

## 626 REVIEW EXERCISES

If  $\int_1^\infty f(x) dx$  converges, then  $F$ , being continuous, is bounded and, by (\*),  $\{a_n\}$  is bounded and therefore convergent. If  $\{a_n\}$  converges, then  $\{a_n\}$  is bounded and, by (\*),  $F$  is bounded. Being increasing,  $F$  is also convergent; i.e.,  $\int_1^\infty f(x) dx$  converges.

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1.  $|x - 2| \leq 3 \implies -1 \leq x \leq 5$ : lub = 5, glb = -1.
2.  $x^2 > 3 \implies x > \sqrt{3}$  or  $x < -\sqrt{3}$ ; no lub, no glb.
3.  $x^2 - x - 2 \leq 0 \implies (x - 2)(x + 1) \leq 0 \implies -1 \leq x \leq 2$ : lub = 2, glb = -1.
4.  $\cos x \leq 1$  for all  $x$ ; no lub, no glb.
5. Since  $e^{-x^2} \leq 1$  for all  $x$ ,  $e^{-x^2} \leq 2$  for all  $x$ ; no lub, no glb.
6.  $\ln x < e \implies 0 < x < e^e$ : lub =  $e^e$ , glb = 0.
7. increasing; bounded below by  $\frac{1}{2}$  and above by  $\frac{2}{3}$ .
8. increasing; bounded below by 0 but not bounded above:  $\frac{n^2 - 1}{n} = n - \frac{1}{n} \rightarrow \infty$  as  $n \rightarrow \infty$ .
9. bounded below by 0 and above by  $\frac{3}{2}$ ; not monotonic
10. increasing; bounded below by  $\frac{4}{5}$  and above by 1.
11.  $\left\{ \frac{2^n}{n^2} \right\} = \left\{ 2, 1, \frac{8}{9}, 1, \frac{32}{25}, \dots \right\}$ ; the sequence is not monotonic.  
 However, it is increasing from  $a_3$  on. The sequence is bounded below by  $\frac{8}{9}$ ; it is not bounded above.
12.  $\left\{ \frac{\sin(n\pi/2)}{n^2} \right\} = \left\{ 1, 0, -\frac{1}{9}, 0, \frac{1}{25}, \dots \right\}$ ; bounded below by  $-\frac{1}{9}$  and above by 1; not monotonic
13. the sequence does not converge;  $n2^{1/n} \rightarrow \infty$  as  $n \rightarrow \infty$
14. converges to 1:  $\frac{n^2 + 3n + 2}{n^2 + 7n + 12} = \frac{1 + \frac{3}{n} + \frac{2}{n^2}}{1 + \frac{7}{n} + \frac{12}{n^2}} \rightarrow 1$ .
15. converges to 1:  $\lim_{n \rightarrow \infty} \frac{1}{n} \ln \left( \frac{n}{n+1} \right) = 0 \implies \lim_{n \rightarrow \infty} \left( \frac{n}{1+n} \right)^{1/n} = 1$

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16. converges to 0:  $\frac{4n^2 + 5n + 1}{n^3 + 1} = \frac{\frac{4}{n} + \frac{5}{n^2} + \frac{1}{n^3}}{1 + \frac{1}{n^3}} \rightarrow 0$ .
17. converges to 0:  $\cos \frac{\pi}{n} \sin \frac{\pi}{n} \rightarrow \cos 0 \sin 0 = 0$  as  $n \rightarrow \infty$
18. diverges:  $(2 + \frac{1}{n})^n > 2^n$  and  $2^n$  diverges.
19. converges to 0:  $0 = [\ln 1]^n \leq [\ln(1 + \frac{1}{n})]^n \leq [\ln 2]^n$ ;  $[\ln 2]^n \rightarrow 0$  as  $n \rightarrow \infty$ .
20. converges to  $\ln 8$ :  $3 \ln 2n - \ln(n^3 + 1) = \ln \frac{8n^3}{n^3 + 1} \rightarrow \ln 8$ .
21. converges to  $\frac{3}{2}$ :  $\frac{3n^2 - 1}{\sqrt{4n^4 + 2n^2 + 3}} = \frac{3 - \frac{1}{n^2}}{\sqrt{4 + \frac{2}{n^2} + \frac{3}{n^4}}} \rightarrow \frac{3}{2}$  as  $n \rightarrow \infty$ .
22. converges to 0:  $\frac{(n^2 + 4)^{\frac{1}{3}}}{2n + 1} = \frac{(1/n + 4/n^3)^{1/3}}{2 + 1/n} \rightarrow 0$  as  $n \rightarrow \infty$ .
23. converges to 0:  $\frac{\pi}{n} \cos \frac{\pi}{n} \rightarrow 0 \cos 0 = 0$  as  $n \rightarrow \infty$ .
24. converges 0:  $(n/\pi) \sin(n\pi) = 0$  for all positive integers  $n$ .
25. converges to 0:  $\int_n^{n+1} e^{-x} dx = [-e^{-x}]_n^{n+1} = e^{-n}(1 - \frac{1}{e}) \rightarrow 0$  as  $n \rightarrow \infty$
26. diverges:  $\int_1^n \frac{1}{\sqrt{x}} dx = [2\sqrt{x}]_1^n = 2\sqrt{n} - 2$  and  $2\sqrt{n} - 2$  diverges.
27. Given  $\epsilon > 0$ . Since  $a_n \rightarrow L$ , there exists a positive integer  $K$  such that if  $n \geq K$ , then  $|a_n - L| < \epsilon$ . Now, if  $n \geq K - 1$ , then  $n + 1 \geq K$  and  $|a_{n+1} - L| < \epsilon$ . Therefore,  $a_{n+1} \rightarrow L$ .
28. Let  $\epsilon > 0$ . Since  $a_n \rightarrow L$ , there is positive integer  $K$  such that if  $n \geq K$ ,

$$|a_n - L| < \frac{\epsilon}{2}.$$

The set  $\{|a_1 - L|, \dots, |a_K - L|\}$  is a finite set so there is a positive integer  $N$  such that if  $n > N$ ,

$$\frac{|a_i - L|}{n} < \frac{\epsilon}{2K}, i = 1, 2, \dots, K.$$

Let  $M = \max\{K, N\}$ . Then, if  $n \geq M$ ,

$$\left| \frac{a_1 + \dots + a_n}{n} - L \right| \leq \sum_{i=1}^K \frac{|a_i - L|}{n} + \sum_{j=K+1}^n \frac{|a_j - L|}{n} < K\left(\frac{\epsilon}{2K}\right) + n\left(\frac{\epsilon}{2n}\right) = \epsilon.$$

Therefore  $m_n \rightarrow L$ .

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29. As an example, let  $a = \frac{\pi}{3}$ . Then

$$\cos \pi/3 = 0.5, \quad \cos \cos 0.5 \cong 0.87758, \dots$$

Using technology (graphing calculator, CAS), we get

$$\cos \cos \dots \cos \pi/3 \rightarrow 0.73910.$$

and  $\cos(0.73910) \cong 0.73910$ .

Hence, numerically, this sequence converges to 0.73910.

30. Let  $f(x) = \sin(\cos x)$  and let  $a = \pi/3$ . Then

$$f(\pi/3) \cong 0.4794, f(f(\pi/3)) \cong 0.7753, \dots$$

After 14 steps, we get  $f(f(\dots f(\pi/3))) \cong 0.6948$  and  $\sin(\cos 0.6948) \cong 0.6948$ .

$$31. \lim_{x \rightarrow \infty} \frac{5x + 2 \ln x}{x + 3 \ln x} = \lim_{x \rightarrow \infty} \frac{5 + 2 \frac{\ln x}{x}}{1 + 3 \frac{\ln x}{x}} = 5; \quad \left( \lim_{x \rightarrow \infty} \frac{\ln x}{x} = 0 \right)$$

$$32. \lim_{x \rightarrow 0} \frac{e^x - 1}{\tan 2x} \stackrel{*}{=} \lim_{x \rightarrow 0} \frac{e^x}{2 \sec^2 2x} = \frac{1}{2}$$

$$33. \lim_{x \rightarrow 0} \frac{\ln(\cos x)}{x^2} \stackrel{*}{=} \lim_{x \rightarrow 0} \frac{-\sin x}{\cos x} = \lim_{x \rightarrow 0} \frac{-1}{2 \cos x} \cdot \frac{\sin x}{x} = -\frac{1}{2}$$

$$34. \text{Set } y = x^{1/(x-1)}. \text{ Then } \ln y = \frac{\ln x}{x-1}, \quad \text{and} \quad \lim_{x \rightarrow 1} \frac{\ln x}{x-1} \stackrel{*}{=} \lim_{x \rightarrow 1} \frac{1}{x} = 1$$

Therefore,  $\lim_{x \rightarrow 1} x^{1/(x-1)} = e$ .

$$35. \lim_{x \rightarrow \infty} \left(1 + \frac{4}{x}\right)^{2x} = \lim_{x \rightarrow \infty} \left[\left(1 + \frac{4}{x}\right)^x\right]^2 = e^8$$

$$36. \lim_{x \rightarrow 0} \frac{e^{2x} - e^{-2x}}{\sin x} \stackrel{*}{=} \lim_{x \rightarrow 0} \frac{2e^{2x} + 2e^{-2x}}{\cos x} = 4$$

$$37. \lim_{x \rightarrow 0^+} x^2 \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x^2}} \stackrel{*}{=} \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{\frac{-2}{x^3}} = \lim_{x \rightarrow 0^+} \frac{-x^2}{2} = 0$$

$$38. \lim_{x \rightarrow \infty} \frac{10^x}{x^{10}} \stackrel{*}{=} \lim_{x \rightarrow \infty} \frac{10^x \ln 10}{10x^9} \stackrel{*}{=} \lim_{x \rightarrow \infty} \frac{10^x (\ln 10)^2}{(10)(9)x^8} \stackrel{*}{=} \dots = \lim_{x \rightarrow \infty} \frac{10^x (\ln 10)^{10}}{10!} \rightarrow \infty$$

$$39. \lim_{x \rightarrow 0} \frac{e^x + e^{-x} - x^2 - 2}{\sin^2 x - x^2} \stackrel{*}{=} \lim_{x \rightarrow 0} \frac{e^x - e^{-x} - 2x}{2 \sin x \cos x - 2x} \stackrel{*}{=} \lim_{x \rightarrow 0} \frac{e^x - e^{-x} - 2}{\sin 2x - 2x} \stackrel{*}{=} \lim_{x \rightarrow 0} \frac{e^x + e^{-x} - 2}{2 \cos 2x - 2} \\ \stackrel{*}{=} \lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{-4 \sin 2x} \stackrel{*}{=} \lim_{x \rightarrow 0} \frac{e^x + e^{-x}}{-8 \cos 2x} = -\frac{1}{4}$$

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$$40. \lim_{x \rightarrow 1} \csc(\pi x) \ln x = \lim_{x \rightarrow 1} \frac{\ln x}{\sin \pi x} \stackrel{*}{=} \lim_{x \rightarrow 1} \frac{1/x}{\pi \cos \pi x} = -\frac{1}{\pi}$$

$$41. \lim_{x \rightarrow \infty} x e^{-x^2} \int_0^x e^{t^2} dt = \lim_{x \rightarrow \infty} \frac{\int_0^x e^{t^2} dt}{\frac{1}{x}} \stackrel{*}{=} \lim_{x \rightarrow \infty} \frac{e^{x^2}}{2x^2 e^{x^2} - e^{x^2}} = \lim_{x \rightarrow \infty} \frac{x^2}{2x^2 - 1} = \frac{1}{2}$$

$$42. \text{ Consider } \ln \frac{e^{-1/x^2}}{x^n} = \left( -\frac{1}{x^2} - n \ln |x| \right).$$

$$\lim_{x \rightarrow 0} \left( -\frac{1}{x^2} - n \ln |x| \right) = \lim_{x \rightarrow 0} -\frac{1 + nx^2 \ln |x|}{x^2} = -\infty$$

$$\text{Therefore } \lim_{x \rightarrow 0} \frac{e^{-1/x^2}}{x^n} \rightarrow 0.$$

$$43. \int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx = \lim_{b \rightarrow \infty} \left[ -2e^{-\sqrt{x}} \right]_1^b = \lim_{b \rightarrow \infty} (-2e^{-\sqrt{b}} + 2e^{-1}) = 2e^{-1}$$

$$44. \int \frac{x}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2} + C$$

$$\int_0^1 \frac{x}{\sqrt{1-x^2}} dx = \lim_{b \rightarrow 1^-} \int_0^b \frac{x}{\sqrt{1-x^2}} dx = \lim_{b \rightarrow 1^-} \left[ -\sqrt{1-x^2} \right]_0^b = \lim_{b \rightarrow 1^-} (-\sqrt{1-b^2} + 1) = 1$$

$$45. \int_0^1 \frac{1}{1-x^2} dx = \lim_{a \rightarrow 1^-} \int_0^a \frac{1}{1-x^2} dx = -\frac{1}{2} \lim_{a \rightarrow 1^-} \int_0^a \left( \frac{1}{x-1} - \frac{1}{x+1} \right) dx \quad (\text{partial fractions})$$

$$= \frac{1}{2} \lim_{a \rightarrow 1^-} \left[ \ln(x+1) - \ln(1-x) \right]_0^a = \frac{1}{2} \lim_{a \rightarrow 1^-} \ln \left[ \frac{a+1}{1-a} \right] = \infty;$$

the integral diverges.

$$46. \int_0^{\pi/2} \sec x dx = \lim_{c \rightarrow \pi/2^-} \int_0^c \sec x dx = \lim_{c \rightarrow \pi/2^-} \left[ \ln(\sec x + \tan x) \right]_0^c = \lim_{c \rightarrow \pi/2^-} \ln(\sec c + \tan c) = \infty;$$

the integral diverges.

$$47. \int_1^{\infty} \frac{\sin(\pi/x)}{x^2} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{\sin(\pi/x)}{x^2} dx = \lim_{b \rightarrow \infty} \left[ \frac{1}{\pi} \cos \pi/x \right]_1^b = \frac{2}{\pi}$$

$$48. \int_0^9 \frac{1}{(x-1)^{2/3}} dx = \lim_{c \rightarrow 1^-} \int_0^c \frac{1}{(x-1)^{2/3}} dx + \lim_{c \rightarrow 1^+} \int_c^9 \frac{1}{(x-1)^{2/3}} dx$$

$$\lim_{c \rightarrow 1^-} \int_0^c \frac{1}{(x-1)^{2/3}} dx = \lim_{c \rightarrow 1^-} 3 \left[ (x-1)^{1/3} \right]_0^c = 3$$

$$\lim_{c \rightarrow 1^+} \int_c^9 \frac{1}{(x-1)^{2/3}} dx = \lim_{c \rightarrow 1^+} 3 \left[ (x-1)^{1/3} \right]_c^9 = 6$$

$$\int_0^9 \frac{1}{(x-1)^{2/3}} = 3 + 6 = 9.$$

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49.  $\int \frac{1}{e^x + e^{-x}} dx = \int \frac{e^x}{e^{2x} + 1} dx = \arctan e^x + C$

$$\int_0^\infty \frac{1}{e^x + e^{-x}} dx = \lim_{c \rightarrow \infty} \int_0^c \frac{1}{e^x + e^{-x}} dx = \lim_{c \rightarrow \infty} \arctan e^x \Big|_0^c = \frac{\pi}{2}$$

50. Set  $u = \ln x$ ,  $du = \frac{1}{x} dx$ ;  $u(2) = \ln 2$ . Then

$$\int_2^\infty \frac{1}{x(\ln x)^k} dx = \int_{\ln 2}^\infty \frac{1}{u^k} du$$

The integral converges if  $k > 1$  and diverges otherwise.

For  $k > 1$ ,

$$\int_2^\infty \frac{1}{x(\ln x)^k} dx = \lim_{c \rightarrow \infty} \int_{\ln 2}^c \frac{1}{u^k} du = \lim_{c \rightarrow \infty} \frac{1}{1-k} u^{1-k} \Big|_{\ln 2}^c = \frac{1}{(k-1)(\ln 2)^{k-1}}$$

51. 
$$\int_0^a \ln(1/x) dx = \int_0^a -\ln x dx = \lim_{c \rightarrow 0^+} \int_c^a -\ln x dx = \lim_{c \rightarrow 0^+} [-x \ln x + x]_c^a$$
  

$$= \lim_{c \rightarrow 0^+} [-a \ln a + a + c \ln c - c] = a \ln(1/a) + a$$

52.  $y = (a^{2/3} - x^{2/3})^{3/2}$ ;  $y' = -x^{-1/3}(a^{2/3} - x^{2/3})^{1/2}$

$$L = \int_0^a \sqrt{1 + (y')^2} dx = \int_0^a a^{1/3} x^{-1/3} dx = \frac{3a}{2}$$

53. For any  $a \in S + T$ ,  $a = s + t$  for some  $s \in S$  and  $t \in T$ . Hence  $a \leq \text{lub}(S) + \text{lub}(T)$ .

Therefore,  $S + T$  is bounded above and  $\text{lub}(S) + \text{lub}(T)$  is an upper bound for  $S + T$ .

Let  $M = \text{lub}(S + T)$  and suppose  $M < \text{lub}(S) + \text{lub}(T)$ . Set  $\epsilon = (\text{lub}(S) + \text{lub}(T)) - M$ . There exist  $s \in S$  and  $t \in T$  such that

$$\text{lub}(S) - s < \epsilon/2, \quad \text{and} \quad \text{lub}(T) - t < \epsilon/2$$

Now,

$$\text{lub}(S) + \text{lub}(T) - (s + t) = \text{lub}(S) - s + \text{lub}(T) - t < \epsilon = \text{lub}(S) + \text{lub}(T) - M$$

which implies  $s + t > M$ , a contradiction. Therefore  $\text{lub}(S) + \text{lub}(T) = \text{lub}(S + T)$ .

54. (a) Since  $S$  is bounded below, there is a number  $b$  such that  $b \leq s$  for every  $s \in S$ . Thus  $b$  is a lower bound for  $S$  and  $B \neq \emptyset$ .

(b) Choose any  $s \in S$ . Then, for any  $b \in B$ ,  $b \leq s$ . Therefore  $B$  is bounded above (each element  $s \in S$  is an upper bound for  $B$ ).

(c) We show first that  $\text{glb}(S)$  is an upper bound for  $B$ . For if not, there is  $b \in B$  such that  $b > \text{glb}(S)$ . Then there is an  $s \in S$  such that  $\text{glb}(S) < s < b$ , which contradicts the fact that  $b$  is a lower bound of  $S$ . It follows that  $\text{lub}(B) \leq \text{glb}(S)$ . If  $\text{lub}(B) < \text{glb}(S)$ , then there exists a number  $a$  such that  $\text{lub}(B) < a < \text{glb}(S)$  which implies that  $a$  is a lower bound for  $S$  and  $a \in B$ . Therefore  $a \leq \text{lub}(B)$ , a contradiction. Thus,  $\text{lub}(B) = \text{glb}(S)$ .

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55. (a) If  $\int_{-\infty}^{\infty} f(x) dx = L$ , then

$$\lim_{c \rightarrow \infty} \int_0^c f(x) dx, \quad \text{and} \quad \lim_{b \rightarrow -\infty} \int_b^0 f(x) dx$$

both exist and

$$\lim_{c \rightarrow \infty} \int_0^c f(x) dx + \lim_{b \rightarrow -\infty} \int_b^0 f(x) dx = L$$

Let  $c = -b$ , then

$$\lim_{c \rightarrow \infty} \int_{-c}^c f(x) dx = L$$

(b) Set  $f(x) = x$ . then  $\lim_{c \rightarrow \infty} \int_{-c}^c x dx = 0$ , but  $\int_{-\infty}^{\infty} x dx$  diverges.

56. (a) Assume that  $f$  is nonnegative on  $(-\infty, \infty)$ .

$$\text{By Exercise 55, } \int_{-\infty}^{\infty} f(x) dx = L \implies \lim_{c \rightarrow \infty} \int_{-c}^c f(x) dx = L$$

Now assume that  $\lim_{c \rightarrow \infty} \int_{-c}^c f(x) dx = L$ . Since  $f$  is nonnegative,

$$\int_0^x f(t) dt \leq L \quad \text{on } [0, \infty).$$

Therefore  $\int_0^x f(t) dt$  is a bounded and nondecreasing function, which implies that

$$\lim_{c \rightarrow \infty} \int_0^c f(x) dx \text{ exists. Similarly, } \lim_{c \rightarrow \infty} \int_{-c}^0 f(x) dx \text{ exists.}$$

Therefore,  $\int_{-\infty}^{\infty} f(x) dx$  exists, and, by the uniqueness of the limit,  $\int_{-\infty}^{\infty} f(x) dx = L$ .

57. Let  $S$  be a set of integers which is bounded above. Then there is an integer  $k \in S$  such that  $k \geq n$  for all  $n \in S$ , for if not,  $S$  is not bounded above. Therefore,  $k$  is an upper bound for  $S$ .

Let  $M = \text{lub}(S)$ . Then  $M \geq k$  since  $k \in S$ . Also  $M \leq k$  since  $k$  is an upper bound for  $S$ . Therefore  $M = k$ ; the least upper bound of  $S$  is an element of  $S$ .

58.  $\text{lub}[L_f(P)] = \int_a^b f(x) dx$ ;  $\text{glb}[U_f(P)] = \int_a^b f(x) dx$ .