Now, by the chain rule,

$$\frac{\partial v_1}{\partial u} = \frac{\partial v_1}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v_1}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial v_1}{\partial z}\frac{\partial z}{\partial u}, \qquad \frac{\partial v_1}{\partial v} = \frac{\partial v_1}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v_1}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial v_1}{\partial z}\frac{\partial z}{\partial v}.$$

Therefore

$$\begin{aligned}(\boldsymbol{\nabla} \times v_1\mathbf{i}) \cdot \mathbf{N} &= \left(\frac{\partial v_1}{\partial u} - \frac{\partial v_1}{\partial x}\frac{\partial x}{\partial u}\right)\frac{\partial x}{\partial v} - \left(\frac{\partial v_1}{\partial v} - \frac{\partial v_1}{\partial x}\frac{\partial x}{\partial v}\right)\frac{\partial x}{\partial u} \\ &= \frac{\partial v_1}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial v_1}{\partial v}\frac{\partial x}{\partial u}\end{aligned}$$

and, as asserted,

$$\iint_S [(\boldsymbol{\nabla} \times v_1\mathbf{i}) \cdot \mathbf{n}]\, d\sigma = \iint_\Gamma \left[\frac{\partial v_1}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial v_1}{\partial v}\frac{\partial x}{\partial u}\right] du\, dv.$$

REVIEW EXERCISES

1. (a) $\mathbf{r}(u) = u\mathbf{i} + u\mathbf{j}, \quad 0 \le u \le 1; \quad \int_C \mathbf{h} \cdot d\mathbf{r} = \int_0^1 (u^3 - u^2)\, du = -\frac{1}{12}$

(b) $\int_C \mathbf{h} \cdot d\mathbf{r} = \int_0^1 (2u^8 - 3u^7)\, du = -\frac{11}{72}$

2. (a) $\int_C \mathbf{h} \cdot d\mathbf{r} = \int_0^{\pi/2} (-\cos^3 u \, \sin u + \sin^3 u \cos u)\, du = 0$

(b) $\int_C \mathbf{h} \cdot d\mathbf{r} = \int_0^{\pi/2} (-3\cos^{11} u \sin u + 3\sin^{11} u \cos u)\, du = 0$

3. Since $\mathbf{h}(x, y) = \nabla f$ where $f(x, y) = x^2y^2 + \frac{1}{2}x^2 - y$,

$$\int_C \mathbf{h}(\mathbf{r}) \cdot \mathbf{dr} = f(2, 4) - f(-1, 2) = \frac{119}{2}$$

for *any* curve C beginning at $(-1, 2)$ and ending at $(2, 4)$.

4. $\mathbf{h}(x, y)$ is a gradient: $\frac{\partial P}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial Q}{\partial x}; \quad \mathbf{h}(x, y) = \nabla \arctan(y/x).$

Therefore the integrals in (a), (b) and (c) all have the same value.

(a) $\mathbf{r}(u) = 2\cos u\,\mathbf{i} + 2\sin u\,\mathbf{j}, \quad 0 \le u \le \frac{3}{4}\pi;$

$$\int_C \mathbf{h} \cdot d\mathbf{r} = \int_0^{3\pi/4} \left[\frac{-2\sin u}{4}(-2\sin u) + \frac{2\cos u}{4}(2\cos u)\right] du = \int_0^{3\pi/4} 1\, du = \frac{3\pi}{4}$$

5. $\mathbf{h}(x,y,z) = \sin y\,\mathbf{i} + xe^{xy}\,\mathbf{j} + \sin z\,\mathbf{k}; \quad \mathbf{r}(u) = u^2\,\mathbf{i} + u\,\mathbf{j} + u^3\,\mathbf{k}, \quad u \in [\,0,3\,]$

$x(u) = u^2 \quad y(u) = u \quad z(u) = u^3, \quad x'(u) = 2u, \quad y'(u) = 1, \quad z'(u) = 3u^2$

$\mathbf{h}(\mathbf{r}(u)) \cdot \mathbf{r}'(u) = 2u\sin u + u^2 e^{u^3} + 3u^2 \sin u^3$

$$\begin{aligned}\int_C \mathbf{h}(\mathbf{r}) \cdot \mathbf{dr} &= \int_0^3 \left(2u\sin u + u^2 e^{u^3} + 3u^2 \sin u^3\right) du \\ &= \left[-2u\cos u + 2\sin u + \tfrac{1}{3}e^{u^3} - \cos u^3\right]_0^3 \\ &= \tfrac{2}{3} - 6\cos 3 + 2\sin 3 + \tfrac{1}{3}e^{27} - \cos 27\end{aligned}$$

6. $\mathbf{h}(x,y,z) = x^2\,\mathbf{i} + xy\,\mathbf{j} + z^2\,\mathbf{k}; \quad \mathbf{r}(u) = \cos u\,\mathbf{i} + \sin u\,\mathbf{j} + u^2\,\mathbf{k}, \quad u \in [0,\,\pi/2]$

$\mathbf{h}(\mathbf{r}(u)) \cdot \mathbf{r}'(u) = 2u^5; \quad \displaystyle\int_C \mathbf{h}(\mathbf{r}) \cdot \mathbf{dr} = \int_0^{\pi/2} 2u^5\,du = \frac{1}{3}\left(\frac{\pi}{2}\right)^6$

7. $\mathbf{F}(x,y,z) = xy\,\mathbf{i} + yz\,\mathbf{j} + xz\,\mathbf{k}; \quad \mathbf{r}(u) = u\,\mathbf{i} + u^2\,\mathbf{j} + u^3\,\mathbf{k}.$

$\mathbf{F}(\mathbf{r}(u)) \cdot \mathbf{r}' = u^3 + 5u^6; \quad W = \displaystyle\int_{-1}^{2} (u^3 + 5u^6)du = \left[\frac{1}{4}u^4 + \frac{5}{7}u^7\right]_{-1}^{2} = \frac{2685}{28}$

8. $\mathbf{F}(x,y) = x\,\mathbf{i} + (y-2)\,\mathbf{j}; \quad \mathbf{r}(u) = (u - \sin u)\,\mathbf{i} + (1 - \cos u)\,\mathbf{j}, \ 0 \le u \le 2\pi.$

$\mathbf{F}(\mathbf{r}(u)) \cdot \mathbf{r}' = u - u\cos u - 2\sin u;$

$W = \displaystyle\int_0^{2\pi} (u - u\cos u - 2\sin u)du = \left[\frac{1}{2}u^2 + u\sin u + 3\cos u\right]_0^{2\pi} = 2\pi^2$

9. A vector equation for the line segment is: $\mathbf{r}(u) = (1+2u)\,\mathbf{i} + 4u\,\mathbf{k}, \quad u \in [\,0,1\,].$

$\mathbf{F}(\mathbf{r}(u)) \cdot \mathbf{r}' = C\,\dfrac{2+20u}{\sqrt{1+4u+20u^2}}; \quad \displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = C\int_0^1 \frac{(20u+2)}{\sqrt{1+4u+20u^2}}\,du = 4C$

10. Suppose that the path C of the object is given by the vector function $\mathbf{r} = \mathbf{r}(u), \ a \le u \le b$. Then

$\mathbf{r}' = \mathbf{v}$ is the velocity of the object and $\mathbf{F} \cdot \mathbf{v} = 0$. The work done by $\mathbf{F}$ is

$$\int_C \mathbf{F}(\mathbf{r}) \cdot \mathbf{dr} = \int_a^b \mathbf{F}(\mathbf{r}(u)) \cdot \mathbf{r}'(u)\,du = \int_a^b \mathbf{F}(\mathbf{r}(u)) \cdot \mathbf{v}(u)\,du = 0.$$

11. $\dfrac{\partial(ye^{xy} + 2x)}{\partial y} = e^{xy} + xye^{xy} = \dfrac{\partial(xe^{xy} - 2y)}{\partial x} \implies \mathbf{h}$ is a gradient.

(a) $\mathbf{h}(\mathbf{r}(u)) \cdot \mathbf{r}' = 3u^2 e^{u^3} - 4u^3 + 2u; \quad \displaystyle\int_C \mathbf{h} \cdot d\mathbf{r} = \int_0^2 \left(3u^2 e^{u^3} - 4u^3 + 2u\right) du = e^8 - 13$

(b) Let $f(x,y) = e^{xy} + x^2 - y^2$. Then $\boldsymbol{\nabla} f = \mathbf{h}$ and $\displaystyle\int_C \mathbf{h} \cdot d\mathbf{r} = f(2,4) - f(0,0) = e^8 - 13$

12. $\dfrac{\partial P}{\partial y} = 4xy + 2 = \dfrac{\partial Q}{\partial x} \implies \mathbf{h}$ is a gradient.

(a) $\mathbf{h}(\mathbf{r}(u)) = \displaystyle\int_C \mathbf{h} \cdot d\mathbf{r} = \int_0^1 \left(6 + 66u + 216u^2 + 576u^3\right) du = \Big[6u + 33u^2 + 72u^3 + 144u^4\Big]_0^1 = 255$

(b) Let $f(x, y) = (x^2y^2 + 2xy)$. Then $\nabla f = \mathbf{h}$ and $\displaystyle\int_C \mathbf{h} \cdot d\mathbf{r} = f(3, 5) - f(0, 1) = 255$

13. $\mathbf{h}(x, y, z) = \nabla f$ where $f(x, y, z) = x^4y^3z^2$.

(a) $\mathbf{h}(\mathbf{r}(u)) = 4u^{15}\,\mathbf{i} + 3u^{14}\,\mathbf{j} + 2u^{13}\,\mathbf{k}$; $\quad \mathbf{r}'(u) = \mathbf{i} + 2u\,\mathbf{j} + 3u^2\,\mathbf{k}$

$$\int_C \mathbf{h}(\mathbf{r}) \cdot \mathbf{dr} = \int_0^1 16\,u^{15}\,du = 1.$$

(b) $\displaystyle\int_C \mathbf{h}(\mathbf{r}) \cdot \mathbf{dr} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(1, 1, 1) - f(0, 0, 0) = 1.$

14. (a) $\mathbf{r}(u) = u\mathbf{i} + 4u\mathbf{j}, \quad 0 \le u \le 2$

$$\int_C y^2\,dx + (x^2 - xy)\,dy = \int_0^2 [16u^2 + 4(u^2 - 4u^2)]\,du = \int_0^2 4u^2\,du = \frac{32}{3}$$

(b) $C_1 : \mathbf{r}(u) = u\mathbf{i}, \quad 0 \le u \le 2; \quad C_2 : \mathbf{r}(u) = 2\mathbf{i} + u\mathbf{j}, \quad 0 \le u \le 8$

$$\int_C y^2\,dx + (x^2 - xy)\,dy = \int_{C_1} y^2\,dx + (x^2 - xy)\,dy + \int_{C_2} y^2\,dx + (x^2 - xy)\,dy = 0 + \int_0^8 (4 - 2u)du = -32$$

(c) $C : \mathbf{r}(u) = u\mathbf{i} + u^3\mathbf{j}, \quad 0 \le u \le 2$

$$\int_C y^2dx + (x^2 - xy)dy = \int_0^2 (3u^4 - 2u^6)\,du = -\frac{608}{35}$$

15. (a) $\mathbf{r}(u) = (1 - u)\mathbf{i} + u\,\mathbf{j}, \quad 0 \le u \le 1.$

$$\begin{aligned}\int_C 2xy^{1/2}\,dx + yx^{1/2}\,dy &= \int_0^1 \Big[2(1 - u)u^{1/2}(-1) + u(1 - u)^{1/2}\Big]\,du \\ &= -2\int_0^1 (1 - u)u^{1/2}\,du + \int_0^1 u(1 - u)^{1/2}\,du \\ &= -\int_0^1 (1 - u)u^{1/2}\,du = -\frac{4}{15}\end{aligned}$$

(b) $\mathbf{r}_1 = \mathbf{i} + u\,\mathbf{j}, \quad 0 \le u \le 1; \quad \mathbf{r}_2 = (1 - u)\,\mathbf{i} + \mathbf{j}$

$$\int_C 2xy^{1/2}dx + yx^{1/2}dy = \int_0^1 u\,du + \int_0^1 -2(1 - u)\,du = -\frac{1}{2}$$

(c) $\mathbf{r} = \cos\,u\,\mathbf{i} + \sin\,u\,\mathbf{j}, \quad 0 \le u \le \pi/2$

$$\int_C 2xy^{1/2}dx + yx^{1/2}dy = \int_0^{\pi/2} \left(-2\sin^{3/2} u\,\cos\,u + \cos^{3/2} u\,\sin\,u\right) du = -\frac{2}{5}$$

16. $\int zdx + xdy + ydz = \int_0^{2\pi} (a^2\cos^2 u - au\sin u + a\sin u)du = \pi a^2 + 2\pi a$

17.
$$\begin{aligned}\int_C ye^{xy}\,dx + \cos x\,dy + \left(\frac{xy}{z}\right)dz &= \int_0^2 \left(u^2 e^{u^3} + 2u\cos u + 3u^2\right) du \\ &= \left[\tfrac{1}{3}\,e^{u^3} + 2u\sin u + 2\cos u + u^3\right]_0^2 \\ &= \frac{1}{3}e^8 + \frac{17}{3} + 4\sin 2 + 2\cos 2\end{aligned}$$

18. $\mathbf{r} = \cos\,u\,\mathbf{i} + \sin\,u\,\mathbf{j};\quad \lambda(x,y) = k;\quad s'(u) = \|\mathbf{r}'\| = 1.$

(a) $M = \int_C \lambda(x,y)\,ds = \int_0^\pi k\,du = k\pi$

By symmetry, $x_M = 0$.

$y_M\,M = \int_C y\lambda(x,y)\,ds = \int_0^\pi k\sin\,u\,du = \Big[-k\cos\,u\Big]_0^\pi = 2k;\quad y_M = \dfrac{2}{\pi}$

(b)
$$\begin{aligned}I &= \int_C \lambda(x,y)R^2(x,y)\,ds = \int_C kx^2\,ds \\ &= \int_0^\pi k\cos^2 u\,du = \frac{k}{2}\int_0^\pi (1 + \sin\,2u)\,du = \frac{1}{2}k\pi\end{aligned}$$

19. (a) Set $C_1 : \mathbf{r}(u) = u\,\mathbf{i} + u^2\,\mathbf{j},\quad 0 \le u \le 1;\quad C_2 : \mathbf{r}(u) = (1-u)\,\mathbf{i} + \sqrt{1-u}\,\mathbf{j},\quad 0 \le u \le 1.$

Then, $C = C_1 + C_2$.

$$\begin{aligned}&\oint_C xy^2\,dx - x^2y\,dy = \int_{C_1} xy^2\,dx - x^2y\,dy + \int_{C_2} xy^2\,dx - x^2y\,dy \\ &= \int_0^1 (u^5 - 2u^5)\,du + \int_0^1 \left[-(1-u)^2 + \tfrac{1}{2}(1-u)^2\right]\,du \\ &= \int_0^1 \left(-u^5\right)du - \tfrac{1}{2}\int_0^1 (1-u)^2\,du = \Big[-\tfrac{1}{6}u^6 + \tfrac{1}{6}(1-u)^3\Big]_0^1 = -\tfrac{1}{3}\end{aligned}$$

(b) $P = xy^2;\quad Q = -x^2y$

$$\oint_C xy^2\,dx - x^2y\,dy = \int_0^1\int_{x^2}^{\sqrt{x}} (-4xy)\,dy\,dx = \int_0^1 \left(2x^2 - 2x^5\right)\,dx = -\frac{1}{3}$$

20. (a) $\oint_C (x^2+y^2)\,dx + (x^2-y^2)\,dy = \iint_\Omega (2x-2y)\,dx\,dy = \int_0^{2\pi}\int_0^1 (2r\cos\theta - 2r\sin\theta)r\,dr\,d\theta = 0$

(b) $\oint_C (x^2+y^2)\,dx + (x^2-y^2)\,dy = \int_0^{2\pi} (-\sin u + \cos 2u\cos u)\,du = 0$

21. $P = x - 2y^2; \quad Q = 2xy$

$$\oint_C (x - 2y^2)\,dx + 2xy\,dy = \int_0^2 \int_0^1 6y\,dy\,dx = 6$$

22. $$\oint_C xy\,dx + \left(\tfrac{1}{2}x^2 + xy\right) dy = \iint_\Omega y\,dx\,dy = \int_{-1}^1 \int_0^{\frac{\sqrt{1-x^2}}{2}} y\,dy\,dx = \int_{-1}^1 \tfrac{1}{8}(1 - x^2)dx = \tfrac{1}{6}$$

23. $P = \ln(x^2 + y^2); \quad Q = \ln(x^2 + y^2); \quad \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = \dfrac{2x - 2y}{x^2 + y^2}$

$$\begin{aligned}\oint_C \ln(x^2 + y^2)\,dx + \ln(x^2 + y^2)\,dy &= \iint_\Omega \frac{2x - 2y}{x^2 + y^2}\,dx\,dy \\ &= \int_0^\pi \int_1^2 \frac{2r\cos\theta - 2r\sin\theta}{r^2}\,r\,dr\,d\theta \\ &= 2\int_0^\pi \int_1^2 (\cos\theta - \sin\theta)\,dr\,d\theta = -4\end{aligned}$$

24. $P = 1/y, \quad Q = 1/x, \quad \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = -\dfrac{1}{x^2} + \dfrac{1}{y^2}$

$$\begin{aligned}\oint_C (1/y)\,dx + (1/x)\,dy = \iint_\Omega \left(-x^{-2} + y^{-2}\right)\,dxdy &= \int_1^4 \int_1^{\sqrt{x}} \left(-x^{-2} + y^{-2}\right)\,dy\,dx \\ &= \int_1^4 \left(-x^{-3/2} - x^{-1/2} + x^{-2} + 1\right) dx = \frac{3}{4}\end{aligned}$$

25. $$\oint y^2 dx = \iint_\Omega -2ydxdy = \int_0^{2\pi} \int_0^{1+\sin\theta} -2r^2 \sin\theta\,dr\,d\theta = \int_0^{2\pi} (-\tfrac{2}{3})(1 + \sin\theta)^3 \sin\theta\,d\theta = -\frac{5\pi}{2}$$

26. $P = e^y \cos x, \; Q = -e^y \sin x, \quad \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = -2e^y \cos x$

$$\oint_C e^y \cos x\,dx - e^y \sin x\,dy = \iint_\Omega (-2e^y \cos x)\,dxdy = \int_0^{\pi/2} \int_0^1 (-2e^y \cos x)\,dy\,dx = 2(1 - e)$$

27. $C_1 : \; \mathbf{r}(u) = -u\,\mathbf{i} + (4 - u^2)\,\mathbf{j}, \; -2 \le u \le 2; \quad C_2 : \; \mathbf{r}(u) = u\,\mathbf{i}, \; -2 \le u \le 2; \quad C = C_1 \cup C_2$

$$\begin{aligned}A = \frac{1}{2}\int_C (-y\,dx + x\,dy) &= \frac{1}{2}\int_{C_1} (-y\,dx + x\,dy) + \frac{1}{2}\int_{C_2} (-y\,dx + x\,dy) \\ &= \frac{1}{2}\int_{-2}^2 -(4 - u^2)(-1)\,du - u(-2u)\,du + \frac{1}{2}\int_{-2}^2 0\,du \\ &= \int_{-2}^2 (4 + u^2)\,du = \frac{32}{3}\end{aligned}$$

28. $C_1: \ \mathbf{r}(u) = (3-2u)\,\mathbf{i} + (1+2u)\,\mathbf{j},\ 0 \le u \le 1; \quad C_2: \ \mathbf{r}(u) = u\,\mathbf{i} + (3/u)\,\mathbf{j},\ 1 \le u \le 3;$

$C = C_1 \cup C_2$

$$\begin{aligned} A &= \frac{1}{2}\int_C (-y\,dx + x\,dy) = \frac{1}{2}\int_{C_1} (-y\,dx + x\,dy) + \frac{1}{2}\int_{C_2} (-y\,dx + x\,dy) \\ &= \frac{1}{2}\int_0^1 [-(1+2u)(-2) + (3-2u)2]\,du + \frac{1}{2}\int_1^3 [-(3/u) + u(-3/u^2)]\,du \\ &= \frac{1}{2}\int_0^1 8\,du + \frac{1}{2}\int_1^3 (-6/u)\,du = 4 - 3\ln 3 \end{aligned}$$

29. By symmetry, it is sufficient to consider the upper part of the sphere: $z = \sqrt{4 - x^2 - y^2}$

$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{4-x^2-y^2}}, \qquad \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{4-x^2-y^2}}$$

Let Ω be the projection of the sphere onto the xy plane, then

$$\begin{aligned} S = 2\iint_\Omega \sqrt{(z_x)^2 + (z_y)^2 + 1}\,dx\,dy &= 2\iint_\Omega \frac{2}{\sqrt{4-x^2-y^2}}\,dx\,dy \\ &= 4\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} \frac{1}{\sqrt{4-r^2}}\,r\,dr\,d\theta \\ &= 4\int_{-\pi/2}^{\pi/2} \left(2 - 2\sqrt{1-\cos^2\theta}\right)\,d\theta = 8(\pi - 2) \end{aligned}$$

30. From $x + y + 2z = 4$, we get $z = \dfrac{4-x-y}{2}$ and $z_x = -\frac{1}{2}, \quad z_y = -\frac{1}{2}$.

$$\text{area of } S = \iint_\Omega \sqrt{z_x^2 + z_y^2 + 1}\,dx\,dy = \sqrt{\frac{3}{2}}\int_0^{2\pi}\int_0^2 r\,dr\,d\theta = 2\sqrt{6}\pi.$$

31. $\dfrac{\partial z}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}}, \quad \dfrac{\partial z}{\partial y} = \dfrac{y}{\sqrt{x^2+y^2}}.$

The projection Ω of the surface onto the xy plane is the disk $x^2 + y^2 \le 9$.

$$S = \iint_\Omega \sqrt{(z_x)^2 + (z_y)^2 + 1}\,dx\,dy = \iint_\Omega \sqrt{2}\,dx\,dy = \int_0^{2\pi}\int_0^3 \sqrt{2}\,r\,dr\,d\theta = 9\,\pi\,\sqrt{2}$$

32.
$$\begin{aligned} A &= 2\iint_\Omega \sqrt{z_x^2 + z_y^2 + 1}\,dxdy = 2\iint_\Omega \sqrt{4x^2 + 4y^2 + 1}\,dxdy \\ &= 2\int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}\,r\,dr\,d\theta = 4\pi\int_0^3 \sqrt{1+4r^2}\,r\,dr \\ &= \frac{\pi}{3}(37^{3/2} - 1) \end{aligned}$$

33. $\iint_S yz\,d\sigma = \sqrt{2}\int_0^{2\pi}\int_0^1 r^2\sin\theta(r\sin\theta+4)\,dr\,d\theta = \dfrac{\sqrt{2}\pi}{4}$

34. $\iint_S xz\,d\sigma = \iint_S x(1-x-y)\,d\sigma = \iint_\Omega x(1-x-y)\sqrt{3}\,dx\,dy = \sqrt{3}\int_0^1\int_0^{1-x} x(1-x-y)dy\,dx =$
$\dfrac{\sqrt{3}}{24}$

35. The cylindrical surface S_1 is parametrized by: $x=u,\ y=2\cos v,\ z=2\sin v,\ 0\le u\le 2,\ 0\le v\le 2\pi$.

$$\mathbf{N}(u,v) = -2\cos v\,\mathbf{i} - 2\sin v\,\mathbf{j},\quad \|\mathbf{N}(u,v)\| = 2$$

$$\iint_{S_1}\left(x^2+y^2+z^2\right)\,d\sigma = \int_0^2\int_0^{2\pi}\left(u^2+4\right)2\,dv\,du = \frac{128\pi}{3}$$

The disc $S_2:\ x=0,\ y^2+z^2$ is parametrized by: $x=0,\ y=u\cos v,\ z=u\sin v,\ 0\le u\le 2,$
$0\le v\le 2\pi$.

$\mathbf{N} = u\,\mathbf{i},\quad \|\mathbf{N}(u,v)\| = u;\qquad \iint_{S_2}\left(x^2+y^2+z^2\right)\,d\sigma = \int_0^2\int_0^{2\pi}\left(0+u^2\right)u\,dv\,du = 8\pi$

The disc $S_3:\ x=2,\ y^2+z^2$ is parametrized by: $x=2,\ y=u\cos v,\ z=u\sin v,\ 0\le u\le 2,$
$0\le v\le 2\pi$.

$\mathbf{N} = u\,\mathbf{i},\quad \|\mathbf{N}(u,v)\| = u;\qquad \iint_{S_2}\left(x^2+y^2+z^2\right)\,d\sigma = \int_0^2\int_0^{2\pi}\left(4+u^2\right)u\,dv\,du = 24\pi$

Thus, $\iint_S\left(x^2+y^2+z^2\right)d\sigma = \dfrac{128\pi}{3}+8\pi+24\pi = \dfrac{224\pi}{3}$

36. The cylindrical surface S is parametrized by:

$$x = 2\cos u,\ y = 2\sin u,\ z = v,\quad 0\le u\le 2\pi,\ 0\le v\le 1.$$

$\mathbf{N}(u,v) = 2\cos u\,\mathbf{i} - 2\sin u\,\mathbf{j},\quad \|\mathbf{N}(u,v)\| = 2;\quad \iint_S xz\,d\sigma = \int_0^1\int_0^{2\pi} 2v\cos u(2)\,dv\,du = 0.$

37. $\nabla\cdot\mathbf{v} = 4x,\quad \nabla\times\mathbf{v} = 2y\mathbf{k}$

38 $\nabla\cdot\mathbf{v} = 0,\quad \nabla\times\mathbf{v} = 0$

39. $\nabla\cdot\mathbf{v} = 1+xy,\quad \nabla\times\mathbf{v} = (xz-x)\mathbf{i} - yz\mathbf{j} + z\mathbf{k}$

40. $\nabla\cdot\mathbf{v} = yz + x\sin xy,\quad \nabla\times\mathbf{v} = x\cos xy\mathbf{i} + (xy - y\cos xy)\mathbf{j} + (y\sin xy - xz)\mathbf{k}$

41. (a) $\nabla \cdot \mathbf{v} = z - x + y$

$$\int_0^1 \int_0^1 \int_0^1 (z - x + y)dzdydx = \frac{1}{2}$$

(b) at $x = 0$, $\mathbf{n} = -\mathbf{i}, \mathbf{v} \cdot \mathbf{n} = 0, \int_0^1 \int_0^1 0dydz = 0$

at $x = 1$, $\mathbf{n} = \mathbf{i}, \mathbf{v} \cdot \mathbf{n} = z, \int_0^1 \int_0^1 zdydz = 1/2$

at $y = 0$, $\mathbf{n} = -\mathbf{j}, \mathbf{v} \cdot \mathbf{n} = xy = 0, \int_0^1 \int_0^1 0dxdz = 0$

at $y = 1$, $\mathbf{n} = \mathbf{j}, \mathbf{v} \cdot \mathbf{n} = -xy = -x, \int_0^1 \int_0^1 -xdxdz = -1/2$

at $z = 0$, $\mathbf{n} = -\mathbf{k}, \mathbf{v} \cdot \mathbf{n} = 0, \int_0^1 \int_0^1 0dydx = 0$

at $z = 1$, $\mathbf{n} = \mathbf{k}, \mathbf{v} \cdot \mathbf{n} = yz, \int_0^1 \int_0^1 ydydx = 1/2$

The sum is $1/2$

42. (a) $\nabla \cdot \mathbf{v} = 3$

$$\iiint_T 3\,dxdydz = \int_0^4 \int_0^{2\pi} \int_0^1 3r\,dr\,d\theta\,dx = 4(2\pi)(\tfrac{3}{2}) = 12\pi$$

(b) at $x = 0$, $\mathbf{n} = -\mathbf{i},\ \mathbf{v} \cdot \mathbf{n} = -z,\ \iint_S -z\,dydz = 0$ (by symmetry)

at $x = 4$, $\mathbf{n} = \mathbf{i},\ \mathbf{v} \cdot \mathbf{n} = 4 + z,\ \iint_S (4 + z)\,dydz = \iint_S 4\,dydz = 4\pi$

for $z = \sqrt{1 - y^2}$, $0 \le x \le 4$, $\mathbf{n} = -y\,\mathbf{j} + \sqrt{1 - y^2}\,\mathbf{k}$ and

$$\mathbf{v} \cdot \mathbf{n} = 1 - 2y^2 - y\sqrt{1 - y^2} + x\sqrt{1 - y^2}$$

$$\int_{-1}^1 \int_0^4 (1 - 2y^2 - y\sqrt{1 - y^2} + x\sqrt{1 - y^2})dxdy = 8 - \frac{16}{3} + 4\pi$$

for $z = -\sqrt{1 - y^2}$, $0 \le x \le 4$, $\mathbf{n} = y\,\mathbf{j} + \sqrt{1 - y^2}\,\mathbf{k}$ and

$$\mathbf{v} \cdot \mathbf{n} = -1 + 2y^2 - y\sqrt{1 - y^2} + x\sqrt{1 - y^2}$$

$$\int_{-1}^1 \int_0^4 (-1 + 2y^2 - y\sqrt{1 - y^2} + x\sqrt{1 - y^2})dxdy = -8 + \frac{16}{3} + 4\pi$$

The sum is 12π

43. The projection of S onto the xy-plane is: $\Omega : x^2 + y^2 \le 9$.

$$\iint_S \mathbf{v}\cdot\mathbf{n}\,d\sigma = \iint_\Omega \left(4x^2 + 2xyz + z^2\right) dx\,dy$$

$$= \iint_\Omega \left(4x^2 + 2xy\left[9 - x^2 - y^2\right] + \left[9 - x^2 - y^2\right]^2\right) dx\,dy$$

$$= \int_0^{2\pi}\int_0^3 \left[4r^2\cos^2\theta + r^2(9 - r^2)\sin 2\theta + (9 - r^2)^2\right] r dr\,d\theta = 324\pi$$

44. On $x = 0$, $\mathbf{n} = -\mathbf{i}$, $\mathbf{v}\cdot\mathbf{n} = -x^2 = 0$, the flux is 0;

on $x = a$, $\mathbf{n} = \mathbf{i}$, $\mathbf{v}\cdot\mathbf{n} = a^2$, the flux is a^4;

on $y = 0$, $\mathbf{n} = -\mathbf{j}$, $\mathbf{v}\cdot\mathbf{n} = xz$, the flux is $\displaystyle\int_0^a\int_0^a xz\,dx\,dz = \frac{1}{4}a^2$;

on $y = a$, $\mathbf{n} = \mathbf{j}$, $\mathbf{v}\cdot\mathbf{n} = -xz$, the flux is $\displaystyle\int_0^a\int_0^a -xz\,dx\,dz = -\frac{1}{4}a^2$;

on $z = 0$, $\mathbf{n} = -\mathbf{k}$, $\mathbf{v}\cdot\mathbf{n} = 0$, the flux is 0;

on $z = a$, $\mathbf{n} = \mathbf{k}$, $\mathbf{v}\cdot\mathbf{n} = a^2$, the flux is a^4.

Hence the total flux is $2a^4$.

45. (a) $(\nabla \times \mathbf{v})\cdot\mathbf{n} = (\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot\left(-\frac{1}{2}x\,\mathbf{i} - \frac{1}{2}y\,\mathbf{j} + \frac{\sqrt{4 - x^2 - y^2}}{2}\,\mathbf{k}\right) = -\frac{1}{2}x - \frac{1}{2}y + \frac{\sqrt{4 - x^2 - y^2}}{2}$

$$\iint_S \left(-\frac{1}{2}x - \frac{1}{2}y + \frac{\sqrt{4 - x^2 - y^2}}{2}\right) d\sigma = \iint_S \left(-\frac{1}{2}x - \frac{1}{2}y + \frac{\sqrt{4 - x^2 - y^2}}{2}\right)\frac{2}{\sqrt{4 - x^2 - y^2}}\,dx\,dy$$

$$= \iint \left(\frac{-x}{\sqrt{4 - x^2 - y^2}} - \frac{-y}{\sqrt{4 - x^2 - y^2}} + 1\right) dx\,dy$$

$$= \int_0^{2\pi}\int_0^2 \left(-\frac{r\cos\theta}{\sqrt{4 - r^2}} - \frac{r\sin\theta}{\sqrt{4 - r^2}} + 1\right) r\,dr\,d\theta = 4\pi$$

(b) $\mathbf{r}(\theta) = 2\cos\theta\,\mathbf{i} + 2\sin\theta\,\mathbf{j}$, $0 \le \theta \le 2\pi$

$$\iint_S [(\nabla \times \mathbf{v})\cdot\mathbf{n}]d\sigma = \oint_C \mathbf{v}(\mathbf{r})\cdot d\mathbf{r} = \int_0^{2\pi} 4\cos^2\theta d\theta = 4\pi$$

46. (a) $\mathbf{v} = z^3\,\mathbf{i} + x\,\mathbf{j} + y^2\,\mathbf{k}; \quad \mathbf{n} = \dfrac{2x\,\mathbf{i} + 2y\,\mathbf{j} + \mathbf{k}}{\sqrt{1+4x^2+4y^2}};$

$$\begin{aligned}\iint_S [(\nabla \times \mathbf{v})\cdot \mathbf{n}]\,d\sigma &= \iint_S \frac{1}{\sqrt{1+4x^2+4y^2}}(4xy + 6yz^2 + 1)\,d\sigma \\ &= \iint_\Omega (4xy + 6yz^2 + 1)\,dx\,dy \\ &= \iint_\Omega [4xy + 6y(9 - x^2 - y^2)^2 + 1]\,dx\,dy \\ &= \int_0^{2\pi}\int_0^3 \left[4r^2\cos\theta\sin\theta + 6r\sin\theta(9-r^2)^2 + 1\right]\,r\,dr\,d\theta \\ &= \int_0^3 2\pi r\,dr = 9\pi\end{aligned}$$

(b) The boundary of the surface is the curve $x^2 + y^2 = 9,\ z = 0;\quad \mathbf{r}(u) = 3\cos\,u\,\mathbf{i} + 3\sin\,u\,\mathbf{j} + 0\,\mathbf{k};$

$\mathbf{v}(\mathbf{r}(u)) = 3\cos\,u\,\mathbf{j} + 9\sin^2 u\,\mathbf{k};\quad \mathbf{r}'(u) = -3\sin u\,\mathbf{i} + 3\cos\,u\,\mathbf{j}$

$$\oint_C \mathbf{v}\cdot d\mathbf{r} = \int_0^{2\pi} 9\cos^2 u\,du = 9\pi.$$