Exercises 1–6. Find the least upper bound (if it exists) and the greatest lower bound (if it exists).

1. $\{x : |x-2| \le 3\}$.

2. $\{x : x^2 > 3\}$.

3. $\{x : x^2 - x - 2 \le 0\}$.

4. $\{x : \cos x \le 1\}$.

5. $\{x : e^{-x^2} \le 2\}$.

6. $\{x : \ln x < e\}$.

Exercises 7–12. Determine the boundedness and monotonicity of the sequence with a_n as indicated.

7. $\dfrac{2n}{3n+1}$.

8. $\dfrac{n^2-1}{n}$.

9. $1 + \dfrac{(-1)^n}{n}$.

10. $\dfrac{4^n}{1+4^n}$.

11. $\dfrac{2^n}{n^2}$.

12. $\dfrac{\sin(n\pi/2)}{n^2}$.

Exercises 13–26. State whether the sequence converges and if it does, find the limit.

13. $n\,2^{1/n}$.

14. $\dfrac{(n+1)(n+2)}{(n+3)(n+4)}$.

15. $\left(\dfrac{n}{1+n}\right)^{1/n}$.

16. $\dfrac{4n^2+5n+1}{n^3+1}$.

17. $\cos(\pi/n)\sin(\pi/n)$.

18. $\left(2+\dfrac{1}{n}\right)^n$.

19. $\left[\ln\left(1+\dfrac{1}{n}\right)\right]^n$.

20. $3\ln 2n - \ln(n^3+1)$.

21. $\dfrac{3n^2-1}{\sqrt{4n^4+2n^2+3}}$.

22. $\dfrac{\sqrt[3]{n^2+4}}{2n+1}$.

23. $(\pi/n)\cos(\pi/n)$.

24. $(n/\pi)\sin(n\pi)$.

25. $\displaystyle\int_n^{n+1} e^{-x}\,dx$.

26. $\displaystyle\int_1^n \frac{1}{\sqrt{x}}\,dx$.

27. Show that, if $a_n \to L$, then $a_{n+1} \to L$.

28. Suppose that the sequence a_n converges to L. Define the sequence m_n by

$$m_n = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Prove that $m_n \to L$.

29. Choose any real number a and form the sequence

$$\cos a, \cos(\cos a), \cos(\cos(\cos a)), \cdots.$$

Convince yourself numerically that this sequence converges to some number L. Determine L and verify that $\cos L = L$. (This is an effective numerical method for solving the equation $\cos x = x$.)

30. Find a numerical solution to the equation $\sin(\cos x) = x$. HINT: Use the method of Exercise 31.

Exercises 31–40. Calculate.

31. $\lim_{x\to\infty} \frac{5x + 2\ln x}{x + 3\ln x}$.

32. $\lim_{x\to 0} \frac{e^x - 1}{\tan 2x}$.

33. $\lim_{x\to 0} \frac{\ln(\cos x)}{x^2}$.

34. $\lim_{x\to 1} x^{1/(x-1)}$.

35. $\lim_{x\to 0} \left(1 + \frac{4}{x}\right)^{2x}$.

36. $\lim_{x\to 0} \frac{e^{2x} - e^{-2x}}{\sin x}$.

37. $\lim_{x\to 0+} x^2 \ln x$.

38. $\lim_{x\to\infty} \frac{(10)^x}{x^{10}}$.

39. $\lim_{x\to 0} \frac{e^x + e^{-x} - x^2 - 2}{\sin^2 x - x^2}$.

40. $\lim_{x\to 1} \csc(\pi x)\ln x$.

41. Calculate $\lim_{x\to\infty} xe^{-x^2} \int_0^x e^{x^2}\,dx$.

42. Let n be a positive integer. Calculate $\lim_{x\to 0} \frac{e^{-1/x^2}}{x^n}$.

Exercises 43–50. Determine whether the integral converges and, if so, evaluate the integral.

43. $\displaystyle\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx.$

44. $\displaystyle\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx.$

45. $\displaystyle\int_0^1 \frac{1}{1-x^2}\,dx.$

46. $\displaystyle\int_0^{\pi/2} \sec x\,dx.$

47. $\displaystyle\int_1^\infty \frac{\sin(\pi/x)}{x^2}\,dx.$

48. $\displaystyle\int_0^9 \frac{1}{(x-1)^{2/3}}\,dx.$

49. $\displaystyle\int_0^\infty \frac{1}{e^x+e^{-x}}\,dx.$

50. $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^k}\,dx.$

51. Evaluate $\displaystyle\int_0^a \ln(1/x)\,dx$ for $a > 0$.

52. Find the length of the curve $y = (a^{2/3} - x^{2/3})^{3/2}$ from $x = 0$ to $x = a$, $a > 0$.

53. Let S and T be nonempty sets of real numbers which are bounded above. Let $S + T$ be the set defined by

$$S + T = \{x + y : x \in S \text{ and } y \in T\}.$$

Prove that lub $(S + T) =$ lub $S +$ lub T.

54. Let S be a nonempty set which is bounded below. Let $B = \{b : b \text{ is a lower bound of } S\}$. Show that (a) B is nonempty; (b) B is bounded above; (c) lub B = glb S.

55. Let f be a function continuous on $(-\infty, \infty)$ and L a real number.

(a) Show that

$$\text{if} \quad \int_{-\infty}^{\infty} f(x)\,dx = L \quad \text{then} \quad \lim_{c\to\infty} \int_{-c}^{c} f(x)\,dx = L.$$

(b) Find an example which shows that the converse of (a) is false.

56. Show that

$$\int_{-\infty}^{\infty} f(x)\,dx = L \quad \text{iff} \quad \lim_{c\to\infty} \int_{-c}^{c} f(x)\,dx = L$$

in the event that f is (a) nonnegative or (b) even.

57. In general the least upper bound of a set of numbers need not be in the set. Show that the least upper bound of a set of integers must be in the set.

58. Let f be a function continuous on $[a,b]$. As usual, denote by $L_f(P)$ and $U_f(P)$ the upper and lower sums that correspond to the partition P. What is the least upper bound of all $L_f(P)$? What is the greatest lower bound of all $U_f(P)$?