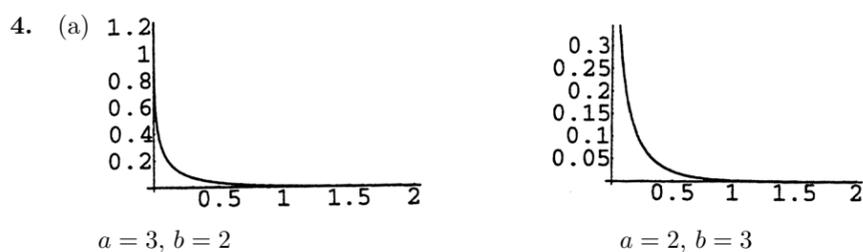


910 REVIEW EXERCISES

(b) $x = ar \cos^3 \theta$, $y = ar \sin^3 \theta$; $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \implies r = 1$ and $x = a \cos^3 \theta$, $y = a \sin^3 \theta$

$$\begin{aligned} A &= \int_{-\frac{\pi}{2}}^0 y(\theta)x'(\theta) d\theta = \int_{-\frac{\pi}{2}}^0 a \sin^3 \theta (3a \cos^2 \theta [-\sin \theta]) d\theta \\ &= 3a^2 \int_0^{\frac{\pi}{2}} \sin^4 \theta \cos^2 \theta d\theta = 3a^2 \int_0^{\frac{\pi}{2}} (\sin^4 \theta - \sin^6 \theta) d\theta \\ &= 3a^2 \left[\frac{3 \cdot 1}{4 \cdot 2} \frac{\pi}{2} - \frac{5 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2} \frac{\pi}{2} \right] \quad (\text{See Exercise 62(b) in 8.3}) \\ &= \frac{3a^2 \pi}{32} \end{aligned}$$

(c) Entire area enclosed: $4 \cdot \frac{3a^2 \pi}{32} = \frac{3a^2 \pi}{8}$



(b) From Problem 2, Jacobian $J = 8abr \cos^7 \theta \sin^7 \theta$

$$A = \int_0^{\frac{\pi}{2}} \int_0^1 8abr \cos^7 \theta \sin^7 \theta d\theta = 4ab \int_0^{\frac{\pi}{2}} \cos^7 \theta \sin^7 \theta d\theta = \frac{ab}{70}$$

REVIEW EXERCISES

1. $\int_0^1 \int_y^{\sqrt{y}} xy^2 dx dy = \int_0^1 \left[\frac{1}{2} x^2 y^2 \right]_y^{\sqrt{y}} dy = \int_0^1 \left(\frac{1}{2} y^3 - \frac{1}{2} y^4 \right) dy = \left[\frac{1}{8} y^4 - \frac{1}{10} y^5 \right]_0^1 = \frac{1}{40}$
2. $\int_0^1 \int_{-y}^y e^{x+y} dx dy = \int_0^1 \left[e^{x+y} \right]_{-y}^y dy = \int_0^1 (e^{2y} - 1) dy = \left[\frac{1}{2} e^{2y} - y \right]_0^1 = \frac{e^2}{2} - \frac{3}{2}$
3. $\int_0^1 \int_x^{3x} 2ye^{x^3} dy dx = \int_0^1 \left[y^2 e^{x^3} \right]_x^{3x} dx = \int_0^1 (9x^2 e^{x^3} - x^2 e^{x^3}) dx = \left[3e^{x^3} - \frac{1}{3} e^{x^3} \right]_0^1 = \frac{8}{3} e - \frac{8}{3}$
4. $\int_1^2 \int_0^{\ln x} x e^y dy dx = \int_1^2 \left[x e^y \right]_0^{\ln x} dx = \int_1^2 x(x-1) dx = \left[\frac{1}{3} x^3 - \frac{1}{2} x^2 \right]_1^2 = \frac{5}{6}$
5. $\int_0^{\pi/4} \int_0^{2 \sin \theta} r \cos \theta dr d\theta = \int_0^{\pi/4} \left[\frac{1}{2} r^2 \cos \theta \right]_0^{2 \sin \theta} d\theta = \int_0^{\pi/4} 2 \sin^2 \theta \cos \theta d\theta = \left[\frac{2}{3} \sin^3 \theta \right]_0^{\pi/4} = \frac{\sqrt{2}}{6}$
6. $\int_{-1}^2 \int_0^4 \int_0^1 xyz dx dy dz = \int_{-1}^2 \int_0^4 \left[\frac{1}{2} x^2 yz \right]_0^1 dy dz = \int_{-1}^2 \int_0^4 \frac{1}{2} yz dy dz = \int_{-1}^2 4z dz = 6$

REVIEW EXERCISES 911

$$\begin{aligned} 7. \int_0^2 \int_0^{2-3x} \int_0^{x+y} x \, dz \, dy \, dx &= \int_0^2 \int_0^{2-3x} [xz]_0^{x+y} \, dy \, dx = \int_0^2 \int_0^{2-3x} (x^2 + xy) \, dy \, dx \\ &= \int_0^2 \left[x^2 y + \frac{1}{2} xy^2 \right]_0^{2-3x} \, dx = \int_0^2 \left(\frac{3}{2} x^3 - 4x^2 + 2x \right) \, dx \\ &= \left[\frac{3}{8} x^4 - \frac{4}{3} x^3 + x^2 \right]_0^2 = -\frac{2}{3} \end{aligned}$$

$$\begin{aligned} 8. \int_0^{\frac{\pi}{2}} \int_z^{\frac{\pi}{2}} \int_0^{\sin z} 3x^2 \sin y \, dx \, dy \, dz &= \int_0^{\frac{\pi}{2}} \int_z^{\frac{\pi}{2}} [x^3 \sin y]_0^{\sin z} \, dy \, dz = \int_0^{\frac{\pi}{2}} \int_z^{\frac{\pi}{2}} \sin^3 z \sin y \, dy \, dz \\ &= \int_0^{\frac{\pi}{2}} \sin^3 z \cos z \, dz = \frac{1}{4} \sin^4 z \Big|_0^{\frac{\pi}{2}} = \frac{1}{4} \end{aligned}$$

$$9. \int_{-\frac{\pi}{2}}^0 \int_0^{2\sin\theta} \int_0^{r^2} r^2 \cos\theta \, dz \, dr \, d\theta = \int_{-\frac{\pi}{2}}^0 \int_0^{2\sin\theta} r^4 \cos\theta \, dr \, d\theta = \int_{-\frac{\pi}{2}}^0 \frac{32}{5} \sin^5\theta \cos\theta \, d\theta = -\frac{16}{15}$$

$$\begin{aligned} 10. \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 \rho^3 \sin\varphi \cos\varphi \, d\rho \, d\theta \, d\varphi &= \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{1}{4} \sin\varphi \cos\varphi \, d\theta \, d\varphi \\ &= \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\pi}{16} \sin 2\varphi \, d\varphi = -\left[\frac{\pi}{32} \cos 2\varphi \right]_{-\frac{\pi}{6}}^{\frac{\pi}{2}} = \frac{3\pi}{64} \end{aligned}$$

$$11. \int_0^1 \int_y^1 e^{x^2} \, dx \, dy = \int_0^1 \int_0^x e^{x^2} \, dy \, dx = \int_0^1 e^{x^2} y \Big|_0^x \, dx = \int_0^1 x e^{x^2} \, dx = \frac{1}{2} e^{x^2} \Big|_0^1 = \frac{e-1}{2}$$

$$12. \int_0^2 \int_{\frac{x}{2}}^1 \cos y^2 \, dy \, dx = \int_0^1 \int_0^{2y} \cos y^2 \, dx \, dy = \int_0^1 x \cos y^2 \Big|_0^{2y} \, dy = \int_0^1 2y \cos y^2 \, dy = \sin y^2 \Big|_0^1 = \sin 1$$

$$13. \int_0^1 \int_0^{\sqrt{1-y^2}} \frac{1}{\sqrt{1-y^2}} \, dx \, dy = \int_0^1 dy = 1$$

$$\begin{aligned} 14. \int_0^1 \int_0^{1-x} y \cos(x+y) \, dy \, dx &= \int_0^1 \int_0^{1-y} y \cos(x+y) \, dx \, dy \\ &= \int_0^1 y \sin(x+y) \Big|_0^{1-y} \, dy = \int_0^1 y(\sin 1 - \sin y) \, dy \\ &= \left[\frac{1}{2} y^2 \sin 1 + y \cos y - \sin y \right]_0^1 = \cos 1 - \frac{1}{2} \sin 1 \end{aligned}$$

$$15. \int_0^1 \int_0^{\sqrt{1-x^2}} xy \, dy \, dx = \int_0^1 \left[\frac{1}{2} xy^2 \right]_0^{\sqrt{1-x^2}} \, dx = \int_0^1 \left(\frac{1}{2} x - \frac{1}{2} x^3 \right) \, dx = \frac{1}{8}$$

$$16. \int_{-\sqrt{3}}^{\sqrt{3}} \int_{y^2/3}^{4-y^2} (x-y) \, dx \, dy = \int_{-\sqrt{3}}^{\sqrt{3}} \left[\frac{x^2}{2} - xy \right]_{y^2/3}^{4-y^2} \, dy = \int_{-\sqrt{3}}^{\sqrt{3}} \left(\frac{4}{9} y^4 + \frac{4}{3} y^3 - 4y^2 - 4y + 8 \right) \, dy = \frac{48\sqrt{3}}{5}$$

$$17. \int_0^2 \int_x^{3x-x^2} (x^2 - xy) \, dy \, dx = \int_0^2 \left[x^2 y - \frac{1}{2} xy^2 \right]_x^{3x-x^2} \, dx = \int_0^2 \left(2x^4 - 2x^3 - \frac{1}{2} x^5 \right) \, dx = -\frac{8}{15}$$

912 REVIEW EXERCISES

$$18. \int_0^2 \int_0^{2-x} x(x-1)e^{xy} dy dx = \int_0^2 [(x-1)e^{xy}]_0^{2-x} dx = \int_0^2 (x-1)e^{2x-x^2} dx - \int_0^2 (x-1) dx$$

$$= -\left[\frac{1}{2}e^{2x-x^2}\right]_0^2 - \left[\frac{1}{2}x^2 - x\right]_0^2 = 0$$

$$19. \int_0^2 \int_0^y \int_0^{\sqrt{4-y^2}} 2xyz dz dx dy = \int_0^2 \int_x^2 xyz^2 \Big|_0^{\sqrt{4-y^2}} dx dy = \int_0^2 \int_0^2 xy(4-y^2) dx dy$$

$$= \int_0^2 \left[\frac{1}{2}y^3(4-y^2)\right] dy = \frac{8}{3}$$

$$20. \iiint_T z dx dy dz = 2 \int_0^1 \int_0^x \int_0^{1-x} z dz dy dx = \int_0^1 \int_0^x (1-x)^2 dy dx = \int_0^1 (1-x)^2 x dx = \frac{1}{12}$$

$$21. \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} xy dz dy dx = \int_0^2 \int_0^{\sqrt{4-x^2}} [xyz]_0^{\sqrt{4-x^2-y^2}} dy dx$$

$$= \int_0^2 \left[-\frac{1}{3}x\sqrt{4-x^2-y^2}\right]_0^{\sqrt{4-x^2}} dx$$

$$= \int_0^2 \frac{1}{3}x(4-x^2)^{\frac{3}{2}} dx = \frac{32}{15}$$

$$22. \iiint_T (x^2 + 2z) dx dy dz = \int_{-2}^2 \int_{x^2}^4 \int_0^{4-y} (x^2 + 2z) dz dy dx$$

$$= \int_{-2}^2 \int_{x^2}^4 (4x^2 - x^2y + 16 - 8y + y^2) dy dx$$

$$= \int_{-2}^2 \left(\frac{1}{6}x^6 - 8x^2 + \frac{64}{3}\right) dx = \frac{2^{10}}{21}$$

$$23. \int_0^2 \int_0^{\sqrt{4-y^2}} e^{\sqrt{x^2+y^2}} dx dy = \int_0^{\pi/2} \int_0^2 e^r r dr d\theta = \frac{\pi}{2} \int_0^2 re^r dr = \frac{\pi}{2} [re^r - e^r]_0^2 = \frac{\pi}{2}(e^2 + 1)$$

$$24. \int_{-1}^1 \int_0^{\sqrt{1-x^2}} \arctan(y/x) dy dx = \int_0^{\pi/2} \int_0^1 r\theta dr d\theta + \int_{\pi/2}^{\pi} \int_0^1 (\theta - \pi)r dr d\theta$$

$$= \int_0^{\pi/2} \frac{\theta}{2} d\theta + \int_{\pi/2}^{\pi} \frac{1}{2}(\theta - \pi) d\theta = 0$$

$$25. V = \int_0^3 \int_0^{2\pi} (9-r^2)r dr d\theta = 2\pi \int_0^3 (9-r^2)r dr = \frac{81\pi}{2}$$

$$26. \int_0^1 \int_{x^2}^{\sqrt{x}} (2-x^2-y^2) dy dx = -\int_0^1 \left(2x^{1/2} - x^{5/2} - \frac{1}{3}x^{3/2} - 2x^2 + x^4 + \frac{1}{3}x^6\right) dx = \frac{52}{105}$$

$$27. V = \int_0^1 \int_0^{1-x} (x^2 + y^2) dy dx = \int_0^1 \left[x^2 - x^3 + \frac{1}{3}(1-x)^3\right] dx = \left[\frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{12}(1-x)^4\right]_0^1 = \frac{1}{6}$$

REVIEW EXERCISES 913

28. $V = \int_0^3 \int_0^{\pi/2} r^2 \sin \theta \, dr \, d\theta = \int_0^{\pi/2} 9 \sin \theta \, d\theta = 9$

29. $M = \int_{-\pi/2}^{\pi/2} \int_0^{\cos x} y \, dy \, dx = \int_{-\pi/2}^{\pi/2} \frac{1}{2} \cos^2 x \, dx = \frac{\pi}{4}$

$x_M M = \int_{-\pi/2}^{\pi/2} \int_0^{\cos x} xy \, dy \, dx = 0$ by symmetry

$y_M M = \int_{-\pi/2}^{\pi/2} \int_0^{\cos x} y^2 \, dy \, dx = \int_{-\pi/2}^{\pi/2} \frac{1}{3} \cos^3 x \, dx = \frac{4}{9}$

The center of mass is: $(0, \frac{16}{9\pi})$

30. $M = \int_0^1 \int_{y^2}^y 2x \, dx \, dy = \int_0^1 (y^2 - y^4) \, dy = \frac{2}{15}$

$x_M M = \int_0^1 \int_{y^2}^y 2x^2 \, dx \, dy = \int_0^1 (\frac{2}{3}y^3 - \frac{2}{3}y^6) \, dy = \frac{1}{14}$

$y_M M = \int_0^1 \int_{y^2}^y 2xy \, dx \, dy = \int_0^1 (y^3 - y^5) \, dy = \frac{1}{12}$

The center of mass is: $(15/28, 5/8)$

31. $M = \int_0^{\pi/2} \int_r^R u^3 \, du \, d\theta = \frac{\pi}{8}(R^4 - r^4)$; (polar coordinates $[u, \theta]$)

By symmetry, $\bar{x} = \bar{y}$.

$x_M M = \int_0^{\pi/2} \int_r^R u^4 \cos \theta \, du \, d\theta = \frac{1}{5}(R^5 - r^5)$; $x_M = \frac{8(R^5 - r^5)}{5\pi(R^4 - r^4)}$

32. $M = \int_0^\pi \int_0^{2(1+\cos \theta)} r^2 \, dr \, d\theta = \int_0^\pi \frac{8}{3}(1 + \cos \theta)^3 \, d\theta = \frac{20}{3}\pi$

$x_M M = \int_0^\pi \int_0^{2(1+\cos \theta)} r^3 \cos \theta \, dr \, d\theta = \int_0^\pi 4(1 + \cos \theta)^4 \cos \theta \, d\theta = 14\pi$

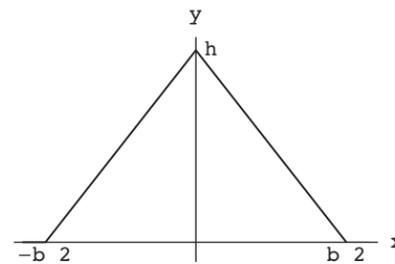
$y_M M = \int_0^\pi \int_0^{2(1+\cos \theta)} r^3 \sin \theta \, dr \, d\theta = \int_0^\pi 4(1 + \cos \theta)^4 \sin \theta \, d\theta = -\frac{4}{5}(1 + \cos \theta)^5 \Big|_0^\pi = \frac{128}{5}$

The center of mass is: $(\frac{21}{10}, \frac{96}{25\pi})$

33. Introduce a coordinate system as shown in the figure.

(a) $A = \frac{1}{2}bh$; by symmetry, $\bar{x} = 0$

$$\begin{aligned} \bar{y} A &= \int_{-b/2}^0 \int_0^{\frac{2h}{b}(x+\frac{b}{2})} y \, dy \, dx + \int_0^{b/2} \int_0^{-\frac{2h}{b}(x-\frac{b}{2})} y \, dy \, dx \\ &= \frac{bh^2}{6} \implies \bar{y} = \frac{h}{3} \end{aligned}$$



(b) $I = \int_{-b/2}^0 \int_0^{\frac{2h}{b}(x+\frac{b}{2})} \lambda y^2 \, dy \, dx + \int_0^{b/2} \int_0^{-\frac{2h}{b}(x-\frac{b}{2})} \lambda y^2 \, dy \, dx = \frac{\lambda bh^3}{12} = \frac{1}{6}Mh^2$

914 REVIEW EXERCISES

$$(c) I = 2 \int_0^{b/2} \int_0^{-\frac{2b}{b}(x-\frac{b}{2})} \lambda x^2 dx dy = \frac{1}{48} \lambda h b^3 = \frac{1}{24} M b^2$$

34. Let $\lambda = \frac{k}{\sqrt{x^2 + y^2}}$

$$(a) M = \int_0^\pi \int_r^R k dr d\theta = k\pi(R - r)$$

By symmetry, $x_M = 0$; $y_M M = \int_0^\pi \int_r^R kr \sin \theta dr d\theta = k(R^2 - r^2)$

The center of mass is: $(0, \frac{R+r}{\pi})$

$$(b) I_x = \int_0^\pi \int_r^R kr^2 \sin^2 \theta dr d\theta = \frac{k}{3}(R^3 - r^3) \int_0^\pi \sin^2 \theta d\theta = \frac{k\pi}{6}(R^3 - r^3)$$

$$(c) I_y = \int_0^\pi \int_r^R kr^2 \cos^2 \theta dr d\theta = \frac{k}{3}(R^3 - r^3) \int_0^\pi \cos^2 \theta d\theta = \frac{k\pi}{6}(R^3 - r^3)$$

35. $V = \int_0^2 \int_0^x \int_0^{2x+2y+1} dz dy dx = \int_0^2 \int_0^x (2x + 2y + 1) dy dx = 10$

36. $V = \int_0^1 \int_{x^2}^x \int_{-1}^{4(x^2+y^2)} dz dy dx = \int_0^1 \int_{x^2}^x (4x^2 + 4y^2 + 1) dy dx$
 $= \int_0^1 \left(x + \frac{16}{3}x^3 - x^2 - 4x^4 - \frac{4}{3}x^6 \right) dx = \frac{107}{210}$

37. The curve of intersection of the two surfaces is the circle: $x^2 + y^2 = 4$, $x = 3$

$$V = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{2x^2+y^2}^{12-x^2-2y^2} dz dy dx = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 3(4 - x^2 - y^2) dy dx$$

 $= 3 \int_0^{2\pi} \int_0^2 (4 - r^2) r dr d\theta$
 $= 3 \int_0^{2\pi} \left[2r^2 - \frac{1}{4}r^4 \right]_0^2 d\theta = 12 \int_0^{2\pi} = 24\pi$

38. $V = \int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^{(2-y-z)/2} dx dz dy = \int_0^1 \int_0^{\pi/2} \frac{2 - r \cos \theta - r \sin \theta}{2} r d\theta dr$
 $= \int_0^1 \frac{1}{2}(\pi - 2r)r dr = \frac{3\pi - 4}{12}$

39. $V = \int_0^{2\pi} \int_0^{\pi/3} \int_{\sec \phi}^2 \rho^2 \sin \phi d\rho d\phi d\theta = \int_0^{2\pi} \int_0^{\pi/3} \left[\frac{1}{3}\rho^3 \right]_{\sec \phi}^2 d\phi d\theta$
 $= \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/3} (8 - \sec^3 \phi) \sin \phi d\phi d\theta$
 $= \frac{1}{3} \int_0^{2\pi} \left[-8 \cos \phi - \frac{1}{2} \sec^2 \phi \right]_0^{\pi/3} d\theta$
 $= \frac{1}{3} \left(\frac{5}{2} \right) (2\pi) = \frac{5\pi}{3}$

REVIEW EXERCISES 915

$$40. V = \int_0^4 \int_0^{(12-3x)/4} \int_0^{16-x^2} dz dy dx = \int_0^4 \int_0^{12-3x} (16-x^2) dy dx = \frac{3}{4} \int_0^4 (64-16x-4x^2+x^3) dx = 80$$

$$41. V = \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^1 \rho^2 \sin \phi d\rho d\phi d\theta = \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \frac{1}{3} \sin \phi d\phi d\theta = \frac{\sqrt{2}\pi}{3}$$

$$42. V = \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^2 \sin \phi d\rho d\phi d\theta = \frac{2-\sqrt{2}}{3} \pi$$

$$43. (a) V = \int_0^1 \int_0^x \int_0^{\sqrt{1-x^2}} dz dy dx + \int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} dz dx dy$$

$$= 2 \int_0^1 \int_0^x \sqrt{1-x^2} dy dx = 2 \int_0^1 x \sqrt{1-x^2} dx = \frac{2}{3}$$

By symmetry, $\bar{x} = \bar{y}$.

$$\bar{x}V = \int_0^1 \int_0^x \int_0^{\sqrt{1-x^2}} x dz dy dx + \int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} x dz dx dy$$

For the first integral:

$$\int_0^1 \int_0^x \int_0^{\sqrt{1-x^2}} x dz dy dx = \int_0^1 \int_0^x x \sqrt{1-x^2} dy dx$$

$$= \int_0^1 x^2 \sqrt{1-x^2} dx = \int_0^{\pi/2} \sin^2 u \cos^2 u du = \frac{\pi}{16}$$

$x = \sin u \quad \uparrow$

For the second integral:

$$\int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} x dz dx dy = \int_0^1 \int_0^y x \sqrt{1-y^2} dx dy = \int_0^1 \frac{1}{2} y^2 \sqrt{1-y^2} dy = \frac{\pi}{32}$$

$$\text{Thus, } \bar{x}V = \frac{3\pi}{32} \implies \bar{x} = \bar{y} = \frac{9\pi}{64}$$

Now calculate \bar{z} :

$$\bar{z}V = \int_0^1 \int_0^x \int_0^{\sqrt{1-x^2}} z dz dy dx + \int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} z dz dx dy;$$

$$\int_0^1 \int_0^x \int_0^{\sqrt{1-x^2}} z dz dy dx = \int_0^1 \int_0^x \frac{1}{2}(1-x^2) dy dx = \frac{1}{2} \int_0^1 (x-x^3) dx = \frac{1}{8}$$

and similarly,

$$\int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} z dz dx dy = \frac{1}{8}.$$

$$\text{Therefore, } \bar{z}V = \frac{1}{4} \implies \bar{z} = \frac{3}{8}$$

916 REVIEW EXERCISES

$$\begin{aligned} \text{(b) } I_z &= \int_0^1 \int_0^x \int_0^{\sqrt{1-x^2}} \lambda (\sqrt{x^2+y^2})^2 dz dy dx + \int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} \lambda (\sqrt{x^2+y^2})^2 dz dx dy; \\ & \int_0^1 \int_0^x \int_0^{\sqrt{1-x^2}} \lambda (\sqrt{x^2+y^2})^2 dz dy dx = \int_0^{\pi/4} \int_0^{\sec\theta} \int_0^{r\sin\theta} \lambda r^3 dz dr d\theta = \frac{3}{20} \lambda \\ \text{and } \int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} \lambda (\sqrt{x^2+y^2})^2 dz dx dy &= \frac{3}{20} \lambda \implies I_z = \frac{3}{10} \lambda \end{aligned}$$

$$44. \text{ (a) } V = \int_0^{2\pi} \int_0^1 (r-r^2)r dr d\theta = 2\pi \int_0^1 (r^2-r^3) dr = \frac{\pi}{6}$$

By symmetry, $\bar{x} = \bar{y} = 0$

$$\bar{z}V = \int_0^{2\pi} \int_0^1 \int_{r^2}^r zr dz dr d\theta = \frac{\pi}{12} \text{ and hence } \bar{z} = \frac{1}{2}$$

$$\text{(b) } I_z = K \int_0^{2\pi} \int_0^1 \int_{r^2}^r r^2 dz dr d\theta = 2\pi K \int_0^1 r^2(r-r^2) dr = \frac{\pi K}{10}$$

Here, K is the density of the mass.

45. Denote polar coordinates by $[u, \theta]$.

$$\text{(a) } M = \int_0^{2\pi} \int_0^r \int_0^h u^3 dz du d\theta = 2\pi h \int_0^r u^3 du = \frac{\pi h r^4}{2}$$

(b) By symmetry, $x_M = y_M = 0$

$$\text{(c) } z_M M = \int_0^{2\pi} \int_0^r \int_0^h u^3 z dz du d\theta = \frac{\pi h^2 r^4}{4} \implies z_M = h/2$$

46. $\lambda = \sqrt{x^2+y^2}$

$$\text{(a) } M = \int_0^{2\pi} \int_0^{\pi/2} \int_0^r \rho^3 \sin^2 \phi d\rho d\phi d\theta = \frac{r^4 \pi}{2} \int_0^{\pi/2} \sin^2 \phi d\phi = \frac{r^4 \pi^2}{8}$$

(b) By symmetry, $x_M = y_M = 0$

$$z_M M = \int_0^{2\pi} \int_0^{\pi/2} \int_0^r \rho^4 \sin^2 \phi \cos \phi d\rho d\phi d\theta = \frac{2r^5 \pi}{15} \implies z_M = \frac{16r}{15\pi}$$

$$47. \text{ (a) } M = \int_0^1 \int_0^{2\pi} \int_r^1 r^2 dz d\theta dr = 2\pi \int_0^1 \int_r^1 r^2 dz dr = 2\pi \int_0^1 r^2(1-r) dr = \frac{\pi}{6}$$

(b) By symmetry, $x_M = y_M = 0$

$$z_M M = \int_0^1 \int_0^{2\pi} \int_r^1 r^2 z dz d\theta dr = \pi \int_0^1 r^2(1-r^2) dr = \frac{2\pi}{15} \implies z_M = \frac{4}{5}$$

$$\text{(c) } I_z = \int_0^1 \int_0^{2\pi} \int_r^1 r^4 dz d\theta dr = \frac{\pi}{15}$$

$$48. \quad J(u, v) = \begin{vmatrix} 2u & 2v \\ -2v & 2u \end{vmatrix} = 4u^2 + 4v^2$$

REVIEW EXERCISES 917

49. $J(u, v) = \begin{vmatrix} e^u \cos v & e^u \sin v \\ -e^u \sin v & e^u \cos v \end{vmatrix} = e^{2u}$

50. $J(u, v, w) = \begin{vmatrix} 2u & 2w & vw \\ 2w & 2v & uw \\ 2v & 2u & uv \end{vmatrix} = (vw - u^2)(4uw - 4v^2)$

51. Set $x = \frac{v-u}{2}$, $y = \frac{v+u}{2} \implies u = y-x$, $v = y+x$, $1 \leq v \leq 2$, $J = -\frac{1}{2}$
 at $x=0$, $y=u$, $y=v \implies u=v$
 at $y=0$, $-x=u$, $x=v \implies u=-v$
 $\iint_{\Omega} \cos\left(\frac{y-x}{y+x}\right) dx dy = \int_1^2 \int_{-v}^v \frac{1}{2} \cos\left(\frac{u}{v}\right) du dv = \int_1^2 v \sin 1 dv = \frac{3}{2} \sin 1$

52. By the hint, $J = \begin{vmatrix} \frac{\cos \theta}{u} & \frac{\sin \theta}{u} & 2r \\ -\frac{r \cos \theta}{u^2} & -\frac{r \sin \theta}{u^2} & 0 \\ -\frac{r \sin \theta}{u} & \frac{r \cos \theta}{u} & 0 \end{vmatrix} = -\frac{2r^3}{u^3}$

$\iiint_T dx dy dz = \int_0^{2\pi} \int_1^2 \int_1^2 \frac{2r^3}{u^3} dr du d\theta = \frac{45\pi}{8}$