Vector Calculus

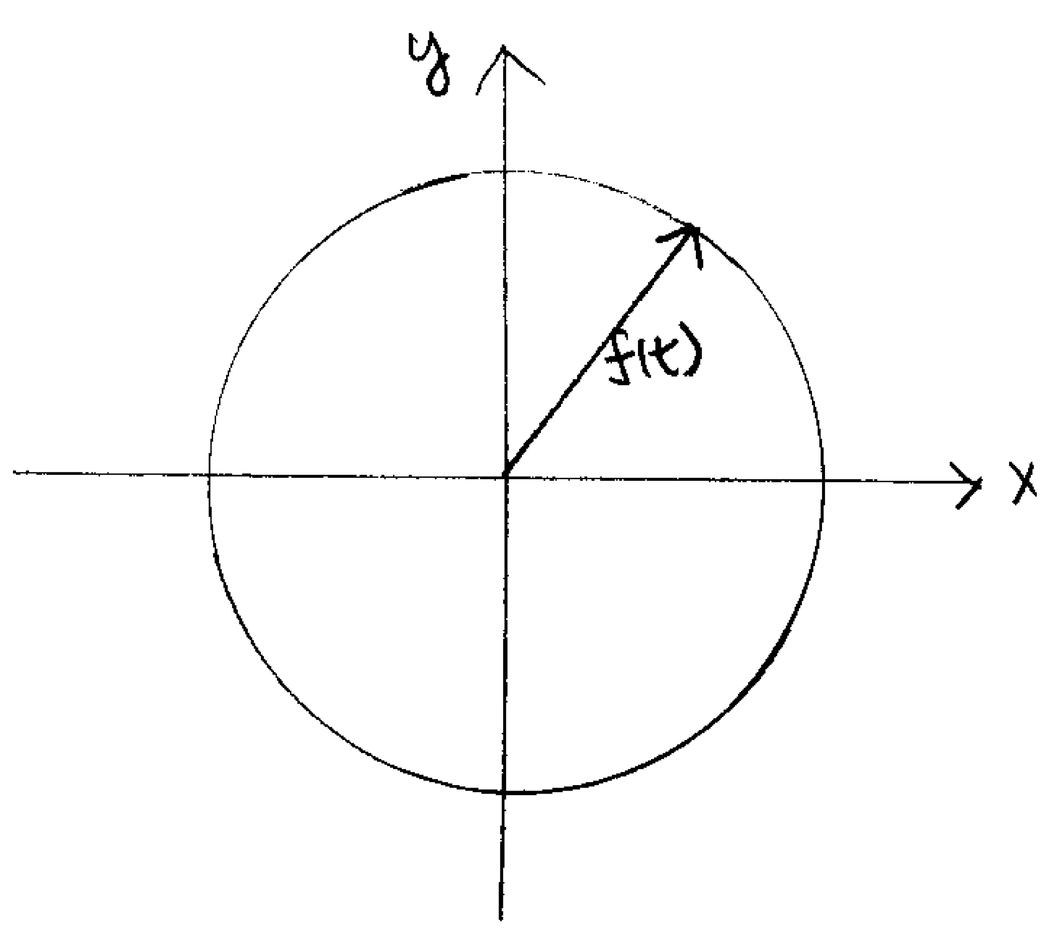
§1 Limit, continuity, Vector Derivative

$\vec{f}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$, vector function

Examples

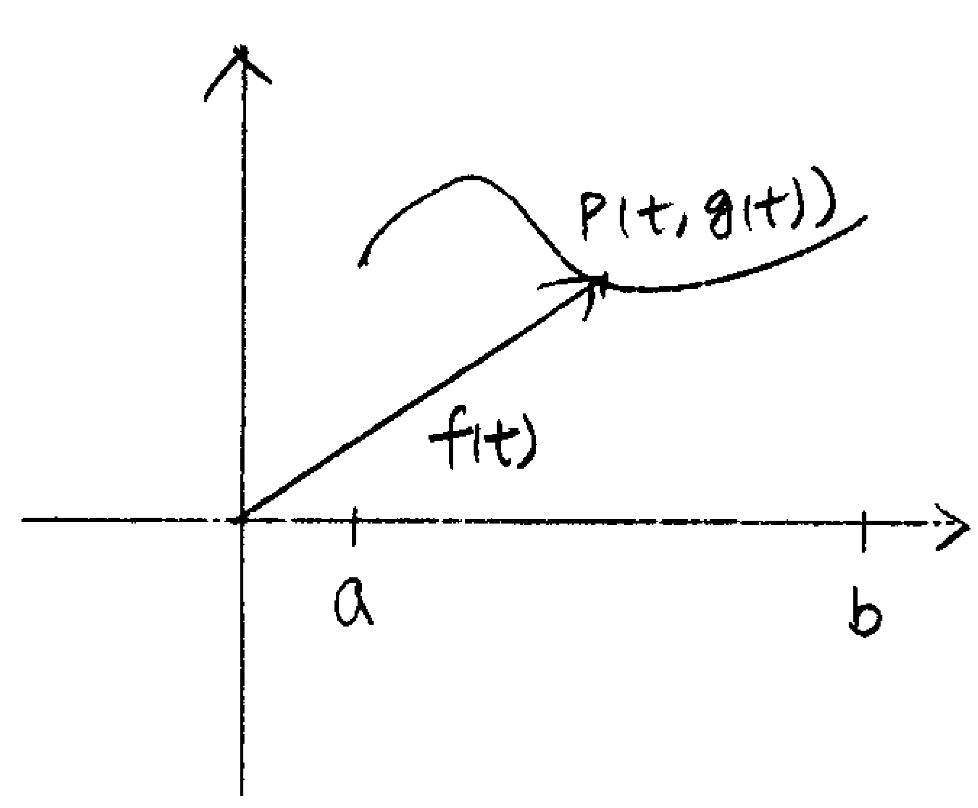
1、 $f_1(t) = \cos t$, $f_2(t) = \sin t$, $f_3(t) = 0$

$\vec{f}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} \Rightarrow \|f(t)\| = \sqrt{\cos^2 t + \sin^2 t} = 1$



2、 $f_1(t) = t$, $f_2(t) = g(t)$, $f_3(t) = 0$, $t \in [a, b]$

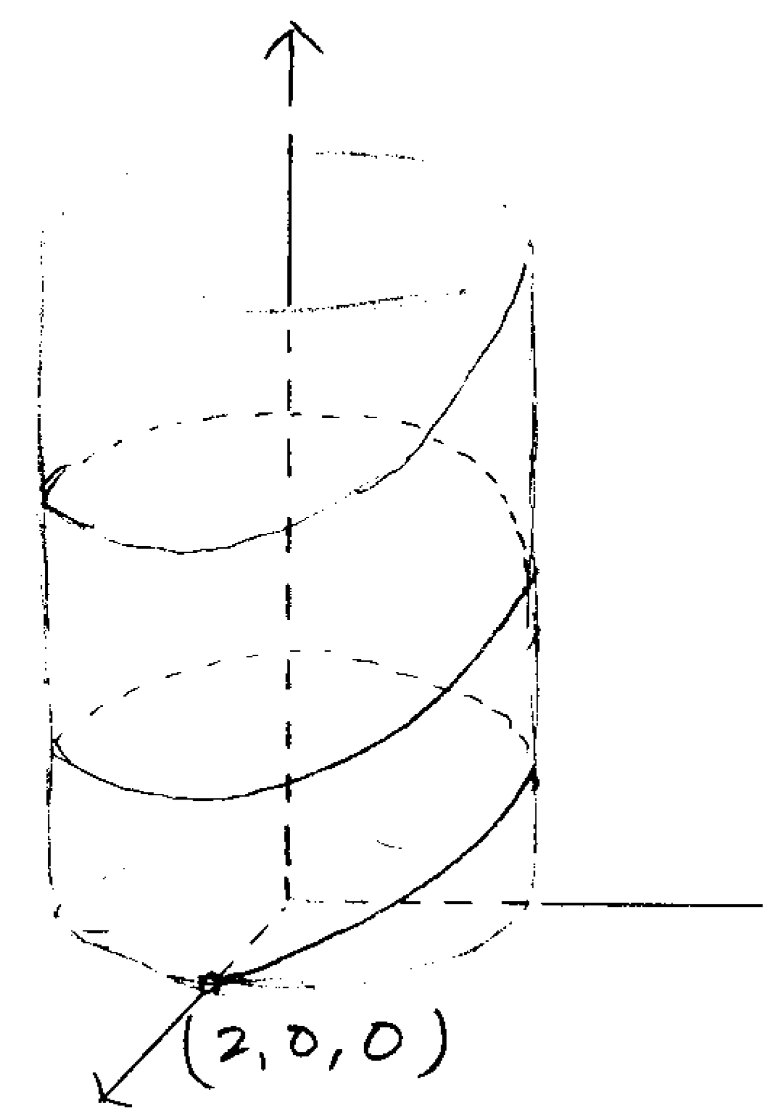
$\Rightarrow \vec{f}(t) = t\vec{i} + g(t)\vec{j}$



3、 $f_1(t) = 2\cos t$, $f_2(t) = 2\sin t$, $f_3(t) = t$, $t \geq 0$

$\vec{f}(t) = 2\cos t\,\vec{i} + 2\sin t\,\vec{j} + t\vec{k}$

$(2\cos t)^2 + (2\sin t)^2 = 4$



Definition 1.1 Limit of a vector function

$$\lim_{t\to t_0} \vec{f}(t) = L \quad \text{provided that} \quad \lim_{t\to t_0} \|\vec{f}(t) - L\| = 0.$$

Fact

If $\lim_{t\to t_0} \vec{f}(t) = L$, then $\lim_{t\to t_0} \|\vec{f}(t)\| = \|L\|$.

<pf>

Note that $0 \le \big| \|\vec{f}(t)\| - \|L\| \big| \le \|\vec{f}(t) - L\|$.

If $\lim_{t\to t_0} \vec{f}(t) = L \Rightarrow \lim_{t\to t_0} \|\vec{f}(t) - L\| = 0 \Rightarrow \lim_{t\to t_0} \big| \|\vec{f}(t)\| - \|L\| \big| = 0$

$\Rightarrow \lim_{t\to t_0} \|\vec{f}(t)\| = \|L\|$ //

Theorem 1.2 Limit Rules

Let $\vec{f}$ & $\vec{g}$ be vector functions, u be a real-valued function.

Suppose that $\vec{f}(t) \to L$, $\vec{g}(t) \to M$, $u(t) \to A$ as $t \to t_0$.

Then $\vec{f}(t) + \vec{g}(t) \to L + M$, $\alpha \vec{f}(t) + \beta \vec{g}(t) \to \alpha L + \beta M$

$u(t)\vec{f}(t) \to AL$, $\vec{f}(t) \cdot \vec{g}(t) \to L \cdot M$, $\vec{f}(t) \times \vec{g}(t) \to L \times M$

<proof>

$$\|\vec{f}(t) + \vec{g}(t) - (L + M)\| = \|(\vec{f}(t) - L) + (\vec{g}(t) - M)\| \le \|\vec{f}(t) - L\| + \|\vec{g}(t) - M\| \to 0$$

$$\|\alpha\vec{f}(t) + \beta\vec{g}(t) - (\alpha L + \beta M)\| = \|\alpha(\vec{f}(t) - L) + \beta(\vec{g}(t) - M)\| \le |\alpha|\,\|\vec{f}(t) - L\| + |\beta|\,\|\vec{g}(t) - M\| \to 0$$

$$\|u(t)\vec{f}(t) - AL\| = \|u(t)\vec{f}(t) - A\vec{f}(t) + A\vec{f}(t) - AL\| \le \|\vec{f}(t)\|\,|u(t) - A| + |A|\,\|\vec{f}(t) - L\|$$

$$\to \|L\| \cdot 0 + |A| \cdot 0 = 0$$

$$\|\vec{f}(t)\vec{g}(t) - L \cdot M\| = |\vec{f}(t)\vec{g}(t) - L\vec{g}(t) + L\vec{g}(t) - LM| \le \|\vec{g}(t)\|\,\|\vec{f}(t) - L\| + \|L\|\,\|\vec{g}(t) - M\|$$

$$\to \|M\|\, 0 + \|L\| \cdot 0 = 0$$

as $t \to t_0$.

$$\|\vec{f}(t)\times\vec{g}(t)-L\times M\| = \|\vec{f}(t)\times\vec{g}(t)-L\times\vec{g}(t)+L\times\vec{g}(t)-L\times M\|$$

$$= \|(\vec{f}(t)-L)\times\vec{g}(t) + L\times(\vec{g}(t)-M)\|$$

$$\le \|(\vec{f}(t)-L)\times\vec{g}(t)\| + \|L\times(\vec{g}(t)-M)\|$$

$$= \|\vec{f}(t)-L\|\|\vec{g}(t)\|\sin\theta_2 + \|L\|\|\vec{g}(t)-M\|\sin\theta_2$$

$$\le \|\vec{f}(t)-L\|\|\vec{g}(t)\| + \|L\|\|\vec{g}(t)-M\|$$

$$\to 0\cdot\|M\| + \|L\|\cdot 0 = 0 \qquad \text{as } t\to t_0. \quad //$$

Theorem 1.3

Let $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$ and $L = L_1\hat{i} + L_2\hat{j} + L_3\hat{k}$

$$\lim_{t\to t_0}\vec{f}(t) = L \iff \lim_{t\to t_0} f_1(t) = L_1,\ \lim_{t\to t_0} f_2(t) = L_2,\ \lim_{t\to t_0} f_2(t) = L_3$$

<proof>

$$\lim_{t\to t_0}\vec{f}(t) = L \iff \lim_{t\to t_0}\|\vec{f}(t)-L\| = 0$$

$$\iff \lim_{t\to t_0}\sqrt{(f_1(t)-L_1)^2+(f_2(t)-L_2)^2+(f_3(t)-L_3)^2} = 0$$

$$\iff \lim_{t\to t_0} f_1(t) = L_1,\ \lim_{t\to t_0} f_2(t) = L_2,\ \lim_{t\to t_0} f_3(t) = L_3 \quad //$$

Example

Find $\lim_{t\to t_0}\vec{f}(t)$ given that $\vec{f}(t) = \cos(t+\pi)\hat{i} + \sin(t+\pi)\hat{j} + e^{-t^2}\hat{k}$

$$\lim_{t\to 0}\vec{f}(t) = \left(\lim_{t\to 0}\cos(t+\pi)\right)\hat{i} + \left(\lim_{t\to 0}\sin(t+\pi)\right)\hat{j} + \left(\lim_{t\to 0} e^{-t^2}\right)\hat{k}$$

$$= -1\cdot\hat{i} + 0\cdot\hat{j} + 1\cdot\hat{k} = -\hat{i}+\hat{k}$$

Definition 1.4

$\vec{f}(t)$ is said to be continuous at t_0 if $\lim_{t \to t_0} \vec{f}(t) = \vec{f}(t_0)$

$\vec{f}(t)$ is said to be differentiable at t_0 if

$$\lim_{h \to 0} \frac{\vec{f}(t_0+h) - \vec{f}(t_0)}{h} \quad \text{exists}$$

Moreover, if the limit exists, say $\vec{f}'(t_0) = \lim_{h \to 0} \frac{\vec{f}(t_0+h) - \vec{f}(t_0)}{h}$

Theorem 1.5

1. Constant functions have derivative $\vec{0}$

2. Functions of the form $\vec{f}(t) = u(t) \cdot \vec{c}$ have derivative $f'(t) = u'(t) \cdot \vec{c}$.

3. If $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$, then $\vec{f}'(t) = f_1'(t)\hat{i} + f_2'(t)\hat{j} + f_3'(t)\hat{k}$

<pf> 1. Let $\vec{f}(t) = \vec{c}$, then

$$\lim_{h \to 0} \frac{\vec{f}(t+h) - \vec{f}(t)}{h} = \lim_{h \to 0} \frac{\vec{c} - \vec{c}}{h} = 0$$

2. $$\lim_{h \to 0} \frac{\vec{f}(t+h) - \vec{f}(t)}{h} = \lim_{h \to 0} \frac{u(t+h) \cdot \vec{c} - u(t) \cdot \vec{c}}{h} = \lim_{h \to 0} \frac{u(t+h) - u(t)}{h} \cdot \vec{c}$$

$$= u'(t) \cdot \vec{c}.$$

3. $$\vec{f}'(t) = \lim_{h \to 0} \frac{\vec{f}(t+h) - \vec{f}(t)}{h} = \lim_{h \to 0} \left(\frac{f_1(t+h) - f_1(t)}{h}\hat{i} + \frac{f_2(t+h) - f_2(t)}{h}\hat{j} + \frac{f_3(t+h) - f_3(t)}{h}\hat{k} \right)$$

$$= \lim_{h \to 0} \frac{f_1(t+h) - f_1(t)}{h}\hat{i} + \lim_{h \to 0} \frac{f_2(t+h) - f_2(t)}{h}\hat{j} + \lim_{h \to 0} \frac{f_3(t+h) - f_3(t)}{h}\hat{k}$$

$$= f_1'(t)\hat{i} + f_2'(t)\hat{j} + f_3'(t)\hat{k}. \qquad //$$

Examples

1. $\vec{f}(t) = t^2 \cdot \vec{a} \Rightarrow \vec{f}'(t) = 2t \cdot \vec{a}$ and $\vec{f}(t) = \sin \pi t (\hat{i} - \hat{j}) \Rightarrow \vec{f}'(t) = \pi \cos \pi t (\hat{i} - \hat{j})$

2. $\vec{f}(t) = t\hat{i} - \sqrt{t}\hat{j} - e^t\hat{k}$, find

(a) the domain of $\vec{f}$ (b) $\vec{f}(0)$ (c) $\vec{f}'(t)$

(d) $\vec{f}'(0)$ (e) $\|\vec{f}(t)\|$ (f) $\vec{f}(t) \cdot \vec{f}'(t)$

(a) $[0, \infty)$

(b) $\vec{f}(0) = 0\,\hat{i} + \sqrt{0}\,\hat{j} - e^{0}\,\hat{k} = -\hat{k}$

(c) $\vec{f}'(t) = \hat{i} + \frac{1}{2\sqrt{t}}\,\hat{j} - e^{t}\,\hat{k}$, only for $t > 0$

(d) $$\vec{f}'(0) = \lim_{h \to 0} \frac{\vec{f}(0+h) - \vec{f}(0)}{h} = \lim_{h \to 0} \frac{h\hat{i} - \sqrt{h}\,\hat{j} - e^{h}\hat{k} + \hat{k}}{h}$$

$$= \lim_{h \to 0} \frac{h}{h}\,\hat{i} - \underset{\substack{\uparrow \\ \text{doesn't exist.}}}{\lim_{h \to 0} \frac{\sqrt{h}}{h}}\,\hat{j} - \lim_{h \to 0} \frac{e^{h} - 1}{h}\,\hat{k}$$

(e) $\|\vec{f}(t)\| = \sqrt{t^2 + \sqrt{t}^{\,2} + (-e^{t})^2} = \sqrt{t^2 + t + e^{2t}}$, for $t \geq 0$.

(f) $\vec{f}(t) \cdot \vec{f}'(t) = (t\hat{i} + \sqrt{t}\,\hat{j} - e^{t}\hat{k}) \cdot (\hat{i} + \frac{1}{2\sqrt{t}}\,\hat{j} - e^{t}\hat{k}) = t + \frac{1}{2} + e^{2t}$, for $t > 0$

3、 Find $\vec{f}''(t)$ for $\vec{f}(t) = t\sin t\,\hat{i} + e^{-t}\hat{j} + t\,\hat{k}$

$\vec{f}'(t) = (\sin t + t\cos t)\,\hat{i} - e^{-t}\hat{j} + \hat{k}$

$\vec{f}''(t) = (\cos t + \cos t - t\sin t)\,\hat{i} + e^{-t}\hat{j} = (2\cos t - t\sin t)\,\hat{i} + e^{-t}\hat{j}$.

Theorem 1.6 (Integration)

For $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$ continuous on $[a, b]$, we set

$\int_a^b \vec{f}(t)\,dt = \left(\int_a^b f_1(t)\,dt\right)\hat{i} + \left(\int_a^b f_2(t)\,dt\right)\hat{j} + \left(\int_a^b f_3(t)\,dt\right)\hat{k}$. Then

1、 $\int_a^b (\vec{f}(t) + \vec{g}(t))\,dt = \int_a^b \vec{f}(t)\,dt + \int_a^b \vec{g}(t)\,dt$

2. $\int_a^b \alpha\,\vec{f}(t)\,dt = \alpha \int_a^b \vec{f}(t)\,dt$

3、 $\int_a^b (\vec{c} \cdot \vec{f}(t))\,dt = \vec{c} \cdot \int_a^b \vec{f}(t)\,dt$ for $\vec{c}$: constant vector.

4. $\left\| \int_a^b \vec{f}(t)\,dt \right\| \leq \int_a^b \|\vec{f}(t)\|\,dt$

<proof> $\left\| \int_a^b \vec{f}(t)\,dt \right\|^2 = \left(\int_a^b \vec{f}(t)\,dt\right) \cdot \left(\int_a^b \vec{f}(t)\,dt\right) = \int_a^b \left(\int_a^b \vec{f}(t)\,dt\right) \cdot \vec{f}(t)\,dt$

$\leq \int_a^b \left\| \int_a^b \vec{f}(t)\,dt \right\| \cdot \|\vec{f}(t)\|\,dt = \left\| \int_a^b \vec{f}(t)\,dt \right\| \cdot \int_a^b \|\vec{f}(t)\|\,dt$

$\Rightarrow \left\| \int_a^b \vec{f}(t)\,dt \right\| \leq \int_a^b \|\vec{f}(t)\|\,dt$.

Proposition 1.7

(1) $(\vec{f}+\vec{g})'(t) = \vec{f}'(t) + \vec{g}'(t)$

(2) $(\alpha \vec{f})'(t) = \alpha \vec{f}'(t)$, $(u \cdot \vec{c})'(t) = u'(t) \cdot \vec{c}$

(3) $(u\vec{f})'(t) = u(t)\,\vec{f}'(t) + u'(t)\,\vec{f}(t)$

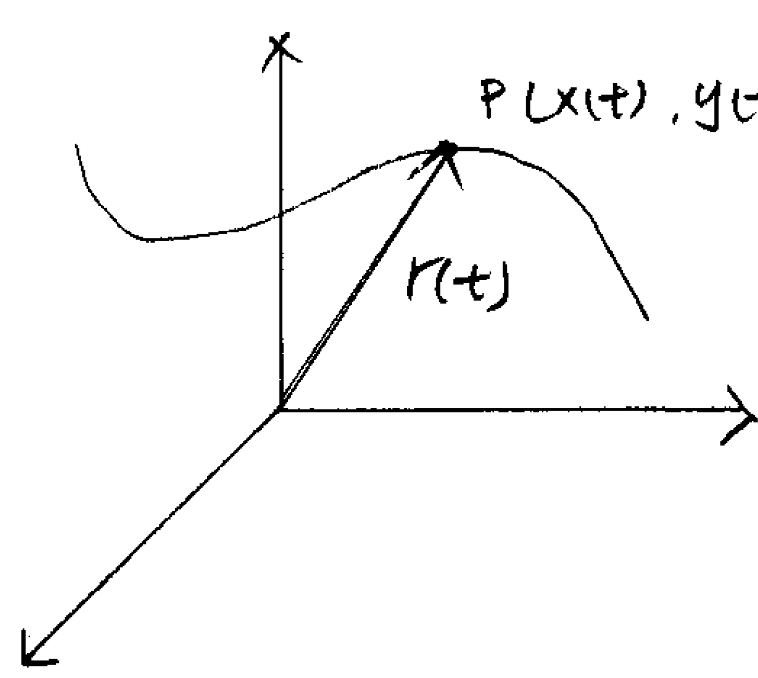
(4) $(\vec{f}\cdot\vec{g})'(t) = (\vec{f}(t)\cdot\vec{g}'(t)) + (\vec{f}'(t)\cdot\vec{g}(t))$

(5) $(\vec{f}\times\vec{g})'(t) = \vec{f}(t)\times\vec{g}'(t) + \vec{f}'(t)\times\vec{g}(t)$

(6) $(\vec{f}\circ u)'(t) = \vec{f}'(u(t))\,u'(t) = u'(t)\,\vec{f}'(u(t))$

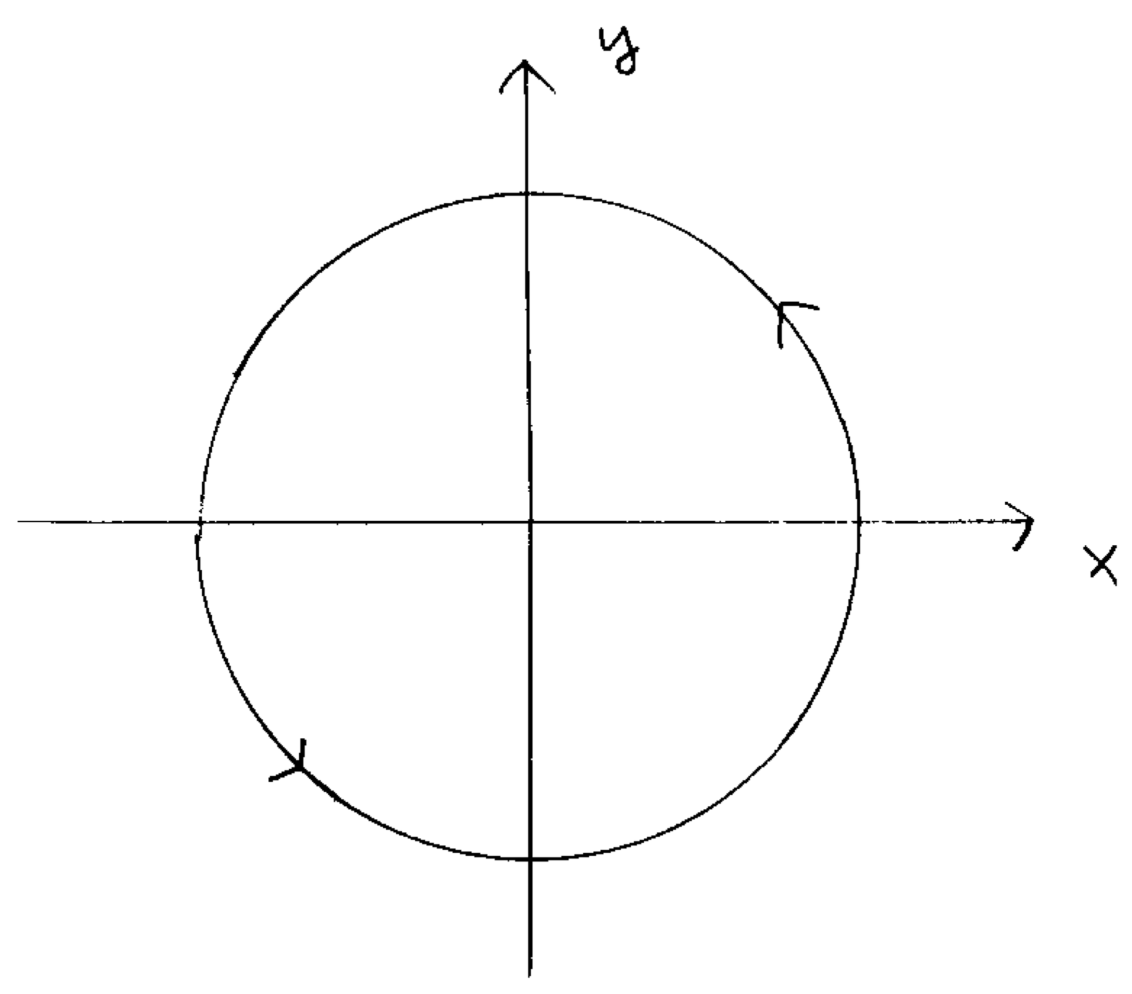
§ 2 Curves

$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$ be a differentiable vector function

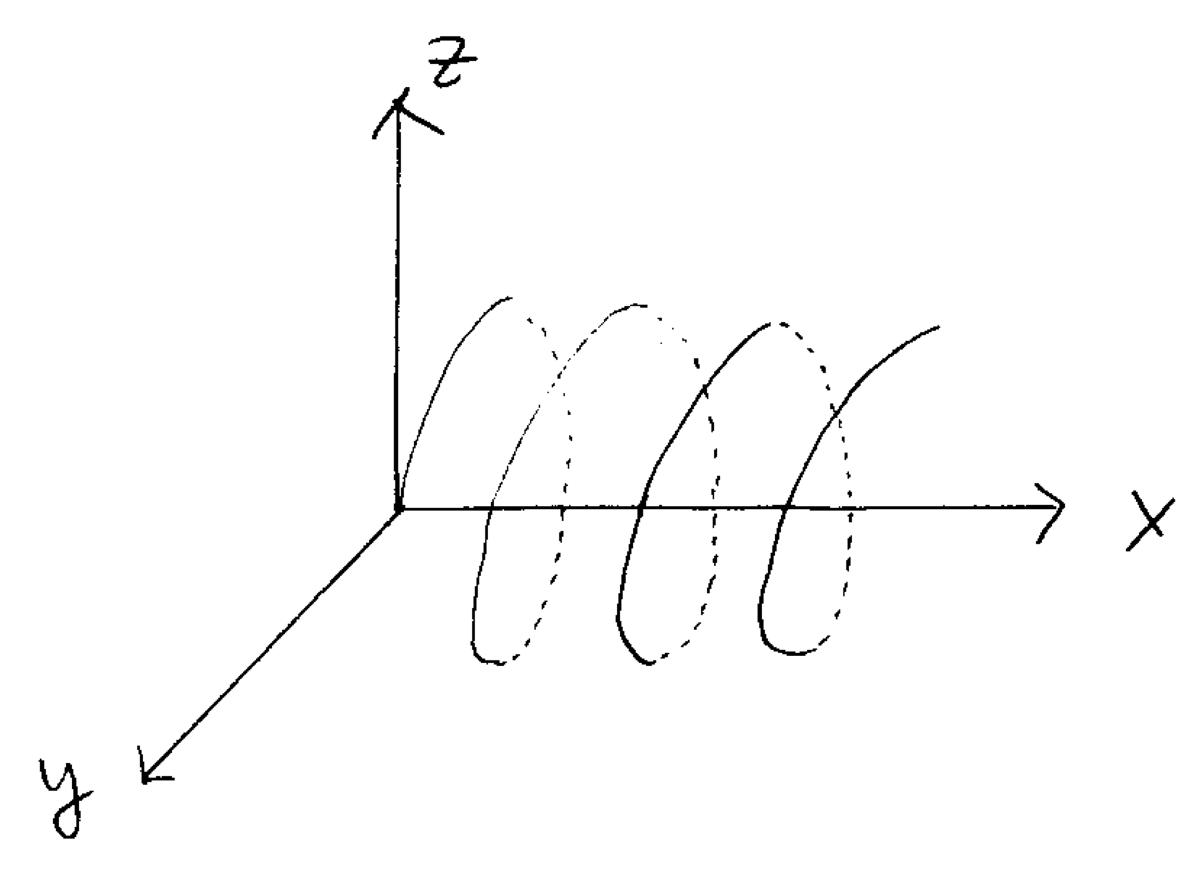


Example

$C_1 : \vec{r}_1(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, $t \in [0, 2\pi]$

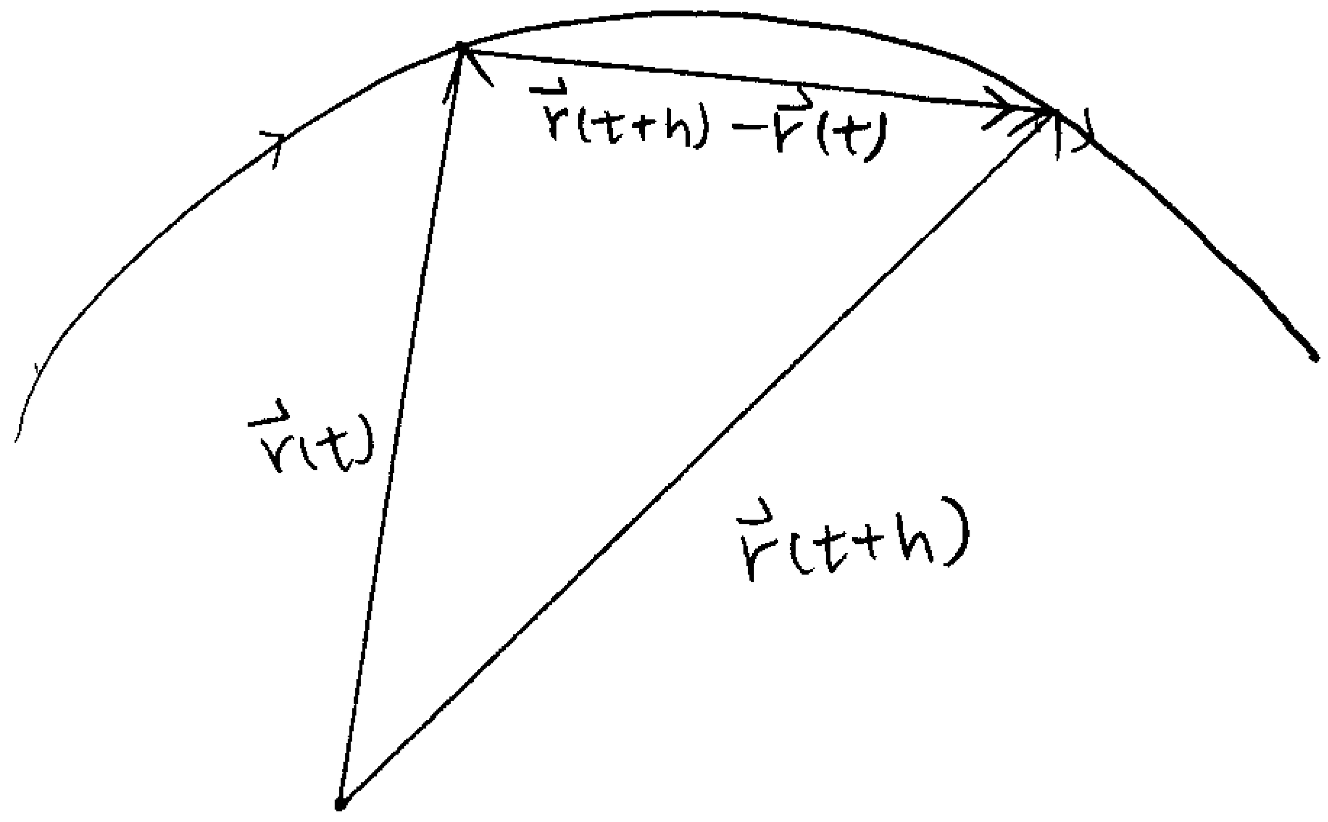


$C_2 : \vec{r}_2(t) = t\,\hat{i} + 2\cos t\,\hat{j} + 3\sin t\,\hat{k}$, $t \geq 0$

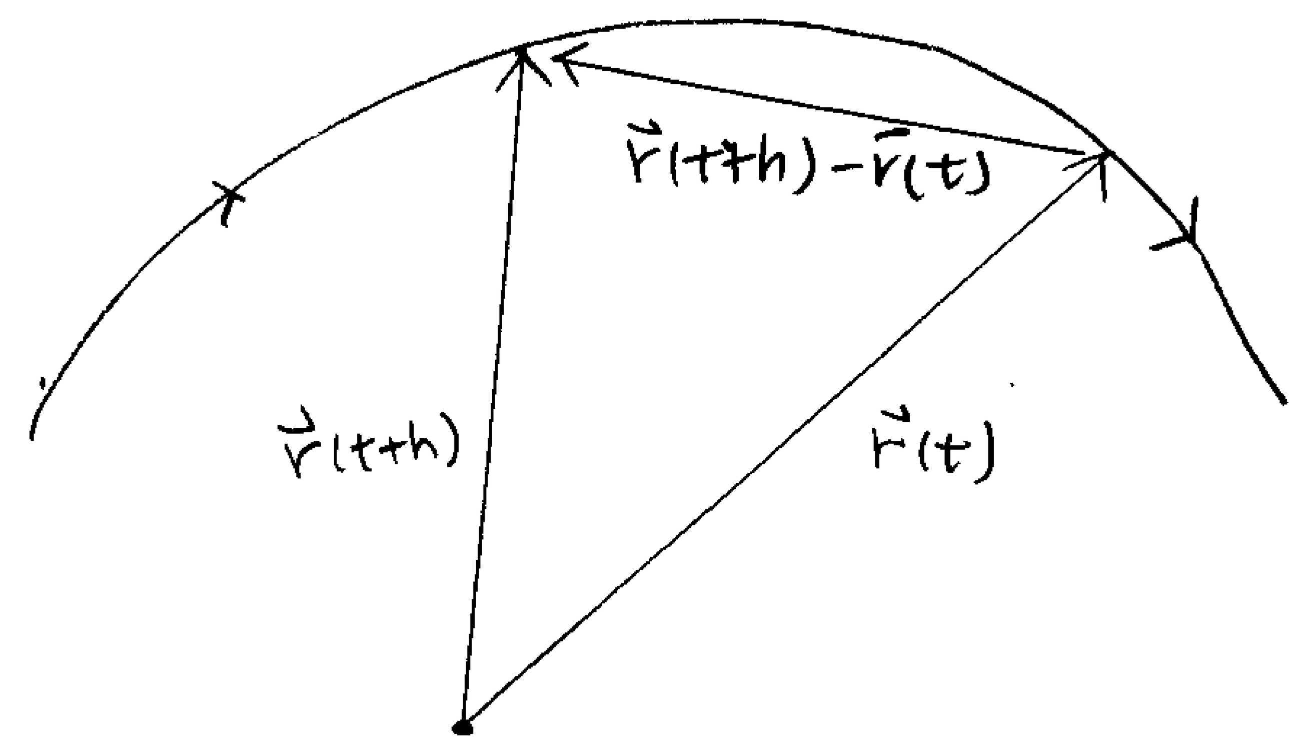


Definition 2.1 Tangent vector

Let $C : \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ be a differentiable curve. The vector $\vec{r}'(t)$, if not $\vec{0}$, is said to be tangent to the curve C at the point $P(x(t), y(t), z(t))$



$h > 0$



$h < 0$

Examples

1. Find a point P on the curve $\vec{r}(t) = (1-2t)\hat{i} + t^2\hat{j} + 2e^{2(t-1)}\hat{k}$ at which the tangent vector $\vec{r}'(t)$ is parallel to the radius vector $\vec{r}(t)$.

sol. $\vec{r}'(t) = -2\hat{i} + 2t\hat{j} + 4e^{2(t-1)}\hat{k}$

$\exists\, \alpha$ s.t. $\vec{r}'(t) = \alpha\vec{r}(t) \Rightarrow -2\hat{i} + 2t\hat{j} + 4e^{2(t-1)}\hat{k} = \alpha\left((1-2t)\hat{i} + t^2\hat{j} + 2e^{2(t-1)}\hat{k}\right)$

$$\Rightarrow \begin{cases} -2 = \alpha - 2\alpha t \\ 2t = \alpha \cdot t^2 \\ 4e^{2(t-1)} = 2\alpha e^{2(t-1)}\hat{k} \end{cases} \Rightarrow \alpha = 2,\ t = 1$$

$\Rightarrow P(-1, 1, 2)$

2. Find a vector tangent to the twisted cubic $r(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$ at the point $P(2,4,8)$, and the parametrize the tangent line at P.

sol. $\vec{r}'(t) = \hat{i} + 2t\hat{j} + 3t^2\hat{k}$.

Note that $P(2,4,8) = \vec{r}(2) \Rightarrow \vec{r}'(2) = \hat{i} + 4\hat{j} + 12\hat{k}$

$\Rightarrow R(u) = (2+u)\hat{i} + (4+4u)\hat{j} + (8+12u)\hat{k}$

3. Show that the circles $C_1 : \vec{r_1}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, $C_2 : \vec{r_2}(u) = \cos u\,\hat{j} + \sin u\,\hat{k}$ intersect at right angles at $P(0,1,0)$ and $Q(0,-1,0)$

<pf>

$$P(0,1,0) = \vec{r_1}\left(\tfrac{\pi}{2}\right) = \vec{r_2}(0)$$

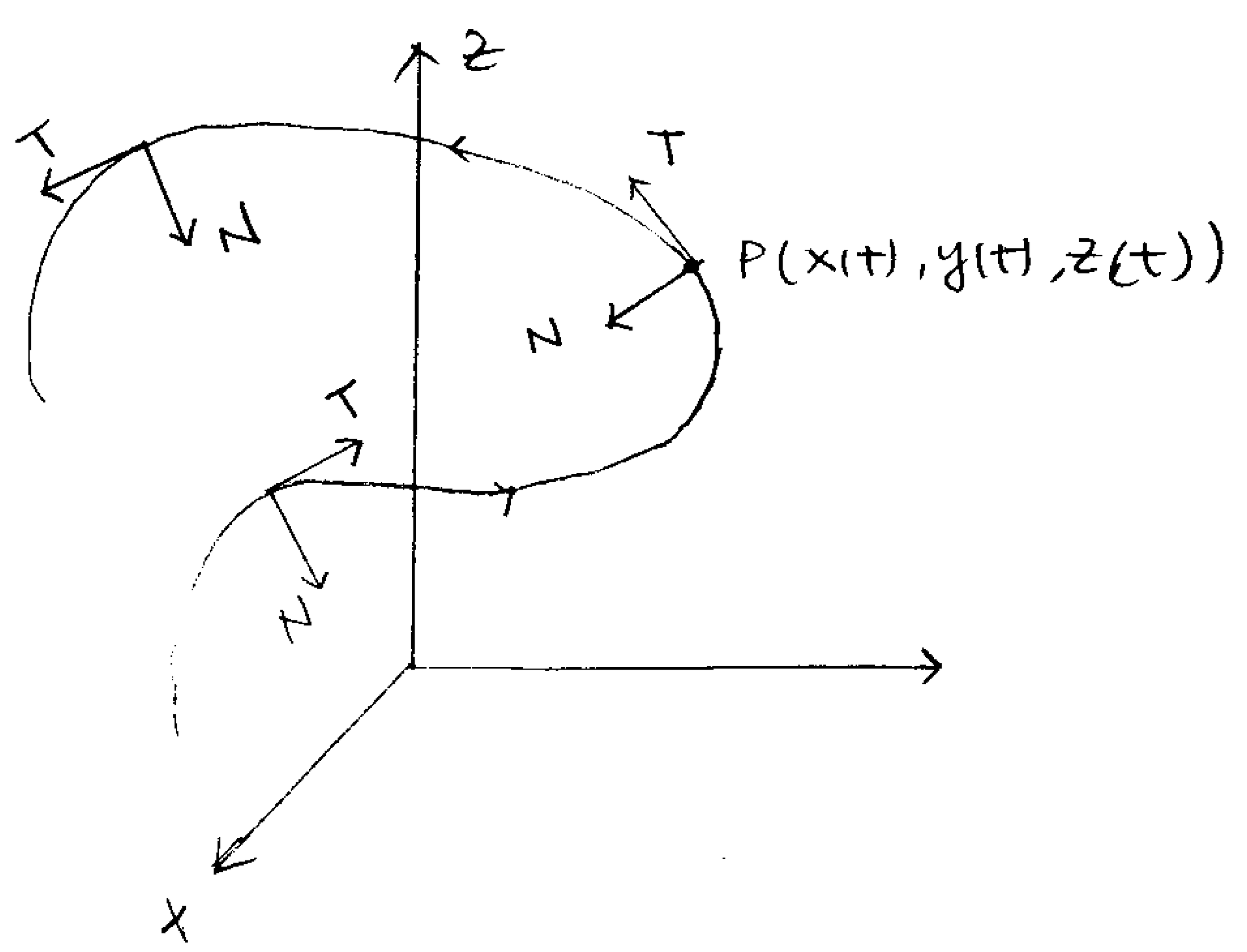
$$Q(0,-1,0) = \vec{r_1}\left(\tfrac{3}{2}\pi\right) = \vec{r_2}(\pi)$$

$$\vec{r_1}'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j} \quad , \quad \vec{r_2}'(u) = -\sin u\,\hat{j} + \cos u\,\hat{k}$$

$$\vec{r_1}'\left(\tfrac{\pi}{2}\right) \cdot \vec{r_2}'(0) = -\hat{i} \cdot \hat{k} = 0$$

$$\vec{r_1}'\left(\tfrac{3}{2}\pi\right) \cdot \vec{r_2}'(\pi) = \hat{i} \cdot (-\hat{k}) = 0 .$$

Let $C : r(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$ be a curve.



$P(x(t), y(t), z(t))$ be point of C

The unit tangent vector $\boxed{T(t) = \dfrac{r'(t)}{\|r'(t)\|}}$

$\Rightarrow \|T(t)\| = 1 \Rightarrow T(t) \cdot T(t) = 1 \Rightarrow T'(t) \cdot T(t) + T(t) \cdot T'(t) = 0 \rightarrow 2\,T(t)\,T'(t) = 0$

$\Rightarrow T(t) \cdot T'(t) = 0 \Rightarrow T'(t) \perp T(t)$

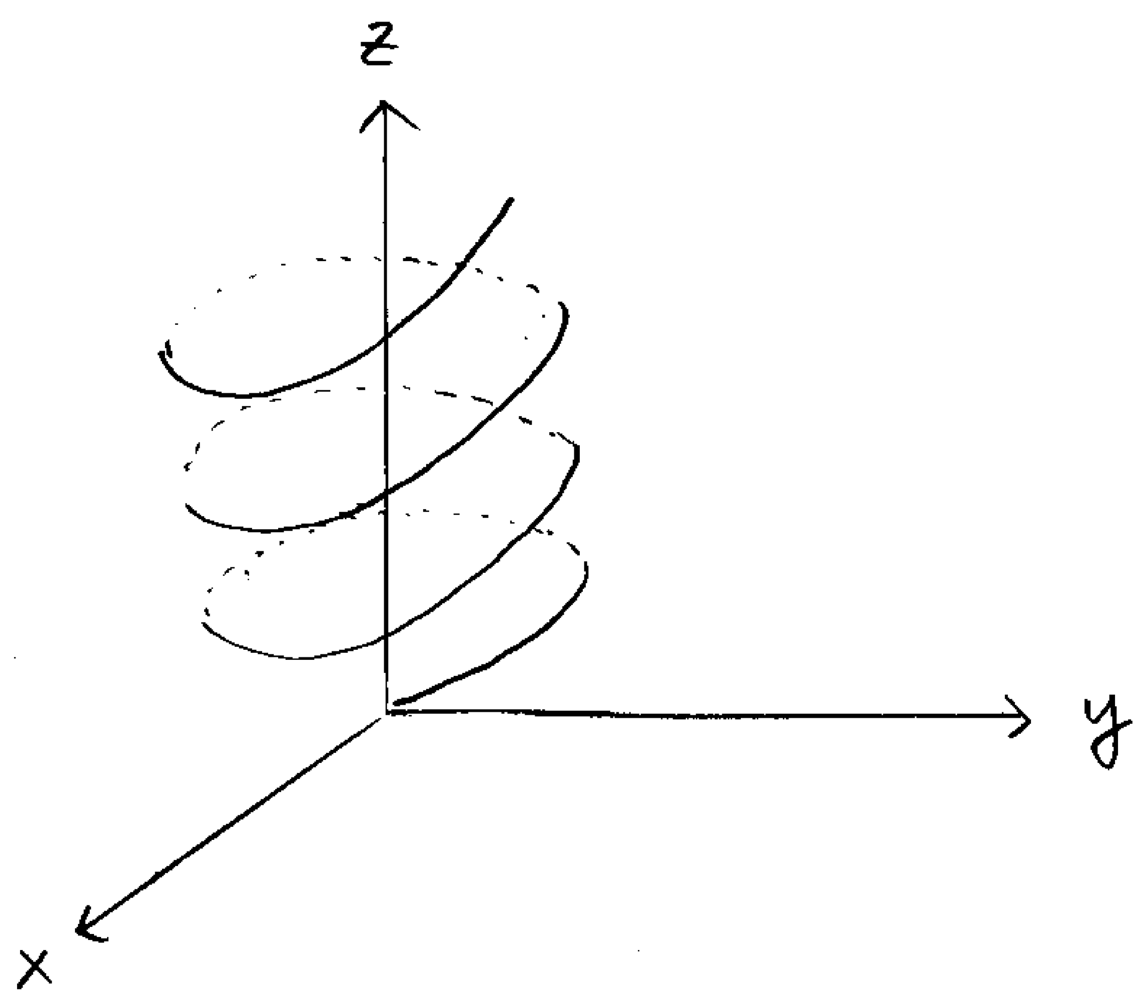
The principal normal vector $\boxed{N(t) = \dfrac{T'(t)}{\|T'(t)\|}}$

The plane determined by the two vectors is called the <u>osculating plane</u>

Example

The simplest parametrization for a circular helix takes the form

$r(t) = a\cos t\, i + a\sin t\, j + bt\, k$ with $a>0$, $b>0$



$$r'(t) = -a\sin t\, i + a\cos t\, j + b\, k$$

$$\| r'(t) \| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2}$$

$$= \sqrt{a^2+b^2}$$

$$T(t) = \frac{-a\sin t\, i + a\cos t\, j + b\, k}{\sqrt{a^2+b^2}}$$

the unit tangent vector.

$$T'(t) = \frac{1}{\sqrt{a^2+b^2}}(-a\cos t\, i - a\sin t\, j)$$

$$\| T'(t) \| = \frac{a}{\sqrt{a^2+b^2}}, \quad N(t) = \frac{1}{\cancel{\sqrt{a^2+b^2}}}(-\cancel{a}\cos t\, i - \cancel{a}\sin t\, j) \cdot \frac{\cancel{\sqrt{a^2+b^2}}}{\cancel{a}}$$

$$= -\cos t\, i - \sin t\, j$$

The osculating plane.

$$T(t) \times N(t) = \begin{vmatrix} i & j & k \\ \frac{-a\sin t}{\sqrt{a^2+b^2}} & \frac{a\cos t}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{-1}{\sqrt{a^2+b^2}} \begin{vmatrix} i & j & k \\ -a\sin t & a\cos t & b \\ \cos t & \sin t & 0 \end{vmatrix}$$

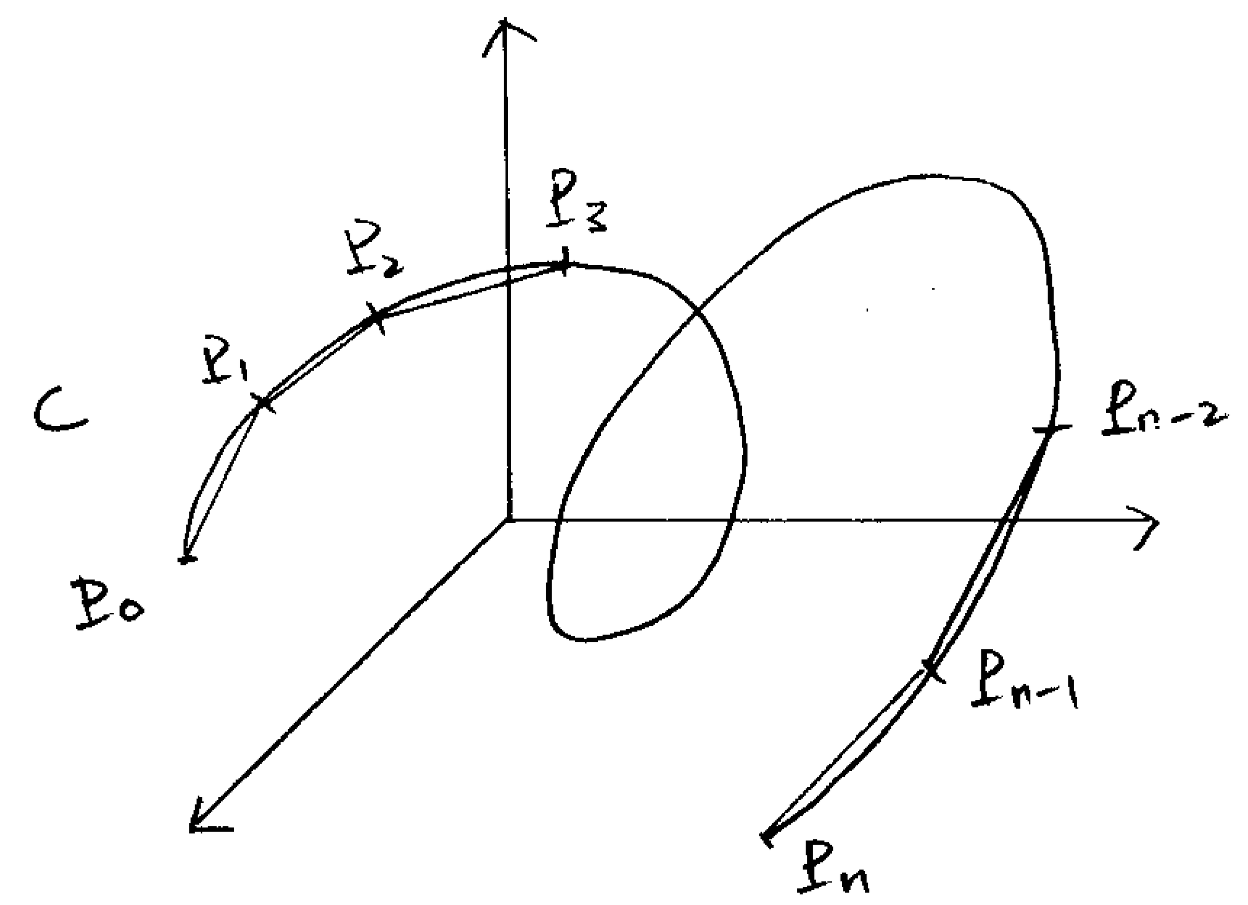
$$= \frac{-1}{\sqrt{a^2+b^2}} \left(b\cos t\, j - a\sin^2 t\, k - a\cos^2 t\, k - b\sin t\, i \right)$$

$$= \frac{-1}{\sqrt{a^2+b^2}} \left(-b\sin t\, i + b\cos t\, j - a\, k \right)$$

$$b\sin t\,(x - a\cos t) - b\cos t\,(y - a\sin t) + a(z - bt) = 0$$

$$\Rightarrow (b\sin t)x - (b\cos t)y + az = abt$$

§3 Arc length and Curvature



$r(t) = x(t)\,i + y(t)\,j + z(t)\,k$ for $t \in [a,b]$

Choosing a finite number of points in $[a,b]$, $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$

$P_i(x(t_i), y(t_i), z(t_i))$, $i = 0, \ldots, n$

$\gamma = \overline{P_0P_1} \cup \cdots \cup \overline{P_{i-1}P_i} \cup \cdots \cup \overline{P_{n-1}P_n}$ a polygon path inscribed in C.

$L(\gamma) = d(P_0, P_1) + d(P_1, P_2) + \cdots + d(P_{n-1}, P_n)$

$\Rightarrow$ $L(\gamma) \leq L(C)$ for each γ inscribed in C.

Definition 3.1

$L(C)$ = the least upper bound of the set of all lengths of polygonal paths inscribed in C.

Theorem 3.2 (Arc length formula)

C be the path traced out by a continuously differentiable vector function $r(t)$, $t \in [a,b]$

$$L(C) = \int_a^b \| r'(t) \| \, dt$$

Examples

1. $r(t) = 2t^{\frac{3}{2}}\hat{i} + 4t\hat{j}$, $t \in [0,1]$

$r'(t) = 3t^{\frac{1}{2}}\hat{i} + 4\hat{j} \Rightarrow \|r'(t)\| = \sqrt{(3t^{\frac{1}{2}})^2 + 4^2} = \sqrt{9t+16}$

$L(c) = \int_0^1 \|r'(t)\|\,dt = \int_0^1 \sqrt{9t+16}\,dt = \frac{1}{9}\left(\frac{2}{3}\right)(9t+16)^{\frac{3}{2}}\Big|_0^1$

$= \frac{250}{27} - \frac{128}{27} = \frac{122}{27}$

2. $r(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j} + t^2\hat{k}$ $t \in [0, \frac{\pi}{2}]$.

$r'(t) = -2\sin t\,\hat{i} + 2\cos t\,\hat{j} + 2t\,\hat{k}$

$\|r'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t + 4t^2} = \sqrt{4+4t^2} = 2\sqrt{1+t^2}$

$L(C) = \int_0^{\frac{\pi}{2}} \|r'(t)\|\,dt = \int_0^{\frac{\pi}{2}} 2\sqrt{1+t^2}\,dt = t\sqrt{1+t^2} + \ln(t+\sqrt{1+t^2})\Big|_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2}\sqrt{1+\frac{\pi^2}{4}} + \ln\left(\frac{\pi}{2} + \sqrt{1+\frac{\pi^2}{4}}\right) \cong 4.158$.

Note $r(0) = 2\hat{i}$ & $r(\frac{\pi}{2}) = 2\hat{j} + \frac{\pi^2}{4}\hat{k}$

$\|r(\frac{\pi}{2}) - r(0)\| = \sqrt{2^2 + 2^2 + \frac{\pi^4}{16}} \cong 3.753$

The curve is about 11% longer than the straight-line distance between the endpoints of the curve.

$C: \ r(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k} \quad , t \in I$

If r is twice differentiable, we can form $r'(t)$ & $r''(t)$.

$r'(t)$: the velocity of the object at time t

$r''(t)$: the acceleration.

$$\boxed{r'(t) = v(t) \qquad \& \qquad r''(t) = a(t)}$$

* $\|v(t)\|$ = the speed at time t

Example

$r(t) = a\cos\omega t\,\hat{i} + a\sin\omega t\,\hat{j} + b\omega t\,\hat{k} \quad (a>0, b>0, \omega>0)$

For each time t, find

(a) the velocity of the particle (b) the speed

(c) the acceleration (d) the magnitude of the acceleration

(e) the angle between the velocity vector and the acceleration vector.

(a) $v(t) = r'(t) = -a\omega\sin\omega t\,\hat{i} + a\omega\cos\omega t\,\hat{j} + b\omega\,\hat{k}$

(b) $\|v(t)\| = \sqrt{a^2\omega^2\sin^2\omega t + a^2\omega^2\cos^2\omega t + b^2\omega^2} = \sqrt{a^2\omega^2 + b^2\omega^2} = \omega\sqrt{a^2+b^2}$

(c) $a(t) = v'(t) = r''(t) = -a\omega^2\cos\omega t\,\hat{i} - a\omega^2\sin\omega t\,\hat{j} = -a\omega^2(\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j})$

(d) $\|a(t)\| = |a\omega^2|\,\|\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}\| = a\omega^2$

(e) $\cos\theta = \dfrac{v(t)\cdot a(t)}{\|v(t)\|\cdot\|a(t)\|} = \dfrac{a^2\omega^3\sin\omega t\cos\omega t - a^2\omega^3\cos\omega t\sin\omega t}{\omega\sqrt{a^2+b^2}\cdot a\omega^2} = 0.$

$\Rightarrow \theta = \frac{1}{2}\pi.$