3. $4\arctan\frac{1}{5} - \arctan\frac{1}{239} < 4\sum_{k=1}^{15}\frac{(-1)^{k-1}}{2k-1}\left(\frac{1}{5}\right)^{2k-1} - \left[\sum_{k=1}^{4}\frac{(-1)^{k-1}}{2k-1}\left(\frac{1}{239}\right)^{2k-1}\right]$

$= 0.785398163397448309616$

$4\arctan\frac{1}{5} - \arctan\frac{1}{239} > 4\sum_{k=1}^{14}\frac{(-1)^{k-1}}{2k-1}\left(\frac{1}{5}\right)^{2k-1} \left[\sum_{k=1}^{3}\frac{(-1)^{k-1}}{2k-1}\left(\frac{1}{239}\right)^{2k-1}\right]$

$= 0.785398163397448306408$

These inequalities imply $3.14159265358979322563 < \pi < 3.14159265358979323846.$

REVIEW EXERCISES

1. $\sum_{k=0}^{\infty}\left(\frac{3}{4}\right)^k = \frac{1}{1-\frac{3}{4}} = 4,$ a geometric series with $r = \frac{3}{4}$.

2. $\sum_{k=0}^{\infty}(-1)^k(\frac{1}{2})^k = \sum_{0}^{\infty}\left(-\frac{1}{2}\right)^k = \frac{1}{1-(-\frac{1}{2})} = \frac{2}{3},$ a geometric series with $r = -\frac{1}{2}$

3. Since $e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!},\quad \sum_{k=0}^{\infty}\frac{(\ln 2)^k}{k!} = e^{\ln 2} = 2$

4. $\sum_{k=1}^{\infty}\frac{1}{k(k+1)} = \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$

5. diverges; limit comparison with $\sum\frac{1}{k}$

6. converges; limit comparison with $\sum\frac{1}{k^2}$

7. converges; root test: $\left(\frac{k+1}{3^k}\right)^{1/k} \to \frac{1}{3}$ as $k\to\infty$, or ratio test: $\frac{k+2}{3^{k+1}}\,\frac{3^k}{k+1} \to \frac{1}{3}$ as $k\to\infty$

8. diverges; ratio test:

$$\frac{(k+1)!}{(k+1)^{(k+1)/2}}\cdot\frac{k^{k/2}}{k!} = \frac{k^{k/2}}{(k+1)^{(k-1)/2}} = \left(\frac{k}{k+1}\right)^{k/2}\sqrt{k+1} \to \infty.$$

9. converges; limit comparison with: $\sum\frac{1}{k^2}$

10. converges; root test: $\left[k\left(\frac{3}{4}\right)^k\right]^{1/k} \to \frac{3}{4}$ as $k\to\infty$

11. converges; ratio test, $\frac{a_{k+1}}{a_k} = \left(\frac{k+1}{k}\right)^e\cdot\frac{1}{e} \to \frac{1}{e} < 1$

12. diverges; ratio test, $\frac{a_{k+1}}{a_k} = \frac{[2(k+1)]!}{2^{k+1}(k+1)!}\cdot\frac{2^k k!}{(2k)!} = 2k+1 \to \infty$

13. converges; basic comparison,

$$\sum \frac{(\arctan k)^2}{1+k^2} \le \frac{\pi^2}{4} \sum \frac{1}{1+k^2} \le \frac{\pi^2}{4} \sum \frac{1}{k^2}$$

14. converges; $\dfrac{2^k+k^4}{3^k} = \dfrac{2^k}{3^k} + \dfrac{k^4}{3^k}$, and each of the series $\sum \dfrac{2^k}{3^k}$ and $\sum \dfrac{k^4}{3^k}$ is convergent.

15. absolutely convergent; basic comparison

$$\sum \left| \frac{(-1)^k}{(k+1)(k+2)} \right| \le \sum \frac{1}{k^2}$$

16. conditionally convergent: $\sum \dfrac{(-1)^k}{2k+1}$ converges by Theorem 11.4.3;

$\sum \left| \dfrac{(-1)^k}{2k+1} \right| = \sum \dfrac{1}{2k+1}$ diverges.

17. absolutely convergent; $\displaystyle\sum_{k=0}^{\infty} \left| \frac{(-1)^k (100)^k}{k!} \right| = \sum_{k=0}^{\infty} \frac{100^k}{k!}$ which converges by the ratio test.

18. conditionally convergent: $\sum \dfrac{(-1)^k}{\sqrt{(k+1)(k+2)}}$ converges by Theorem 11.4.3;

$\sum \left| \dfrac{(-1)^k}{\sqrt{(k+1)(k+2)}} \right| = \sum \dfrac{1}{\sqrt{(k+1)(k+2)}}$ diverges – limit comparison with $\sum \dfrac{1}{k}$.

19. converges conditionally; Theorem 12.5.3: $\sum \dfrac{\ln k}{\sqrt{k}}$ diverges by the integral test.

20. absolutely convergent; limit comparison with $\sum \dfrac{1}{k^{3/2}}$

21. diverges; limit comparison with $\sum \dfrac{1}{k}$: $\dfrac{1}{k} - \dfrac{1}{k+1} - \dfrac{1}{k+2} = \dfrac{2-k^2}{k(k+1)(k+2)}$

22. $\displaystyle 1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \cdots = \sum_{k=0}^{\infty} \left(\frac{1}{2k+1} - \frac{1}{(2k+2)^2} \right) = \sum_{k=0}^{\infty} \frac{4k^2+6k+3}{(2k+1)(2k+2)^2}$

diverges; limit comparison with $\sum \dfrac{1}{k}$.

23. $\displaystyle e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. Therefore,

$$xe^{2x^2} = x \sum_{k=0}^{\infty} \frac{(2x^2)^k}{k!} = \sum_{k=0}^{\infty} \frac{2^k}{k!} x^{2k+1}$$

24. $\displaystyle \ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k$. Therefore,

$$\ln(1+x^2) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^{2k}$$

25. $\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$. Therefore,

$$\sqrt{x}\arctan\left(\sqrt{x}\right) = x^{1/2}\sum_{0}^{\infty} \frac{(-1)^k x^{(2k+1)/2}}{2k+1} = x^{\frac{1}{2}}\sum_{0}^{\infty} \frac{(-1)^k x^{k+\frac{1}{2}}}{2k+1} = \sum_{0}^{\infty}(-1)^k \frac{x^{k+1}}{2k+1}$$

26. $a^x = e^{x\ln a}$. Therefore,

$$a^x = \sum_{k=0}^{\infty} \frac{(\ln a)^k}{k!}\, x^k$$

27. $\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\, x^k$. Therefore,

$$\ln(1+x^2) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\,(x^2)^k \quad \text{and} \quad \ln(1-x^2) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\,(-x^2)^k.$$

$$f(x) = x\ln\frac{1+x^2}{1-x^2} = x[\ln(1+x^2) - \ln(1-x^2)] = x\left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\,(x^2)^k - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\,(-x^2)^k\right)$$

$$= 2\sum_{k=0}^{\infty} \frac{x^{4k+3}}{2k+1}$$

28. $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$. Therefore,

$$(x+x^2)\sin x^2 = (x+x^2)\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(x^2)^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(x^{4k+3} + x^{4k+4}\right).$$

29. $f(x) = (1-x)^{1/3}$ $\qquad f(0) = 1$

$f'(x) = -\frac{1}{3}(1-x)^{-\frac{2}{3}}$ $\qquad f'(0) = -\frac{1}{3}$

$f''(x) = -\frac{2}{9}(1-x)^{-\frac{5}{3}}$ $\qquad f''(0) = -\frac{2}{9}$

$f'''(x) = -\frac{10}{27}(1-x)^{-\frac{7}{3}}$ $\qquad f'''(0) = -\frac{10}{27}$

$P_3(x) = 1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3$

30. $f(x) = \arcsin x$ $\qquad f(0) = 0$

$f'(x) = \frac{1}{\sqrt{1-x^2}}$ $\qquad f'(0) = 1$

$f''(x) = \frac{x}{(1-x^2)^{3/2}}$ $\qquad f''(0) = 0$

$f'''(x) = \frac{3x^2}{(1-x^2)^{5/2}} + \frac{1}{(1-x^2)^{3/2}}$ $\qquad f'''(0) = 1$

$f^{(4)}(x) = \frac{15x^3}{(1-x^2)^{7/2}} + \frac{9x}{(1-x^2)^{5/2}}$ $\qquad f^{(4)}(0) = 0$

$P_4(x) = 1 + \frac{1}{6}\,x^3$

31. $\left[-\frac{1}{5}, \frac{1}{5}\right)$; ratio test: $\dfrac{b_{k+1}}{b_k} = \dfrac{5k}{k+1}|x| \to 5|x| \Longrightarrow r = \dfrac{1}{5}$

At $x = -\frac{1}{5}$, $\sum \dfrac{(-1)^k}{k}$ converges; at $x = \frac{1}{5}$, $\sum \dfrac{1}{k}$ diverges.

32. $(-3, 3)$; ratio test: $\dfrac{b_{k+1}}{b_k} = \dfrac{1}{3}|x| \implies r = 3$

at $x = -3$, $\sum \dfrac{(-1)^k}{3^k}(-3)^{k+1} = \sum 3(-1)^{2k+1}$ diverges;

at $x = 3$, $\sum \dfrac{(-1)^k}{3^k} 3^{k+1} = \sum 3(-1)^k$ diverges

33. $(-\infty, \infty)$; ratio test: $\dfrac{b_{k+1}}{b_k} = \dfrac{2|x-1|^2}{(2k+2)(2k+1)} \to 0 \Longrightarrow r = \infty$

34. $(0, 4)$; ratio test: $\dfrac{b_{k+1}}{b_k} = \dfrac{1}{2}|x-2| \implies r = 2$

at $x = 0$, $\sum \dfrac{1}{2^k}(-2)^k = \sum (-1)^k$ diverges;

at $x = 4$, $\sum \dfrac{1}{2^k} 2^k = \sum 1$ diverges

35. $(-9, 9)$; ratio test: $\dfrac{b_{k+1}}{b_k} = \dfrac{k+1}{9k}|x| \to \dfrac{1}{9}|x| \Longrightarrow r = 9$

At $x = -9$, $\sum k$ diverges; at $x = 9$, $\sum (-1)^k k$ diverges

36. $(-1, 1)$; ratio test: $\dfrac{b_{k+1}}{b_k} = \dfrac{(k+1)(2k+1)}{k(2k+3)}|x|^2 \to |x|^2 \Longrightarrow r = 1$

at $x = 1$, $\sum \dfrac{k}{2k+1}$ diverges $\left(\frac{k}{2k+1} \to \frac{1}{2} \text{ as } k \to \infty\right)$;

at $x = -1$, $\sum \dfrac{-k}{2k+1}$ diverges for the same reason.

37. $(-4, -2]$; ratio test: $\dfrac{b_{k+1}}{b_k} = \dfrac{\sqrt{k}}{\sqrt{k+1}}|x+3| \to |x+3| \Longrightarrow r = 1$

at $x = -2$, $\sum \dfrac{(-1)^k}{\sqrt{k}}$ converges;

at $x = -4$, $\sum \dfrac{1}{\sqrt{k}}$ diverges

38. diverges except at $x = -1$: $\dfrac{b_{k+1}}{b_k} = (k+1)|x+1| \to \infty \Longrightarrow r = 0$

39. $f(x) = e^{-2x} = e^{-2(x+1)+2} = e^2 \cdot e^{-2(x+1)} = e^2 \displaystyle\sum_0^\infty \frac{[-2(x+1)]^k}{k!} = e^2 \sum_0^\infty \frac{(-1)^k\, 2^k}{k!}(x+1)^k$; $r = \infty$.

40. $\sin 2x = \sum_{0}^{\infty} \frac{(-1)^k 2^{2k}}{(2k)!}\left(x - \frac{\pi}{4}\right)^{2k}; \quad r = \infty.$

41. $f(x) = \ln x = \ln[1 + (x-1)] = \sum_{1}^{\infty} \frac{(-1)^{k+1}}{k}(x-1)^k; \quad r = 1$

42. $\sqrt{x+1} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \sum_{3}^{\infty}(-1)^{k+1}\frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2k-3)}{2^k\, k!}x^k, \quad r = 1$

43. $\frac{1}{1+x^4} = \sum_{k=0}^{\infty}(-1)^k x^{4k}$

$$\int_0^{1/2} \frac{1}{1+x^4}\,dx = \sum_{k=0}^{\infty}(-1)^k \int_0^{1/2} x^{4k}dx = \sum_{k=0}^{\infty}(-1)^k \frac{1}{4k+1}\frac{1}{2^{4k+1}}$$

This is an alternating series with decreasing terms and the third term $\frac{1}{9(2^9)} \approx 0.0002$. Hence

$$\int_0^{1/2} \frac{1}{1+x^4}dx \approx \frac{1}{2} - \frac{1}{5(2^5)} \approx 0.4938$$

44. $e^x = \sum_{k=0}^{n} \frac{x^k}{k!}$ with remainder $|R_n(x)| \le \max|f^{(n+1)}(t)|\frac{|x|^{n+1}}{(n+1)!}.$

$$R_n(2/3) < 3\frac{(2/3)^{n+1}}{(n+1)!} < 0.01 \quad \text{iff} \left(\frac{3}{2}\right)^{n+1}(n+1)! > 300 \implies n = 4.$$

Therfore $\quad e^{2/3} \approx 1 + \frac{2}{3} + \frac{1}{2}(2/3)^2 + \frac{1}{6}(2/3)^3 + \frac{1}{24}(2/3)^4 \approx 1.9465$

45. $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}; \quad \frac{1-\cos x}{x^2} = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k)!}x^{2(k-1)}$

$$\int_0^1 \frac{1-\cos x}{x^2}\,dx = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k)!}\frac{1}{2k-1}$$

This is an alternating series with decreasing terms and the 4th term $\frac{1}{8!(7)} < 0.0001$. Therefore

$$\int_0^1 \frac{1-\cos x}{x^2}dx \approx \sum_{k=1}^{3}\frac{(-1)^{k+1}}{(2k)!}\frac{1}{2k-1} \approx 0.4864$$

46. $\sin x = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1}; \quad x\sin x^4 = \sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(2k+1)!}x^{8k+5}.$

$$\int_0^1 x\sin x^4\,dx = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\frac{1}{8k+6}$$

This is an alternating series with decreasing terms and the 3th term $\dfrac{1}{5!(24)} < 0.01$. Therefore

$$\int_0^1 x \sin x^4 \, dx \approx \frac{1}{6} - \frac{1}{3!(14)} \approx 0.155$$

47. Let $g(x) = \sin x$ and $a = \pi/4$. Then $\sin x = \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}(x - \pi/4) - \dfrac{\sqrt{2}}{2(2!)}(x - \pi/4)^2 - \dfrac{\sqrt{2}}{2(3!)}(x - \pi/4)^3 + \cdots$

$$\begin{aligned} |R_n(x)| &= \frac{|g^{(n+1)}(c)|}{(n+1)!} \left| \left(x - \frac{\pi}{4}\right)^{n+1} \right| \\ &\le \frac{|\,(x - \pi/4)\,|^{n+1}}{(n+1)!} \qquad (g^{(n+1)}(c) = \pm \sin c \text{ or } \pm \cos c) \end{aligned}$$

Now, $48^\circ = \dfrac{48\pi}{180}$ radians. We want to find the smallest positive integer n such that $|R_n(48\pi/180 - \pi/4)| < 0.0001$.

$$|R_n(48\pi/180 - \pi/4)| \le \left(\frac{\pi}{60}\right)^{n+1} \frac{1}{(n+1)!} \cong \frac{(0.05236)^{n+1}}{(n+1)!} < 0.0001 \quad \Longrightarrow \quad n \ge 2.$$

$\sin x \cong P_2(x) = \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}\left(x - \dfrac{\pi}{4}\right) - \dfrac{\sqrt{2}}{4}\left(x - \dfrac{\pi}{4}\right)^2; \quad \sin 48^\circ \cong \dfrac{\sqrt{2}}{2}\left[1 + \dfrac{\pi}{60} - \dfrac{1}{2}\left(\dfrac{\pi}{60}\right)^2\right] \cong 0.7432$

48. $\displaystyle\int_0^1 x^2 e^{-x^2} \, dx = \sum_{k=0}^{\infty} \int_0^1 \frac{(-1)^k}{k!} x^{2k+2} \, dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{2k+3}$

This is an alternating series with decreasing terms and the 6th term $\dfrac{1}{5!(13)} < 0.001$. Therefore

$$\int_0^1 x^2 e^{-x^2} \, dx \approx \sum_{k=0}^{4} \frac{(-1)^k}{k!} \frac{1}{2k+3} \approx 0.1900$$

49. For the sine function, $x - \frac{1}{6}x^3 + \frac{1}{120}x^5 = P_5 = P_6$. Therefore, for $x \in [0, \pi/4]$ we have

$$|\sin x - P_5(x)| = \left| \frac{f^{(7)}(c)}{7!} x^7 \right| \le \frac{1}{7!}\left(\frac{\pi}{4}\right)^7 < 0.000037$$

$$\left(|f^{(7)}(c)| = \cos c \le 1\right) \uparrow$$

50. For the cosine function, $1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 = P_6 = P_7$. Therefore, for $x \in [0, \pi/4]$ we have

$$|\cos x - P_6(x)| = \left| \frac{f^{(8)}(c)}{8!} x^8 \right| \le \frac{1}{8!}\left(\frac{\pi}{4}\right)^8 < 0.0000036$$

$$\left(|f^{(8)}(c)| = \cos c \le 1\right) \uparrow$$

51. $\sum_{k=1}^{\infty} a_k = \int_1^{\infty} xe^{-x}\,dx = \frac{2}{e}$

52. Let $\epsilon > 0$. For each positive integer n, set $a_n = x_n - \frac{\epsilon}{2^{n+1}}$ and $b_n = x_n + \frac{\epsilon}{2^{n+1}}$. Then

$$b_n - a_n = \frac{\epsilon}{2^n} \quad \text{and} \quad \sum_{n=1}^{\infty}(b_n - a_n) = \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \frac{\epsilon}{2} < \epsilon$$

53. If $\sum_{k=1}^{\infty}(a_{k+1} - a_k)$ converges, then the sequence of partial sums $s_n = a_{n+1} - a_1$ converges.

Therefore, the sequence a_k converges.

If the sequence a_k converges, then the sequence $s_n = a_{n+1} - a_1$ converges which implies that the series $\sum_{k=1}^{\infty}(a_{k+1} - a_k)$ converges.

54. (a) $a_k = \sum_{n=0}^{\infty}(\frac{1}{k})^n = \frac{1}{1 - 1/k} = \frac{k}{k-1}$ and $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{k}{k-1}$.

The series diverges because $a_k = \frac{k}{k-1} \not\to 0$.

(b) $a_k = \sum_{n=1}^{\infty}(\frac{1}{k})^n = \frac{1/k}{1 - 1/k} = \frac{1}{k-1}$ and $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{k-1}$.

The series diverges; limit comparison with $\sum \frac{1}{k}$.

(c) $a_k = \sum_{n=2}^{\infty}(\frac{1}{k})^n = \frac{1/k^2}{1 - 1/k} = \frac{1}{k(k-1)}$ and $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{k(k-1)}$.

The series converges; limit comparison with $\sum \frac{1}{k^2}$.