30. $\dfrac{\partial f}{\partial x} = \dfrac{y}{z} - e^z \implies f(x,y,z) = \dfrac{xy}{z} - xe^z + g(y,z)$

$\dfrac{\partial f}{\partial y} = \dfrac{x}{z} + \dfrac{\partial g}{\partial y} = \dfrac{x}{z} + 1 \implies f(x,y,z) = \dfrac{xy}{z} + y - xe^z + h(z)$

$\dfrac{\partial f}{\partial z} = -\dfrac{xy}{z^2} - xe^z + h'(z) = -xe^z - \dfrac{xy}{z^2} \implies f(x,y,z) = \dfrac{xy}{z} - xe^z + y + C$

31. $\mathbf{F}(\mathbf{r}) = \boldsymbol{\nabla}\left(\dfrac{GmM}{r}\right)$

32.
$$\mathbf{h}(\mathbf{r}) = \begin{cases} \boldsymbol{\nabla}\left(\dfrac{k}{n+2}r^{n+2}\right), & n \neq 2 \\ \boldsymbol{\nabla}\left(k\ln r\right), & n = -2. \end{cases}$$

REVIEW EXERCISES

1. $\nabla f(x,y) = (4x - 4y)\mathbf{i} + (3y^2 - 4x)\mathbf{j}$

2. $\nabla f(x,y) = \dfrac{y^3 - x^2y}{(x^2+y^2)^2}\mathbf{i} + \dfrac{x^3 - xy^2}{(x^2+y^2)^2}\mathbf{j}$

3. $\nabla f(x,y) = (ye^{xy}\tan 2x + 2e^{xy}\sec^2 2x)\,\mathbf{i} + xe^{xy}\tan 2x\,\mathbf{j}$

4. $\nabla f = \dfrac{1}{x^2+y^2+z^2}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$

5. $\nabla f(x,y) = 2xe^{-yz}\sec z\,\mathbf{i}, -zx^2e^{-yz}\sec z\mathbf{j} - (x^2ye^{-yz}\sec z - x^2e^{-yz}\sec z\tan z)\mathbf{k}$

6. $\nabla f(x,y) = ye^{-3z}\cos xy\,\mathbf{i} + e^{-3z}(x\cos xy + \sin y)\,\mathbf{j}, -3e^{-3z}(\sin xy - \cos y)\,\mathbf{k}$

7. $\nabla f(x,y) = (2x - 2y)\,\mathbf{i} - 2x\,\mathbf{j}, \quad \nabla f(1,-2) = 6\,\mathbf{i} - 2\,\mathbf{j}; \qquad \mathbf{u_a} = \dfrac{1}{\sqrt{5}}\,\mathbf{i} + \dfrac{2}{\sqrt{5}}\,\mathbf{j};$

$f'_{\mathbf{u_a}}(1,-2) = \nabla f(1,-2) \cdot \mathbf{u_a} = \dfrac{2}{\sqrt{5}}.$

8. $\nabla f(x,y) = (e^{xy} + xye^{xy})\,\mathbf{i} + x^2e^{xy}\,\mathbf{j}, \quad \nabla f(2,0) = \mathbf{i} + 4\mathbf{j}; \qquad \mathbf{u_a} = \dfrac{1}{2}\mathbf{i} + \dfrac{\sqrt{3}}{2}\mathbf{j}$

$f'_{\mathbf{u_a}}(2,0) = \nabla f(2,0) \cdot \mathbf{u_a} = \dfrac{1}{2} + 2\sqrt{3}.$

9. $\nabla f(x,y,z) = (y^2 + 6xz)\,\mathbf{i} + (2xy + 2z)\,\mathbf{j} + (2y + 3x^2)\,\mathbf{k}, \quad \nabla f(1,-2,3) = 22\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k};$

$\mathbf{u_a} = \dfrac{1}{3}\mathbf{i} - \dfrac{2}{3}\mathbf{j} + \dfrac{2}{3}\mathbf{k}; f'_{\mathbf{u_a}}(1,-2,3) = \nabla f(1,-2,3) \cdot \mathbf{u_a} = \dfrac{16}{3}.$

10. $\nabla f(x,y,z) = \dfrac{2x}{x^2+y^2+z^2}\,\mathbf{i} + \dfrac{2y}{x^2+y^2+z^2}\,\mathbf{j} + \dfrac{2z}{x^2+y^2+z^2}\,\mathbf{k}, \quad \nabla f(1,2,3) = \dfrac{1}{7}(\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k});$

$\mathbf{u_a} = \dfrac{1}{\sqrt{3}}\,\mathbf{i} - \dfrac{1}{\sqrt{3}}\,\mathbf{j} + \dfrac{1}{\sqrt{3}}\,\mathbf{k}; \qquad f'_{\mathbf{u_a}}(1,2,3) = \nabla f(1,2,3) \cdot \mathbf{u_a} = \dfrac{2}{7\sqrt{3}}.$

11. $\nabla f(x,y) = (6x - 2y^2)\,\mathbf{i} - 4xy\,\mathbf{j}, \quad \nabla f(3,-2) = 10\,\mathbf{i} + 24\,\mathbf{j};$

$\mathbf{a} = (0,0) - (3,-2) = (-3,2) = -3\,\mathbf{i} + 2\,\mathbf{j}, \quad \mathbf{u_a} = \dfrac{-3}{\sqrt{13}}\,\mathbf{i} + \dfrac{2}{\sqrt{13}}\,\mathbf{j};$

$f'_{\mathbf{u_a}}(3,-2) = \nabla f(3,-2) \cdot \mathbf{u_a} = \dfrac{18}{\sqrt{13}}.$

12. $\nabla f(x,y,z) = (y^2 z - 3yz)\,\mathbf{i} + (2xyz - 3xz)\,\mathbf{j} + (xy^2 - 3xy)\,\mathbf{k}, \quad \nabla f(1,-1,2) = 8\,\mathbf{i} - 10\,\mathbf{j} + 4\,\mathbf{k};$

$\mathbf{r}'(t) = \mathbf{i} - \pi \sin \pi t\,\mathbf{j} + 2e^{t-1}\,\mathbf{k}, \quad \mathbf{a} = \mathbf{r}'(1) = \mathbf{i} + 2\mathbf{k}, \quad \mathbf{u_a} = \dfrac{1}{\sqrt{5}}\,\mathbf{i} + \dfrac{2}{\sqrt{5}}\,\mathbf{k};$

$f'_{\mathbf{u_a}}(1,-1,2) = \nabla f(1,-1,2) \cdot \mathbf{u_a} = \dfrac{16}{\sqrt{5}}.$

13. $\nabla f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}\,(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}), \quad \nabla f(3,-1,4) = \dfrac{1}{\sqrt{26}}(3\,\mathbf{i} - \mathbf{j} + 4\,\mathbf{k});$

$\mathbf{a} = \pm(4\,\mathbf{i} - 3\,\mathbf{j} + \mathbf{k}), \quad \mathbf{u_a} = \pm\dfrac{1}{\sqrt{26}}(4\,\mathbf{i} - 3\,\mathbf{j} + \mathbf{k}); \qquad f'_{\mathbf{u_a}}(3,-1,4) = \nabla f(3,-1,4) \cdot \mathbf{u_a} = \pm\dfrac{19}{26}.$

14. $\nabla f(x,y) = 2e^{2x}(\cos y - \sin y)\,\mathbf{i} - e^{2x}(\sin y + \cos y)\,\mathbf{j}, \quad \nabla f\left(\tfrac{1}{2}, -\tfrac{1}{2}\pi\right) = 2e\,\mathbf{i} + e\,\mathbf{j};$

maximum directional derivative: $\|\nabla f\left(\tfrac{1}{2}, -\tfrac{1}{2}\pi\right)\| = e\sqrt{5}.$

15. $\nabla f(x,y,z) = \cos xyz(yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}), \quad \nabla f(\tfrac{1}{2}, \tfrac{1}{3}, \pi) = \tfrac{\pi\sqrt{3}}{6}\,\mathbf{i} + \tfrac{\pi\sqrt{3}}{4}\,\mathbf{j} + \tfrac{\sqrt{3}}{12}\,\mathbf{k};$

minimum directional derivative: $f'_{\mathbf{u}} = -\|\nabla f(\tfrac{1}{2}, \tfrac{1}{3}, \pi)\| = -\tfrac{\sqrt{39\pi^2+3}}{12}$

16. Let $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$ be the path of the particle. $\nabla I(x,y) = -2x\,\mathbf{i} - 6y\,\mathbf{j}$. Then

$$x'(t) = -2x(t), \quad y'(t) = -6y(t) \quad \Longrightarrow \quad x(t) = C_1 e^{-2t}, \quad y(t) = C_2 e^{-6t}.$$

$$\mathbf{r}(0) = (4,3) \quad \Longrightarrow \quad C_1 = 4, \quad C_2 = 3.$$

Therefore the path of the particle is: $\mathbf{r}(t) = 4e^{-2t}\,\mathbf{i} + 3e^{-6t}\,\mathbf{j},\ t \geq 0,$ or, $y = \tfrac{3}{64}x^3,\ 0 < x \leq 4$

17. Let $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$ be the path of the particle. $\nabla T = -e^{-x}\cos y\,\mathbf{i} - e^{-x}\sin y\,\mathbf{j}$. Then

$$x'(t) = -e^{-x(t)}\cos y(t), \quad y'(t) = -e^{-x(t)}\sin y(t) \quad \Longrightarrow \quad \frac{y'(t)}{x'(t)} = \tan y(t) \quad \Longrightarrow \quad \frac{dy}{dx} = \tan y$$

The solution is $\sin y = Ce^x$. Since $\mathbf{r}(0) = 0$, $C = 0$ and $y = 0$. The particle moves to the right the x-axis.

18. $\nabla z = 8x\,\mathbf{i} + 2y\,\mathbf{j}; \quad \mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}.$

$$x'(t) = -8x(t), \quad y'(t) = -2y(t) \quad \Longrightarrow \quad x(t) = C_1 e^{-8t}, \quad y(t) = C_2 e^{-2t}.$$

(a) $\mathbf{r}(0) = (1,1) \quad \Longrightarrow \quad C_1 = 1,\ C_2 = 1; \quad x = e^{-8t},\ y = e^{-2t}$ or $x = y^4$.

(b) $\mathbf{r}(0) = (1,-2) \quad \Longrightarrow \quad C_1 = 1,\ C_2 = -2; \quad x = e^{-8t},\ y = -2e^{-2t}$ or $x = y^4/16$.

19. $\nabla f(x,y) = e^x \arctan y\,\mathbf{i} + e^x \dfrac{1}{1+y^2}\,\mathbf{j};\quad \nabla f(0,1) = \dfrac{\pi}{4}\,\mathbf{i} + \dfrac{1}{2}\,\mathbf{j}.$

$\mathbf{u} = \dfrac{\nabla f(0,1)}{\|\nabla f(0,1)\|} = \dfrac{1}{\sqrt{4+\pi^2}}(\pi\,\mathbf{i} + 2\,\mathbf{j});\quad \text{rate: } \|\nabla f(0,1)\| = \dfrac{\sqrt{\pi^2+4}}{4}$

20. $\nabla f(x,y,z) = \dfrac{1}{(y+z)^2}[(y+z)\,\mathbf{i} + (z-x)\,\mathbf{j} - (x+y)\,\mathbf{k}];\quad \nabla f(-1,1,3) = \dfrac{1}{4}\,\mathbf{i} + \dfrac{1}{4}\,\mathbf{j}.$

$\mathbf{u} = \dfrac{\nabla f(-1,1,3)}{\|\nabla f(-1,1,3)\|} = \tfrac{1}{2}\sqrt{2}\,\mathbf{i} + \tfrac{1}{2}\sqrt{2}\,\mathbf{j};\quad \text{rate: } \|\nabla f(-1,1,3)\| = \tfrac{1}{4}\sqrt{2}$

21. rate: $\dfrac{df}{dt} = \nabla f \cdot \mathbf{r}' = \left(4x\,\mathbf{i} - 9y^2\,\mathbf{j}\right) \cdot \left(\dfrac{1}{2}t^{-1/2}\,\mathbf{i} + 2e^{2t}\,\mathbf{j}\right) = 2 - 18e^{6t}$

22. $f(\mathbf{r}(t)) = \sin t^2 + \cos t^2,\quad \text{rate: } \ f'(\mathbf{r}(t)) = 2t\cos t^2 - 2t\sin t^2$

23. rate: $\dfrac{df}{dt} = \nabla f \cdot \mathbf{r}' = \left[\left(\dfrac{1}{y} + \dfrac{z}{x^2}\right)\mathbf{i} - \dfrac{x}{y^2}\,\mathbf{j} - \dfrac{1}{x}\,\mathbf{k}\right] \cdot (\cos\,t\,\mathbf{i} - \sin\,t\,\mathbf{j} + \sec^2 t\,\mathbf{k}) = \dfrac{1-\sin t}{\cos^2 t}$

24. $\dfrac{du}{dt} = \nabla u \cdot \mathbf{r}' = \dfrac{1}{1+x^2y^2}(y\,\mathbf{i} + x\,\mathbf{j}) \cdot (\sec^2 t\,\mathbf{i} + 2e^{2t}\,\mathbf{j}) = \dfrac{e^{2t}}{1+e^{4t}\tan^2 t}(\sec^2 t + 2\tan t)$

25. $\dfrac{du}{dt} = \nabla u \cdot \mathbf{r}' = \left[(3y^2 - 2x)\,\mathbf{i} + 6xy\,\mathbf{j}\right] \cdot [(2t+2)\,\mathbf{i} + 3\,\mathbf{j}] = 104t^3 + 150t^2 - 8t$

26. $u(\mathbf{r}(t)) = \dfrac{1}{\sqrt{1+t^2}},\quad \dfrac{du}{dt} = \dfrac{-t}{(1+t^2)^{3/2}}$

27. area $A = \frac{1}{2}x(t)y(t)\sin\theta(t)$

$\dfrac{dA}{dt} = 0 = \tfrac{1}{2}y(t)x'(t)\sin\theta(t) + \tfrac{1}{2}x(t)y'(t)\sin\theta(t) + \tfrac{1}{2}\theta'(t)x(t)y(t)\cos\theta(t) = 0$

At $x = 4,\ y = 5,\ \theta = \pi/3,\ \dfrac{dx}{dt} = \dfrac{dy}{dt} = 2,$ we have

$$5\frac{d\theta}{dt} + 2\sqrt{3} + \frac{5\sqrt{3}}{2} = 0 \quad \Longrightarrow \quad \frac{d\theta}{dt} = -\frac{9\sqrt{3}}{10}.$$

28. $V = \pi r^2 h;\quad \dfrac{dV}{dt} = 2\pi rh\,\dfrac{dr}{dt} + \pi r^2\dfrac{dh}{dt}$

Measure in centimeters: at $r = 12,\ h = 1000,\ \dfrac{dr}{dt} = 4,\ \dfrac{dh}{dt} = 150,$

$$\frac{dV}{dt} = 2\pi(12)(1000)(4) + \pi(144)(150) = 117{,}600\pi \text{ cu.cm/yr} \cong 0.37 \text{ cu m/yr}.$$

29. $\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y};\quad \dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y}$

$\dfrac{\partial u}{\partial s}\dfrac{\partial u}{\partial t} = \left(\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y}\right)\left(\dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y}\right) = \left(\dfrac{\partial u}{\partial x}\right)^2 - \left(\dfrac{\partial u}{\partial y}\right)^2$

30. $\dfrac{\partial u}{\partial s} = u_x e^s \cos t + u_y e^s \sin t$

$\dfrac{\partial^2 u}{\partial s^2} = u_{xx} e^{2s} \cos^2 t + u_{xy} e^{2s} \sin t \cos t + u_x e^s \cos t + u_y e^s \sin t + u_{yx} e^{2s} \cos t \sin t + u_{yy} e^{2s} \sin^2 t$

$\dfrac{\partial u}{\partial t} = -u_x e^s \sin t + u_y e^s \cos t$

$\dfrac{\partial^2 u}{\partial t^2} = u_{xx} e^{2s} \sin^2 t - u_{xy} e^{2s} \sin t \cos t - u_x e^s \cos t - u_y e^s \sin t - u_{yx} e^{2s} \cos t \sin t + u_{yy} e^{2s} \cos^2 t$

$\dfrac{\partial^2 u}{\partial s^2} + \dfrac{\partial^2 u}{\partial t^2} = e^{2s}(u_{xx} + u_{yy}) \quad \Longrightarrow \quad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = e^{-2s}\left[\dfrac{\partial^2 u}{\partial s^2} + \dfrac{\partial^2 u}{\partial t^2}\right]$

31. $\nabla f(x,y) = (3x^2 - 6xy)\,\mathbf{i} + (-3x^2 + 2y)\,\mathbf{j}$; $\nabla f(1,-1) = \mathbf{N} = 9\,\mathbf{i} - 5\,\mathbf{j}$
normal line: $x = 1 + 9t, \quad y = -1 - 5t$; tangent line: $x = 1 + 5t, \quad y = -1 + 9t$

32. $\nabla f(x,y) = -\pi y \sin \pi xy\,\mathbf{i} - \pi x \sin \pi xy\,\mathbf{j}$; $\nabla f(1/3, 2) = -\pi\sqrt{3}\,\mathbf{i} - \dfrac{\pi\sqrt{3}}{6}\,\mathbf{j}$, take N = 6**i**+ **j**;
normal line: $x = 1/3 + 6t, \quad y = 2 + t$; tangent line: $x = 1/3 + t, \quad y = 2 - 6t$

33. Set $f(x,y,z) = x^{1/2} + y^{1/2} - z$
$\nabla f(x,y,z) = \dfrac{1}{2\sqrt{x}}\,\mathbf{i} + \dfrac{1}{2\sqrt{y}}\,\mathbf{j} - \mathbf{k}$; $\nabla f(1,1,2) = \frac{1}{2}\,\mathbf{i} + \frac{1}{2}\,\mathbf{j} - \mathbf{k}$. Take $\mathbf{N} = \mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$.
tangent plane: $(x-1) + (y-1) - 2(z-2) = 0$; normal line: $x = 1 + t, \quad y = 1 + t, \quad z = 2 - 2t$

34. Set $f(x,y,z) = x^2 + y^2 + z^2$.
$\nabla f(x,y,z) = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$; $\nabla f(1,2,-2) = 2\,\mathbf{i} + 4\,\mathbf{j} - 4\,\mathbf{k}$. Take $\mathbf{N} = \mathbf{i} + 2\,\mathbf{j} - 2\,\mathbf{k}$.
tangent plane: $(x-1) + 2(y-2) - 2(z+2) = 0$; normal line: $x = 1 + t, \ y = 2 + 2t, \ z = -2 - 2t$

35. Set $f(x,y,z) = z^3 + xyz - 2$.
$\nabla f(x,y,z) = yz\,\mathbf{i} + xz\,\mathbf{j} + (3z^2 + xy)\,\mathbf{k}$; $\nabla f(1,1,1) = \mathbf{i} + \mathbf{j} + 4\mathbf{k}$.
tangent plane: $(x-1) + (y-1) + 4(z-1) = 0$; normal line: $x = 1 + t; \ y = 1 + t; \ z = 1 + 4t$

36. Set $f(x,y,z) = e^{3x} \sin 3y - z$.
$\nabla f(x,y,z) = 3e^{3x} \sin 3y\,\mathbf{i} + 3e^{3x} \cos 3y\,\mathbf{j} - \mathbf{k}$; $\nabla f(0, \pi/6, 1) = 3\mathbf{i} - \mathbf{k}$.
tangent plane: $3(x-0) - (z-1) = 0$ or $3x - z + 1 = 0$; normal line: $x = 3t, \ y = \pi/6, \ z = 1 - t$

37. The point $(2,2,1)$ is on each hyperboloid. Set $f(x,y,z) = x^2 + 2y^2 - 4z^2$, $g(x,y,z) = 4x^2 - y^2 + 2z^2$.
$\nabla f = 2x\,\mathbf{i} + 4y\,\mathbf{j} - 8z\,\mathbf{k}$, $\nabla f(2,2,1) = (4, 8, -8)$; $\nabla g = 8x\,\mathbf{i} - 2y\,\mathbf{j} + 4z\,\mathbf{k}$, $\nabla g(2,2,1) = (16, -4, 4)$.
Since $\nabla f(2,2,1) \cdot \nabla g(2,2,1) = 0$, the hyperboloids are mutually perpendicular at $(2,2,1)$.

P1: PBU/OVY P2: PBU/OVY QC: PBU/OVY T1: PBU
JWDD027-16 JWDD027-Salas-v1 December 7, 2006 16:36

38. Set $f(x,y,z)=x^2+y^2+z^2$. At each point (x_0,y_0,z_0), $\nabla f(x_0,y_0,z_0)=2x_0\,\mathbf{i}+2y_0\,\mathbf{j}+2z_0\,\mathbf{k}$.
normal line to the sphere: $x=x_0+x_0t$, $y=y_0+y_0t$, $z=z_0+z_0t$. At $t=-1$, $x=y=z=0$.

39. $\nabla f(x,y)=(2xy-2y)\,\mathbf{i}+(x^2-2x+4y-15)\,\mathbf{j}=\mathbf{0}$ at $(5,0)$, $(-3,0)$, $(1,4)$.
$f_{xx}=2y$, $f_{xy}=2x-2$, $f_{yy}=4$.

point	A	B	C	D	result
$(5,0)$	0	8	4	-64	saddle
$(-3,0)$	0	-8	4	-64	saddle
$(1,4)$	8	0	4	32	loc. min.

$f(1,4)=-34$

40. $\nabla f(x,y)=(6x-3y^2)\,\mathbf{i}+(3y^2+6y-6xy)\,\mathbf{j}=\mathbf{0}$ at $(0,0)$, $(2,2)$, $(\frac{1}{2},-1)$.
$f_{xx}=6$, $f_{xy}=-6y$, $f_{yy}=6y-6x+6$.

point	A	B	C	D	result
$(0,0)$	6	0	6	36	loc. min.
$(2,2)$	6	-12	6	-108	saddle
$(\frac{1}{2},-1)$	6	6	-3	-54	saddle

$f(0,0)=0$

41. $\nabla f(x,y)=(3x^2-18y)\,\mathbf{i}+(3y^2-18x)\,\mathbf{j}=\mathbf{0}$ at $(0,0)$, $(6,6)$.
$f_{xx}=6x$, $f_{xy}=-18$, $f_{yy}=6y$.

point	A	B	C	D	result
$(0,0)$	0	-18	0	-18^2	saddle
$(6,6)$	36	-18	36	>0	loc. min.

$f(6,6)=-216$

42. $\nabla f(x,y)=(3x^2-12x)\,\mathbf{i}+(2y+1)\,\mathbf{j}=\mathbf{0}$ at $(0,-\frac{1}{2})$, $(4,-\frac{1}{2})$.
$f_{xx}=6x-12$, $f_{xy}=0$, $f_{yy}=2$.

point	A	B	C	D	result
$(0,-\frac{1}{2})$	-12	0	2	-24	saddle
$(4,-\frac{1}{2})$	12	0	2	24	loc. min.

$f(4,-\frac{1}{2})=-\frac{145}{4}$

43. $\nabla f(x,y) = (1-2xy+y^2)\,\mathbf{i} + (-1-x^2+2xy)\,\mathbf{j} = \mathbf{0}$ at $(1,1),\ (-1,-1)$.

$f_{xx} = -2y, \quad f_{xy} = -2x+2y, \quad f_{yy} = 2x.$

point	A	B	C	D	result
$(1,1)$	-2	0	2	-4	saddle
$(-1,-1)$	2	0	-2	-4	saddle

44. $\nabla f(x,y) = e^{-(x^2+y^2)/2}\left[(y^2-x^2y^2)\,\mathbf{i} + (2xy-xy^3)\,\mathbf{j}\right] = \mathbf{0}$ at $(\pm 1, \pm\sqrt{2}),\ (x,0),\ x$ any real number.

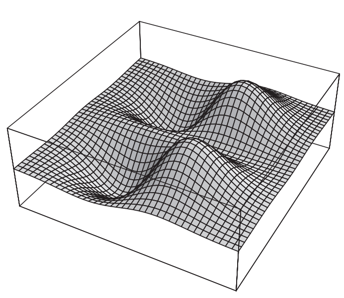
$$f_{xx} = e^{-(x^2+y^2)/2}(-3xy^2+x^3y^2), \quad f_{xy} = e^{-(x^2+y^2)/2}(2y-y^3-2x^2y+x^2y^3),$$

$$f_{yy} = e^{-(x^2+y^2)/2}(2x-5xy^2+xy^4).$$

point	A	B	C	D	result
$(1,\sqrt{2})$	$-4e^{-3/2}$	0	$-4e^{-3/2}$	$16e^{-3}$	loc. max
$(1,-\sqrt{2})$	$-4e^{-3/2}$	0	$-4e^{-3/2}$	$16e^{-3}$	loc. max
$(-1,\sqrt{2})$	$4e^{-3/2}$	0	$4e^{-3/2}$	$16e^{-3}$	loc. min
$(-1,-\sqrt{2})$	$4e^{-3/2}$	0	$4e^{-3/2}$	$16e^{-3}$	loc. min

local maxima: $f(1,\sqrt{2}) = f(1,-\sqrt{2}) = 2e^{-3/2}$; local minima: $f(-1,\sqrt{2}) = f(-1,-\sqrt{2}) = -2e^{-3/2}$.
At $(x,0)$, $D=0$ and $f(x,0)\equiv 0$. For $x<0$, $f(x,y) < f(x,0)$ for all $y \neq 0$; for $x>0$, $f(x,y) > f(x,0)$ for all $y>0$; $(0,0)$ is a saddle point.

Here is a graph of the surface.



45. $\nabla f = (2x-2)\,\mathbf{i} + (2y+2)\,\mathbf{j} = \mathbf{0}$ at $(1,-1)$ in D; $\quad f(1,-1)=0$

Next we consider the boundary of D. We parametrize the circle by:

$C:\ \mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}, \quad t\in[0,2\pi]$

The values of f on the boundary are given by the function

$$F(t) = f(\mathbf{r}(t)) = 6 - 4\cos t + 4\sin t, \quad t\in[0,2\pi]$$

$$F'(t) = 4\sin t + 4\cos t: \quad F'(t)=0 \implies \sin t = -\cos t \implies t = \frac{3}{4}\pi,\ \frac{7}{4}\pi$$

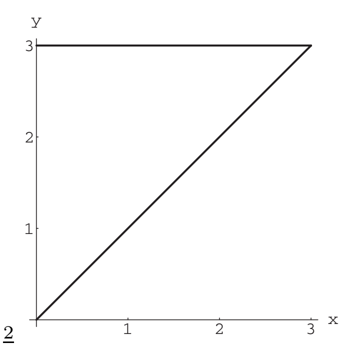
Evaluating F at the endpoints and critical numbers, we have:

$$F(0) = F(2\pi) = f(2,0) = 2; \quad F\left(\tfrac{3}{4}\pi\right) = f\left(-\sqrt{2}, \sqrt{2}\right) = 6 + 4\sqrt{2};$$

$$F\left(\tfrac{7}{4}\pi\right) = f\left(\sqrt{2}, -\sqrt{2}\right) = 6 - 4\sqrt{2}.$$

f takes on its absolute maximum of $6 + 4\sqrt{2}$ at $\left(-\sqrt{2}, \sqrt{2}\right)$; f takes on its absolute minimum of 0 at $(1,-1)$.

46. $\nabla f(x,y) = (4x-4)\,\mathbf{i} + (2y-4)\,\mathbf{j} = \mathbf{0}$ at $(1,2)$ on the boundry of D; no critical points in D.
Next we consider the boundary of D. We parametrize each side of the triangle:



$C_1: \ \mathbf{r}_1(t) = t\,\mathbf{i} + 2t\,\mathbf{j}, \quad t \in [0,1]$

$C_2: \ \mathbf{r}_2(t) = (1-t)\,\mathbf{i} + 2\,\mathbf{j}, \quad t \in [0,1]$

$C_3: \ \mathbf{r}_3(t) = (2-t)\,\mathbf{j}, \quad t \in [0,2]$

Now,

$f_1(t) = f(\mathbf{r}_1(t)) = 6t^2 - 8t + 3, \quad t \in [0,1];$ critical number: $t = \frac{2}{3}$

$f_2(t) = f(\mathbf{r}_2(t)) = 2t^2 - 3+, \quad t \in [0,1];$ critical number

$f_3(t) = f(\mathbf{r}_3(t)) = t^2 - 1, \quad t \in [0,2];$ critical number

Evaluating these functions at the endpoints of their domains and at the critical numbers, we find that:

$f_1(0) = f_3(2) = f(0,0) = 3; \qquad f_1(2/3) = f(2/3, 4/3) = -\frac{7}{3}; \qquad f_1(1) = f_2(0) = f(1,2) = -3;$

$f_2(1) = f_3(0) = f(0,2) = -1.$

f takes on its absolute maximum of 3 at $(0,0)$ and its absolute minimum of -3 at $(1,2)$.

47. $\nabla f(x,y) = (8x - y)\,\mathbf{i} + (-x + 2y + 1)\,\mathbf{j} = \mathbf{0}$ at $(-1/15, -8/15)$ in D; $f(-1/15, -8/15) = -4/15$.
On the boundary of $D: \ x = \cos t, \ \ y = 2\sin t$. Set

$$F(t) = f(\cos t, 2\sin t) = 4 + 2\sin t - 2\sin t\cos t, \quad 0 \le t \le 2\pi.$$

Then

$$F'(t) = 2\cos t - 4\cos^2 t + 2 = -2(2\cos t + 1)(\cos t - 1); \quad F'(t) = 0 \quad \Longrightarrow \quad t = \frac{2\pi}{3}, \ \frac{4\pi}{3}.$$

Evaluating F at the endpoints of the interval and at the critical points, we get

$$F(0) = F(2\pi) = f(1,0) = 4, \ F(2\pi/3) = f(-1/2, \sqrt{3}) = 4 + \frac{3\sqrt{3}}{2},$$

$$F(4\pi/3) = f(-1/2, -\sqrt{3}) = 4 - \frac{3\sqrt{3}}{2} > -\frac{4}{15}$$

f takes on its absolute maximum of 2 at $(0,1)$; f takes on its absolute minimum of $-4/15$ at $(-1/15, -8/15)$.

48. $\nabla f(x,y) = 4x^3\,\mathbf{i} + 6y^2\,\mathbf{j} = \mathbf{0}$ at $(0,0)$ in D; $f(0,0) = 0$.

On the boundary of D: $x = \cos t$, $y = \sin t$. Set

$$F(t) = f(\cos t, \sin t) = \cos^4 t + 2\sin^3 t, \quad 0 \le t \le 2\pi.$$

Then

$F'(t) = 4\cos^3 t \sin t + 6\sin^2 t \cos t = 2\sin t \cos t(2\sin t - 1)(\sin t + 2);$

$F'(t) = 0 \implies t = \pi/6,\ \pi/2,\ 5\pi/6,\ \pi,\ 3\pi/2$

Evaluating F at the endpoints of the interval and at the critical points, we get

$$F(0) = F(2\pi) = f(1,0) = 1,\ F(\pi/6) = f(\sqrt{3}/2, 1/2) = 13/16,\ F(\pi/2) = f(0,1) = 2,$$

$$F(5\pi/6) = f(-\sqrt{3}/2, 1/2) = 13/16,\ F(\pi) = f(-1,0) = 1,\ F(3\pi/2) = f(0,-1) = -2.$$

f takes on its absolute maximum of 2 at $(0,1)$; f takes on its absolute minimum of -2 at $(0,-1)$.

49. Set $f(x,y,z) = D^2 = (x-1)^2 + (y+2)^2 + (z-3)^2$, $g(x,y) = 3x + 2y - z - 5$.

$\nabla f = 2(x-1)\,\mathbf{i} + 2(y+2)\,\mathbf{j} + 2(z-3)\,\mathbf{k}$, $\nabla g = 3\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$.

Set $\nabla f = \lambda \nabla g$:

$2(x-1) = 3\lambda \implies x = \frac{3}{2}\lambda + 1,$

$2(y+2) = 2\lambda \implies y = \lambda - 2,$

$2(z-3) = -\lambda \implies z = -\frac{1}{2}\lambda + 3.$

Substituting these values in $3x + 2y - z = 5$ gives $\lambda = \dfrac{9}{7} \implies x = \dfrac{41}{14},\ y = -\dfrac{5}{7},\ z = \dfrac{33}{14}$.

The point on the plane that is closest to $(1,-2,3)$ is $(41/14, -5/7, 33/14)$. The distance from the point to the plane is $\dfrac{9}{\sqrt{14}}$.

50. Set $f(x,y,z) = 3x - 2y + z$, $g(x,y,z) = x^2 + y^2 + z^2 - 14$,

$\nabla f = 3\,\mathbf{i} - 2\,\mathbf{j} + \mathbf{k}$, $\nabla g = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$.

Set $\nabla f = \lambda \nabla g$:

$$3 = 2\lambda x \implies x = 3/2\lambda, \quad -2 = 2\lambda y \implies y = -1/\lambda, \quad 1 = 2\lambda z \implies z = 1/2\lambda.$$

Substituting these values in $x^2 + y^2 + z^2 = 14$ gives $\lambda = \pm\frac{1}{2} \implies x = 3,\ y = -2,\ z = 1$ or $x = -3,\ y = 2,\ z = -1$. Evaluating f: $f(3,-2,1) = 14$, $f(-3,2,-1) = -14$. The maximum value of f on the sphere is 14.

51. Set $f(x,y,z) = x + y - z$, $g(x,y,z) = x^2 + y^2 + 4z^2 - 4$,

$\nabla f = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\nabla g = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 8z\,\mathbf{k}$.

Set $\nabla f = \lambda \nabla g$:

$$1 = 2\lambda x \implies x = 1/2\lambda, \quad 1 = 2\lambda y \implies y = 1/2\lambda, \quad -1 = 8\lambda z \implies z = -1/8\lambda.$$

Substituting these values in $x^2 + y^2 + 4z^2 = 4$ gives $\lambda = \pm\frac{3}{8} \implies x = 4/3,\ y = 4/3,\ z = -1/3$ or $x = -4/3,\ y = -4/3,\ z = 1/3$. Evaluating f: $f(\frac{4}{3}, \frac{4}{3}, -\frac{1}{3}) = 3$, $f(-\frac{4}{3}, -\frac{4}{3}, \frac{1}{3}) = -3$. The maximum value of f is 3, the minimum value is -3.

52. Let the length, width and height be x, y, z respectively. Then the total cost is

$$f(x,y,z) = \tfrac{1}{2}\,xy + \tfrac{1}{2}\,xz + \tfrac{1}{2}\,yz + \tfrac{1}{10}\,xy = \tfrac{3}{5}\,xy + \tfrac{1}{2}\,xz + \tfrac{1}{2}\,yz$$

with the condition

$$g(x,y,z) = xyz - 16 = 0.$$

Note first that $xyz = 16 \implies x \neq 0, \quad y \neq 0 \quad z \neq 0.$

$$\nabla f = \left(\tfrac{3}{5}\,y + \tfrac{1}{2}\,z\right)\mathbf{i} + \left(\tfrac{3}{5}\,x + \tfrac{1}{2}\,z\right)\mathbf{j} + \left(\tfrac{1}{2}\,y + \tfrac{1}{2}\,x\right)\mathbf{k}, \qquad \nabla g = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$$

$$\nabla f = \lambda\,\nabla g \implies \tfrac{3}{5}\,y + \tfrac{1}{2}\,z = \lambda yz, \quad \tfrac{3}{5}\,x + \tfrac{1}{2}\,z = \lambda xz, \quad \tfrac{1}{2}\,x + \tfrac{1}{2}\,y = \lambda xy$$

Multiply the first equation by x, the second equation by y and subtract. This gives:

$$\tfrac{1}{2}(xz - yz) = 0 \implies z(x-y) = 0 \implies y = x$$

Substituting $y = x$ in the third equation yields $x = \lambda x^2 \implies x = \dfrac{1}{\lambda}$.

Substituting $x = y = \dfrac{1}{\lambda}$ in the first equation yields $z = \dfrac{6}{5\lambda}$.

Finally, substituting these values for x, y and z into the equation $xyz = 16$, we get $\lambda = \dfrac{\sqrt[3]{3}}{2\sqrt[3]{5}}$.

Therefore, $x = y = \dfrac{2\sqrt[3]{5}}{\sqrt[3]{3}} = \dfrac{10}{\sqrt[3]{75}} \cong 2.37$ and $z = \dfrac{6}{5}x = \dfrac{12}{\sqrt[3]{75}} \cong 2.85$.

53. $df = (9x^2 - 10xy^2 + 2)\,dx + (-10x^2y - 1)\,dy$

54. $df = (2xy\sec^2 x^2 - 2y^2)\,dx + (\tan x^2 - 4xy)\,dy$

55. $df = \dfrac{y^2z + z^2y}{(x+y+z)^2}\,dx + \dfrac{xz^2 + zx^2}{(x+y+z)^2}\,dy + \dfrac{x^2y + y^2x}{(x+y+z)^2}\,dz$

56. $df = -\dfrac{z}{y^2 + xz}\,dx + \left(ze^{yz} - \dfrac{2y}{y^2 + xz}\right)dy + \left(ye^{yz} - \dfrac{x}{y^2 + xz}\right)dz$

57. Set $f(x,y,z) = e^x\sqrt{y + z^3}$. Then

$$df = e^x\sqrt{y + z^3}\,\Delta x + \frac{e^x}{2}\frac{1}{\sqrt{y+z^3}}\,\Delta y + \frac{e^x}{2}\frac{3z^2}{\sqrt{y+z^3}}\,\Delta z.$$

With $x = 0,\ y = 15,\ z = 1,\ \Delta x = 0.02,\ \Delta y = 0.2,\ \Delta z = 0.01,\ df = 4\,\Delta x + \tfrac{1}{8}\,\Delta y + \tfrac{3}{8}\,\Delta z \cong 0.1088.$

Therefore, $e^{0.02}\sqrt{15.2 + (1.01)^3} \cong e^0\sqrt{15+1} + 0.1088 = 4.1088.$

58. Set $f(x,y) = x^{1/3}\cos^2 y$. Then

$$df = \tfrac{1}{3}\,x^{-2/3}\cos^2 y\,\Delta x - 2x^{1/3}\cos y\,\sin y\,\Delta y.$$

With $x = 64,\ y = 30^\circ = \pi/6,\ \Delta x = 0.5,\ \Delta y = -2^\circ = -\dfrac{\pi}{90},\ df = \dfrac{1}{64}\,\Delta x - 2\sqrt{3}\,\Delta y \cong 0.1287.$

Therefore, $(64.5)^{1/3}\cos^2(28^\circ) \cong 64^{1/3}\cos^2(30^\circ) + 0.1287 = 3.1287.$

59. $V = \pi r^2 h; \quad r = 5\,\text{ft.}, \quad h = 22\,\text{ft.}, \quad \Delta r = 0.01\,\text{in.} = \frac{1}{1200}\,\text{ft.}, \quad \Delta h = 0.01 = \frac{1}{1200}$

$$dV = 2\pi rh\,\Delta r + \pi r^2\,\Delta h$$

Using the values given above,

$$dV = 2\pi(5)(22)\,\frac{1}{1200} + \pi(25)\,\frac{1}{1200} \cong 0.6414\,\text{cu. ft.} \cong 1108.35\,\text{cu. in.}; \quad \frac{1108.35}{231} \cong 4.80.$$

Approximately 4.80 gallons will be needed.

60. $\dfrac{\partial P}{\partial y} = 12x^2y - 8x = \dfrac{\partial Q}{\partial x}$; the vector function is a gradient.

$\dfrac{\partial f}{\partial x} = 6x^2y^2 - 8xy + 2x, \quad f(x,y) = 2x^3y^2 - 4x^2y + x^2 + \phi(y),$

$\dfrac{\partial f}{\partial y} = 4x^3y - 4x^2 + \phi'(y) = 4x^3y - 4x^2 - 8.$

Thus, $\phi'(y) = -8,\ \phi(y) = -8y + C,$ and $f(x,y) = 2x^3y^2 - 4x^2y + x^2 - 8y + C.$

61. $\dfrac{\partial P}{\partial y} = 2x - \sin x = \dfrac{\partial Q}{\partial x}$; the vector function is a gradient.

$\dfrac{\partial f}{\partial x} = 2xy + 3 - y\sin x, \quad f(x,y) = x^2y + 3x + y\cos x + \phi(y),$

$\dfrac{\partial f}{\partial y} = x^2 + \cos x + \phi'(y) = x^2 + 2y + 1 + \cos x.$

Thus, $\phi'(y) = 2y + 1,\ \phi(y) = y^2 + y + C,$ and $f(x,y) = x^2y + 3x + y\cos x + y^2 + y + C.$

62. $\dfrac{\partial P}{\partial y} = 2xy + 4y; \quad \dfrac{\partial Q}{\partial x} = -2xy + 2; \quad \dfrac{\partial P}{\partial y} \neq \dfrac{\partial Q}{\partial x};$ the vector function is not a gradient.

63. $\dfrac{\partial P}{\partial y} = e^y \sin z = \dfrac{\partial Q}{\partial x}, \quad \dfrac{\partial P}{\partial z} = e^y \cos z = \dfrac{\partial R}{\partial x}, \quad \dfrac{\partial Q}{\partial z} = xe^y \cos z = \dfrac{\partial R}{\partial y};$
the vector function is a gradient.

$f(x,y,z) = \displaystyle\int (e^y \sin z + 2x)\,dx = xe^y \sin z + x^2 + \phi(y,z),$

$f_y = xe^y \sin z + \dfrac{\partial \phi}{\partial y} = xe^y \sin z - y^2 \implies \dfrac{\partial \phi}{\partial y} = -y^2 \implies \phi = -\dfrac{1}{3}y^3 + \psi(z),$

$f(x,y,z) = xe^y \sin z + x^2 - \dfrac{1}{3}y^3 + \psi(z), \quad f_z = xe^y \cos z + \psi'(z) = xe^y \cos z \implies \psi'(x) = 0 \implies$
$\psi(x) = C$

Therefore $f(x,y,z) = xe^y \sin z + x^2 - \frac{1}{3}y^3 + C.$