**Exercises 21–28.** Find the Taylor series expansion in powers of $x$.

**21.** $f(x) = xe^{2x^2}$.

**22.** $f(x) = \ln(1 + x^2)$.

**23.** $f(x) = \sqrt{x} \arctan \sqrt{x}$.

**24.** $f(x) = a^x, a > 0$.

**25.** $f(x) = x \ln\left(\dfrac{1 + x^2}{1 - x^2}\right)$.

**26.** $f(x) = (x + x^2) \sin x^2$.

**27.** $f(x) = (1 - x)^{1/3}$ up to $x^3$.

**28.** $f(x) = \arcsin x$ up to $x^4$.

**Exercises 29–36.** Find the interval of convergence.

**29.** $\sum \dfrac{5^k}{k} x^k$.

**30.** $\sum \dfrac{(-1)^k}{3^k} x^{k+1}$.

**31.** $\sum \dfrac{2^k}{(2k)!} (x - 1)^{2k}$.

**32.** $\sum \dfrac{1}{2^k} (x - 2)^k$.

**33.** $\sum \dfrac{(-1)^k k}{3^{2k}} x^k$.

**34.** $\sum \dfrac{k}{2k + 1} x^{2k+1}$.

**35.** $\sum \dfrac{(-1)^k}{\sqrt{k}} (x + 3)^k$.

**36.** $\sum \dfrac{k!}{2} (x + 1)^k$.

**Exercises 37–40.** Find the Taylor series expansion of $f$ and give the radius of convergence.

**37.** $f(x) = e^{-2x}$ in powers of $(x + 1)$.

**38.** $f(x) = \sin 2x$ in powers of $(x - \pi/4)$.

**39.** $f(x) = \ln x$ in powers of $(x - 1)$.

**40.** $f(x) = \sqrt{x + 1}$ in powers of $x$.

**Exercises 41–46.** Estimate within the accuracy indicated from a series expansion.

**41.** $\displaystyle\int_0^{1/2} \frac{dx}{1+x^4}$, 0.01. **42.** $e^{2/3}$, 0.01.

**43.** $\sqrt[3]{68}$, 0.01. **44.** $\displaystyle\int_0^1 x \sin x^4 \, dx$, 0.01.

**45.** $\sin 48°$, 0.0001. **46.** $\displaystyle\int_0^1 x^2 e^{-x^2} \, dx$, 0.001.

**47.** Use the Lagrange form of the remainder to show that the approximation

$$\sin x \cong x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5$$

is accurate to four decimal places for $0 \leq x \leq \pi/4$.

**48.** Use the Lagrange form of the remainder to show that the approximation

$$\cos x \cong 1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \tfrac{1}{720}x^6$$

is accurate to five decimal places for $0 \leq x \leq \pi/4$.

**49.** Find the sum of the series

$$\sum_{k=1}^{\infty} a_k \quad \text{given that } a_k = \int_k^{k+1} x e^{-x} \, dx.$$

**50.** Show that every sequence of real numbers can be covered by a sequence of open intervals of arbitrarily small total length; namely, show that if $x_1, x_2, x_3, \ldots$ is a sequence of real numbers and $\epsilon$ is positive, then there exists a sequence of open intervals $(a_n, b_n)$ with $a_n < x_n < b_n$ such that

$$\sum_{n=1}^{\infty} (b_n - a_n) < \epsilon.$$

**51.** Prove that the series $\sum_{k=1}^{\infty}(a_{k+1} - a_k)$ converges iff the sequence $a_k$ converges.

**52.** Determine whether or not the series $\sum_{k=2}^{\infty} a_k$ converges or diverges. If it converges, find the sum.

(a) $a_k = \sum_{n=0}^{\infty}\left(\frac{1}{k}\right)^n$.

(b) $a_k = \sum_{n=1}^{\infty}\left(\frac{1}{k}\right)^n$.

(c) $a_k = \sum_{n=2}^{\infty}\left(\frac{1}{k}\right)^n$.