P1: PBU/OVY P2: PBU/OVY QC: PBU/OVY T1: PBU
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3. $4\arctan\frac{1}{5} - \arctan\frac{1}{239} < 4\sum_{k=1}^{15}\frac{(-1)^{k-1}}{2k-1}\left(\frac{1}{5}\right)^{2k-1} - \left[\sum_{k=1}^{4}\frac{(-1)^{k-1}}{2k-1}\left(\frac{1}{239}\right)^{2k-1}\right]$

$= 0.785398163397448309616$

$4\arctan\frac{1}{5} - \arctan\frac{1}{239} > 4\sum_{k=1}^{14}\frac{(-1)^{k-1}}{2k-1}\left(\frac{1}{5}\right)^{2k-1} \left[\sum_{k=1}^{3}\frac{(-1)^{k-1}}{2k-1}\left(\frac{1}{239}\right)^{2k-1}\right]$

$= 0.785398163397448306408$

These inequalities imply $3.14159265358979322563 < \pi < 3.14159265358979323846.$

REVIEW EXERCISES

1. $\sum_{k=0}^{\infty}\left(\frac{3}{4}\right)^k = \frac{1}{1-\frac{3}{4}} = 4,$ a geometric series with $r = \frac{3}{4}$.

2. $\sum_{k=0}^{\infty}(-1)^k(\frac{1}{2})^k = \sum_{0}^{\infty}\left(-\frac{1}{2}\right)^k = \frac{1}{1-(-\frac{1}{2})} = \frac{2}{3},$ a geometric series with $r = -\frac{1}{2}$

3. Since $e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}, \quad \sum_{k=0}^{\infty}\frac{(\ln 2)^k}{k!} = e^{\ln 2} = 2$

4. $\sum_{k=1}^{\infty}\frac{1}{k(k+1)} = \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$

5. diverges; limit comparison with $\sum\frac{1}{k}$

6. converges; limit comparison with $\sum\frac{1}{k^2}$

7. converges; root test: $\left(\frac{k+1}{3^k}\right)^{1/k} \to \frac{1}{3}$ as $k\to\infty$, or ratio test: $\frac{k+2}{3^{k+1}}\frac{3^k}{k+1} \to \frac{1}{3}$ as $k\to\infty$

8. diverges; ratio test:

$$\frac{(k+1)!}{(k+1)^{(k+1)/2}}\cdot\frac{k^{k/2}}{k!} = \frac{k^{k/2}}{(k+1)^{(k-1)/2}} = \left(\frac{k}{k+1}\right)^{k/2}\sqrt{k+1} \to \infty.$$

9. converges; limit comparison with: $\sum\frac{1}{k^2}$

10. converges; root test: $\left[k\left(\frac{3}{4}\right)^k\right]^{1/k} \to \frac{3}{4}$ as $k\to\infty$

11. converges; ratio test, $\frac{a_{k+1}}{a_k} = \left(\frac{k+1}{k}\right)^e\cdot\frac{1}{e} \to \frac{1}{e} < 1$

12. diverges; ratio test, $\frac{a_{k+1}}{a_k} = \frac{[2(k+1)]!}{2^{k+1}(k+1)!}\cdot\frac{2^k k!}{(2k)!} = 2k+1 \to \infty$

13. converges; basic comparison,

$$\sum \frac{(\arctan k)^2}{1+k^2} \le \frac{\pi^2}{4} \sum \frac{1}{1+k^2} \le \frac{\pi^2}{4} \sum \frac{1}{k^2}$$

14. converges; $\dfrac{2^k + k^4}{3^k} = \dfrac{2^k}{3^k} + \dfrac{k^4}{3^k}$, and each of the series $\sum \dfrac{2^k}{3^k}$ and $\sum \dfrac{k^4}{3^k}$ is convergent.

15. absolutely convergent; basic comparison

$$\sum \left| \frac{(-1)^k}{(k+1)(k+2)} \right| \le \sum \frac{1}{k^2}$$

16. conditionally convergent: $\sum \dfrac{(-1)^k}{2k+1}$ converges by Theorem 11.4.3;

$\sum \left| \dfrac{(-1)^k}{2k+1} \right| = \sum \dfrac{1}{2k+1}$ diverges.

17. absolutely convergent; $\displaystyle\sum_{k=0}^{\infty} \left| \frac{(-1)^k (100)^k}{k!} \right| = \sum_{k=0}^{\infty} \frac{100^k}{k!}$ which converges by the ratio test.

18. conditionally convergent: $\sum \dfrac{(-1)^k}{\sqrt{(k+1)(k+2)}}$ converges by Theorem 11.4.3;

$\sum \left| \dfrac{(-1)^k}{\sqrt{(k+1)(k+2)}} \right| = \sum \dfrac{1}{\sqrt{(k+1)(k+2)}}$ diverges – limit comparison with $\sum \dfrac{1}{k}$.

19. converges conditionally; Theorem 12.5.3: $\sum \dfrac{\ln k}{\sqrt{k}}$ diverges by the integral test.

20. absolutely convergent; limit comparison with $\sum \dfrac{1}{k^{3/2}}$

21. diverges; limit comparison with $\sum \dfrac{1}{k}$: $\dfrac{1}{k} - \dfrac{1}{k+1} - \dfrac{1}{k+2} = \dfrac{2-k^2}{k(k+1)(k+2)}$

22. $\displaystyle 1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \cdots = \sum_{k=0}^{\infty} \left(\frac{1}{2k+1} - \frac{1}{(2k+2)^2} \right) = \sum_{k=0}^{\infty} \frac{4k^2+6k+3}{(2k+1)(2k+2)^2}$

diverges; limit comparison with $\sum \dfrac{1}{k}$.

23. $\displaystyle e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. Therefore,

$$xe^{2x^2} = x \sum_{k=0}^{\infty} \frac{(2x^2)^k}{k!} = \sum_{k=0}^{\infty} \frac{2^k}{k!} x^{2k+1}$$

24. $\displaystyle \ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k$. Therefore,

$$\ln(1+x^2) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^{2k}$$