- **Polygon:** With n sides has angles that sum up to $\boxed{(n-2)180°}$

pentagon n=5, hexagon n=6, octagon n=8

$\alpha + \beta + \gamma = 180°$

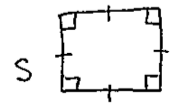
sum = 360°



sum = 3 (180°)
= 540°

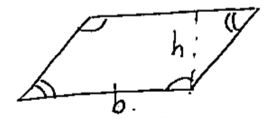
- **Quadrilaterals: Polygon with 4 Sides**

Square



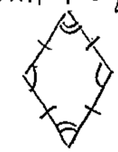
area = s^2

Parallelogram

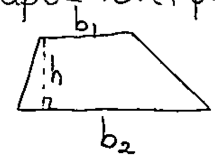


area = bh

Rhombus
(Parallelogram with 4 equal sides)

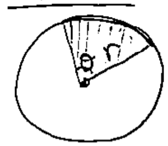


Trapezoid: (1 pair parallel, 1 pair nonparallel)



area = $\frac{1}{2}(b_1 + b_2)h$

- **Circles**

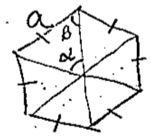


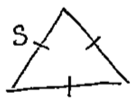
- Circumference = $2\pi r$
- area = πr^2
- area of sector = $\frac{1}{2} r^2$ (θ radians)
  $= \left(\frac{\theta \text{ degrees}}{360°}\right)(\pi r^2)$

- arc length = r (θ radians)
  $= \left(\frac{\theta \text{ degrees}}{360°}\right)(2\pi r)$

ex 17 An equilateral triangle and a regular hexagon have the same perimeter. The area of the hexagon is $72\sqrt{3}$. Area of the triangle?

Soln:

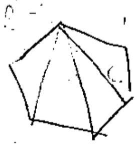




Triangle area $= \frac{1}{2}s^2 \sin 60° = \frac{s^2\sqrt{3}}{4}$

Perimeter $3s = 6a \Rightarrow s = 2a$

$\alpha = \frac{360°}{6} = 60°$



Hexagon angles add to $4(180°)$

$\beta = \frac{4(180°)}{6 \cdot 2} = 60° \Rightarrow$ 6 Equilateral triangles

Hexagon Area:

$\Rightarrow 6\left(\frac{a^2\sqrt{3}}{4}\right) = 72\sqrt{3}$

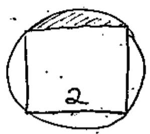
$a^2 = 48$

Area triangle

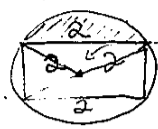
$= \frac{s^2\sqrt{3}}{4} = a^2\sqrt{3} = \boxed{48\sqrt{3}}$

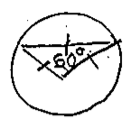
(2a)

ex 18 The rectangle with one side of length 2 is inscribed in a circle of radius 2. What's the area of the shaded part?



Soln:



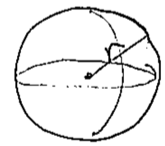


$\frac{1}{2}r^2\theta$ − area of equilateral triangle

$= \frac{1}{2}2^2\left(\frac{\pi}{3}\right) - \frac{2^2\sqrt{3}}{4} = \boxed{0.362}$

Solid Geometry

Sphere

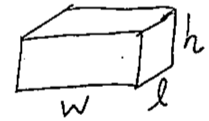


Volume $= \frac{4}{3}\pi r^3$

$SA = 4\pi r^2$

surface area

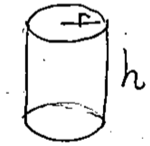
Box



$V = wlh$

$SA = 2(wl + lh + wh)$

Right Circular Cylinder



$V = \pi r^2 h$

$SA = 2(\pi r^2) + (2\pi r)h$

Right Circular Cone

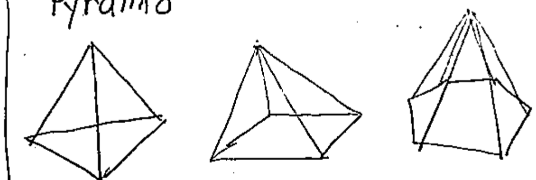
$V = \frac{1}{3}\pi r^2 h$

Lateral area $= \frac{1}{2}cl$

(c = circumference)

SA = Lateral area + πr^2

Pyramid

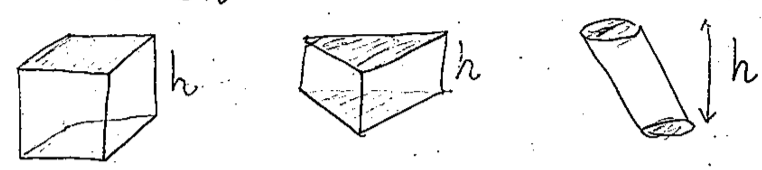


Volume $= \frac{1}{3}Bh$

where B = area of base

Uniform Solid: the cross-section is always the same

Volume = Bh



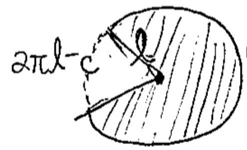
B = area of base

Equation of a sphere centered at (h,k,l), radius r

$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$

• Why is the lateral area of a right circular cone with base circumference c and slant height ℓ equal $\frac{1}{2}c\ell$?

c = circumf.

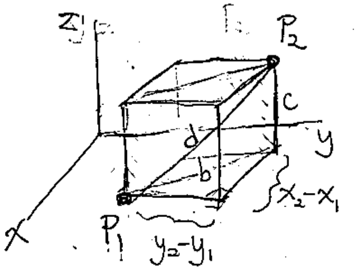


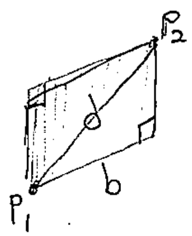
area $= \pi\ell^2 - \underbrace{\text{area of sector}}$

$\frac{1}{2}\ell^2\theta = \frac{1}{2}\ell(\underset{\sim}{\ell\theta})$ arc length

$$\text{Area} = \pi\ell^2 - \frac{1}{2}\ell(2\pi\ell - c)$$
$$= \pi\ell^2 - \pi\ell^2 + \boxed{\frac{1}{2}c\ell}$$

• The distance formula in 3D: Apply the Pythag. Thm twice





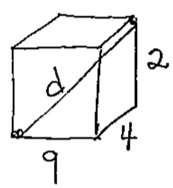
$\Big\} c = (z_2 - z_1)$

$$d(P_1,P_2) = \sqrt{b^2+c^2}$$

$$b^2 = (x_2-x_1)^2 + (y_2-y_1)^2$$

$$\Rightarrow d(P_1,P_2) = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

ex 19

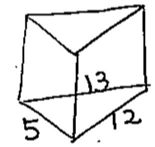


Box. $d = ?$ (the diagonal)

Soln: $d = \sqrt{2^2+4^2+9^2} = \sqrt{101}$

(like #41) $\boxed{\text{distance} = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}}$

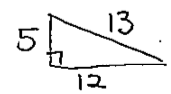
ex 20



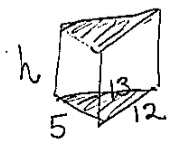
Triangle with sides of lengths 5, 13, 12 forms the base of this right uniform solid. Surface area = 360. Volume?

Soln: $SA = 2B + \text{side areas}$ $\quad V = Bh$. h? B?

5-12-13 triangle (Pythagorean triplet). Right triangle!



$B = \frac{1}{2}5(12) = 30$



$SA = 2\overset{30}{B} + 5h + 13h + 12h = 360$

$\Rightarrow h = 10$

$V = Bh = 30(10) = \boxed{300}$

ex 21 If the volume of a right circular cylinder is reduced by 15% by reducing its height by 5%, by what percent must the radius of the base be reduced?

Soln: $\boxed{\text{"Reduce } r \text{ by } p\%\text{" means to decrease the radius to } r - \frac{p}{100}r = (1-\frac{p}{100})r}$

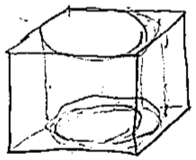
	Before	After
Volume	$\frac{1}{3}\pi r^2 h$	$0.85(\frac{1}{3}\pi r^2 h)$
height	h	$0.95h$
radius	r	$r_2 = (1-\frac{p}{100})r$

Volume after: $0.85\,\cancel{\tfrac{1}{3}\pi}r^2\cancel{h} = \cancel{\tfrac{1}{3}\pi}r_2^2(0.95\cancel{h})$

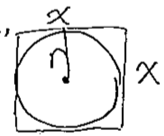
$$(1-\tfrac{p}{100})\cancel{r} = r_2 = \sqrt{\frac{0.85}{0.95}}\,\cancel{r}$$

$$1 - \frac{p}{100} = \sqrt{\frac{0.85}{0.95}} \Rightarrow p = \boxed{5.4\%}$$

(ex) 22 Right Circular cone inscribed in cube with edge of length x. Volume of cylinder
to volume of cube? ratio of



Soln,

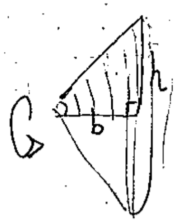


$r = \frac{x}{2}$

$$\frac{V_{cylinder}}{V_{cube}} = \frac{\pi\left(\frac{x}{2}\right)^2 x}{x^3} = \boxed{\frac{\pi}{4}}$$

(ex) 23 When a right triangle of area 4 is rotated $360°$ about its longer leg, the solid has volume 16. What's the volume when the same right triangle is rotated about its shorter leg?

Soln:



$V_1 = \frac{1}{3}\pi b^2 h = 16$, $\frac{1}{2}bh = 4$

$\Rightarrow bh = 8$

$\frac{1}{3}\pi\left(\frac{8^2}{h^{\cancel{2}}}\right)\cancel{h} = 16$ $\quad b = \frac{8}{h} = 1.91$

$h = 4.19$

$V_2 = \frac{1}{3}\pi h^2 b$

$= \frac{1}{3}\pi \underbrace{(bh)}_{8} h \approx \boxed{35.1}$