30. $\dfrac{\partial f}{\partial x} = \dfrac{y}{z} - e^z \implies f(x,y,z) = \dfrac{xy}{z} - xe^z + g(y,z)$

$\dfrac{\partial f}{\partial y} = \dfrac{x}{z} + \dfrac{\partial g}{\partial y} = \dfrac{x}{z} + 1 \implies f(x,y,z) = \dfrac{xy}{z} + y - xe^z + h(z)$

$\dfrac{\partial f}{\partial z} = -\dfrac{xy}{z^2} - xe^z + h'(z) = -xe^z - \dfrac{xy}{z^2} \implies f(x,y,z) = \dfrac{xy}{z} - xe^z + y + C$

31. $\mathbf{F}(\mathbf{r}) = \boldsymbol{\nabla}\left(\dfrac{GmM}{r}\right)$

32.
$$\mathbf{h}(\mathbf{r}) = \begin{cases} \boldsymbol{\nabla}\left(\dfrac{k}{n+2}r^{n+2}\right), & n \neq 2 \\ \boldsymbol{\nabla}\,(k \ln r)\,, & n = -2. \end{cases}$$

REVIEW EXERCISES

1. $\nabla f(x,y) = (4x - 4y)\mathbf{i} + (3y^2 - 4x)\mathbf{j}$

2. $\nabla f(x,y) = \dfrac{y^3 - x^2y}{(x^2+y^2)^2}\mathbf{i} + \dfrac{x^3 - xy^2}{(x^2+y^2)^2}\mathbf{j}$

3. $\nabla f(x,y) = (ye^{xy}\tan 2x + 2e^{xy}\sec^2 2x)\,\mathbf{i} + xe^{xy}\tan 2x\,\mathbf{j}$

4. $\nabla f = \dfrac{1}{x^2+y^2+z^2}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$

5. $\nabla f(x,y) = 2xe^{-yz}\sec z\,\mathbf{i}, -zx^2e^{-yz}\sec z\mathbf{j} - (x^2ye^{-yz}\sec z - x^2e^{-yz}\sec z\tan z)\mathbf{k}$

6. $\nabla f(x,y) = ye^{-3z}\cos xy\,\mathbf{i} + e^{-3z}(x\cos xy + \sin y)\,\mathbf{j}, -3e^{-3z}(\sin xy - \cos y)\,\mathbf{k}$

7. $\nabla f(x,y) = (2x - 2y)\,\mathbf{i} - 2x\,\mathbf{j}, \quad \nabla f(1,-2) = 6\,\mathbf{i} - 2\,\mathbf{j}; \qquad \mathbf{u_a} = \dfrac{1}{\sqrt{5}}\,\mathbf{i} + \dfrac{2}{\sqrt{5}}\,\mathbf{j};$

$f'_{\mathbf{u_a}}(1,-2) = \nabla f(1,-2)\cdot\mathbf{u_a} = \dfrac{2}{\sqrt{5}}.$

8. $\nabla f(x,y) = (e^{xy} + xye^{xy})\,\mathbf{i} + x^2e^{xy}\,\mathbf{j}, \quad \nabla f(2,0) = \mathbf{i} + 4\mathbf{j}; \qquad \mathbf{u_a} = \dfrac{1}{2}\mathbf{i} + \dfrac{\sqrt{3}}{2}\mathbf{j}$

$f'_{\mathbf{u_a}}(2,0) = \nabla f(2,0)\cdot\mathbf{u_a} = \dfrac{1}{2} + 2\sqrt{3}.$

9. $\nabla f(x,y,z) = (y^2 + 6xz)\,\mathbf{i} + (2xy + 2z)\,\mathbf{j} + (2y + 3x^2)\,\mathbf{k}, \quad \nabla f(1,-2,3) = 22\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k};$

$\mathbf{u_a} = \dfrac{1}{3}\mathbf{i} - \dfrac{2}{3}\mathbf{j} + \dfrac{2}{3}\mathbf{k};\ f'_{\mathbf{u_a}}(1,-2,3) = \nabla f(1,-2,3)\cdot\mathbf{u_a} = \dfrac{16}{3}.$

10. $\nabla f(x,y,z) = \dfrac{2x}{x^2+y^2+z^2}\,\mathbf{i} + \dfrac{2y}{x^2+y^2+z^2}\,\mathbf{j} + \dfrac{2z}{x^2+y^2+z^2}\,\mathbf{k}, \quad \nabla f(1,2,3) = \dfrac{1}{7}(\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k});$

$\mathbf{u_a} = \dfrac{1}{\sqrt{3}}\,\mathbf{i} - \dfrac{1}{\sqrt{3}}\,\mathbf{j} + \dfrac{1}{\sqrt{3}}\,\mathbf{k}; \qquad f'_{\mathbf{u_a}}(1,2,3) = \nabla f(1,2,3)\cdot\mathbf{u_a} = \dfrac{2}{7\sqrt{3}}.$

11. $\nabla f(x,y) = (6x - 2y^2)\,\mathbf{i} - 4xy\,\mathbf{j}, \quad \nabla f(3,-2) = 10\,\mathbf{i} + 24\,\mathbf{j};$

$\mathbf{a} = (0,0) - (3,-2) = (-3,2) = -3\,\mathbf{i} + 2\,\mathbf{j}, \quad \mathbf{u_a} = \dfrac{-3}{\sqrt{13}}\,\mathbf{i} + \dfrac{2}{\sqrt{13}}\,\mathbf{j};$

$f'_{\mathbf{u_a}}(3,-2) = \nabla f(3,-2) \cdot \mathbf{u_a} = \dfrac{18}{\sqrt{13}}.$

12. $\nabla f(x,y,z) = (y^2z - 3yz)\,\mathbf{i} + (2xyz - 3xz)\,\mathbf{j} + (xy^2 - 3xy)\,\mathbf{k}, \quad \nabla f(1,-1,2) = 8\,\mathbf{i} - 10\,\mathbf{j} + 4\,\mathbf{k};$

$\mathbf{r}'(t) = \mathbf{i} - \pi \sin \pi t\,\mathbf{j} + 2e^{t-1}\,\mathbf{k}, \quad \mathbf{a} = \mathbf{r}'(1) = \mathbf{i} + 2\mathbf{k}, \quad \mathbf{u_a} = \dfrac{1}{\sqrt{5}}\,\mathbf{i} + \dfrac{2}{\sqrt{5}}\,\mathbf{k};$

$f'_{\mathbf{u_a}}(1,-1,2) = \nabla f(1,-1,2) \cdot \mathbf{u_a} = \dfrac{16}{\sqrt{5}}.$

13. $\nabla f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}\,(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}), \quad \nabla f(3,-1,4) = \dfrac{1}{\sqrt{26}}(3\,\mathbf{i} - \mathbf{j} + 4\,\mathbf{k});$

$\mathbf{a} = \pm(4\,\mathbf{i} - 3\,\mathbf{j} + \mathbf{k}), \quad \mathbf{u_a} = \pm\dfrac{1}{\sqrt{26}}(4\,\mathbf{i} - 3\,\mathbf{j} + \mathbf{k}); \qquad f'_{\mathbf{u_a}}(3,-1,4) = \nabla f(3,-1,4) \cdot \mathbf{u_a} = \pm\dfrac{19}{26}.$

14. $\nabla f(x,y) = 2e^{2x}(\cos y - \sin y)\,\mathbf{i} - e^{2x}(\sin y + \cos y)\,\mathbf{j}, \quad \nabla f\left(\frac{1}{2}, -\frac{1}{2}\pi\right) = 2e\,\mathbf{i} + e\,\mathbf{j};$

maximum directional derivative: $\|\nabla f\left(\frac{1}{2}, -\frac{1}{2}\pi\right)\| = e\sqrt{5}.$

15. $\nabla f(x,y,z) = \cos xyz(yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}), \quad \nabla f(\frac{1}{2}, \frac{1}{3}, \pi) = \frac{\pi\sqrt{3}}{6}\,\mathbf{i} + \frac{\pi\sqrt{3}}{4}\,\mathbf{j} + \frac{\sqrt{3}}{12}\,\mathbf{k};$

minimum directional derivative: $f'_{\mathbf{u}} = -\|\nabla f(\frac{1}{2}, \frac{1}{3}, \pi)\| = -\frac{\sqrt{39\pi^2+3}}{12}$

16. Let $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$ be the path of the particle. $\nabla I(x,y) = -2x\,\mathbf{i} - 6y\,\mathbf{j}$. Then

$$x'(t) = -2x(t), \quad y'(t) = -6y(t) \quad \Longrightarrow \quad x(t) = C_1e^{-2t}, \quad y(t) = C_2e^{-6t}.$$

$$\mathbf{r}(0) = (4,3) \quad \Longrightarrow \quad C_1 = 4, \quad C_2 = 3.$$

Therefore the path of the particle is: $\mathbf{r}(t) = 4e^{-2t}\,\mathbf{i} + 3e^{-6t}\,\mathbf{j},\ t \ge 0,$ or, $y = \frac{3}{64}x^3,\ 0 < x \le 4$

17. Let $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$ be the path of the particle. $\nabla T = -e^{-x}\cos y\,\mathbf{i} - e^{-x}\sin y\,\mathbf{j}$. Then

$$x'(t) = -e^{-x(t)}\cos y(t), \quad y'(t) = -e^{-x(t)}\sin y(t) \quad \Longrightarrow \quad \frac{y'(t)}{x'(t)} = \tan y(t) \quad \Longrightarrow \quad \frac{dy}{dx} = \tan y$$

The solution is $\sin y = Ce^x$. Since $\mathbf{r}(0) = 0$, $C = 0$ and $y = 0$. The particle moves to the right the x-axis.

18. $\nabla z = 8x\,\mathbf{i} + 2y\,\mathbf{j}; \quad \mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}.$

$$x'(t) = -8x(t), \quad y'(t) = -2y(t) \quad \Longrightarrow \quad x(t) = C_1e^{-8t}, \quad y(t) = C_2e^{-2t}.$$

(a) $\mathbf{r}(0) = (1,1) \quad \Longrightarrow \quad C_1 = 1,\ C_2 = 1; \quad x = e^{-8t},\ y = e^{-2t}$ or $x = y^4.$

(b) $\mathbf{r}(0) = (1,-2) \quad \Longrightarrow \quad C_1 = 1,\ C_2 = -2; \quad x = e^{-8t},\ y = -2e^{-2t}$ or $x = y^4/16.$

19. $\nabla f(x,y) = e^x \arctan y\,\mathbf{i} + e^x \dfrac{1}{1+y^2}\,\mathbf{j}; \quad \nabla f(0,1) = \dfrac{\pi}{4}\,\mathbf{i} + \dfrac{1}{2}\,\mathbf{j}.$

$\mathbf{u} = \dfrac{\nabla f(0,1)}{\|\nabla f(0,1)\|} = \dfrac{1}{\sqrt{4+\pi^2}}(\pi\,\mathbf{i} + 2\,\mathbf{j}); \quad \text{rate: } \|\nabla f(0,1)\| = \dfrac{\sqrt{\pi^2+4}}{4}$

20. $\nabla f(x,y,z) = \dfrac{1}{(y+z)^2}[(y+z)\,\mathbf{i} + (z-x)\,\mathbf{j} - (x+y)\,\mathbf{k}]; \quad \nabla f(-1,1,3) = \dfrac{1}{4}\,\mathbf{i} + \dfrac{1}{4}\,\mathbf{j}.$

$\mathbf{u} = \dfrac{\nabla f(-1,1,3)}{\|\nabla f(-1,1,3)\|} = \tfrac{1}{2}\sqrt{2}\,\mathbf{i} + \tfrac{1}{2}\sqrt{2}\,\mathbf{j}; \quad \text{rate: } \|\nabla f(-1,1,3)\| = \tfrac{1}{4}\sqrt{2}$

21. rate: $\dfrac{df}{dt} = \nabla f \cdot \mathbf{r}' = \left(4x\,\mathbf{i} - 9y^2\,\mathbf{j}\right) \cdot \left(\dfrac{1}{2}t^{-1/2}\,\mathbf{i} + 2e^{2t}\,\mathbf{j}\right) = 2 - 18e^{6t}$

22. $f(\mathbf{r}(t)) = \sin t^2 + \cos t^2, \quad \text{rate: } \quad f'(\mathbf{r}(t)) = 2t\cos t^2 - 2t\sin t^2$

23. rate: $\dfrac{df}{dt} = \nabla f \cdot \mathbf{r}' = \left[\left(\dfrac{1}{y} + \dfrac{z}{x^2}\right)\mathbf{i} - \dfrac{x}{y^2}\,\mathbf{j} - \dfrac{1}{x}\,\mathbf{k}\right] \cdot (\cos t\,\mathbf{i} - \sin t\,\mathbf{j} + \sec^2 t\,\mathbf{k}) = \dfrac{1-\sin t}{\cos^2 t}$

24. $\dfrac{du}{dt} = \nabla u \cdot \mathbf{r}' = \dfrac{1}{1+x^2y^2}(y\,\mathbf{i} + x\,\mathbf{j}) \cdot (\sec^2 t\,\mathbf{i} + 2e^{2t}\,\mathbf{j}) = \dfrac{e^{2t}}{1+e^{4t}\tan^2 t}(\sec^2 t + 2\tan t)$

25. $\dfrac{du}{dt} = \nabla u \cdot \mathbf{r}' = \left[(3y^2 - 2x)\,\mathbf{i} + 6xy\,\mathbf{j}\right] \cdot [(2t+2)\,\mathbf{i} + 3\,\mathbf{j}] = 104t^3 + 150t^2 - 8t$

26. $u(\mathbf{r}(t)) = \dfrac{1}{\sqrt{1+t^2}}, \quad \dfrac{du}{dt} = \dfrac{-t}{(1+t^2)^{3/2}}$

27. area $A = \tfrac{1}{2}x(t)y(t)\sin\theta(t)$

$\dfrac{dA}{dt} = 0 = \tfrac{1}{2}y(t)x'(t)\sin\theta(t) + \tfrac{1}{2}x(t)y'(t)\sin\theta(t) + \tfrac{1}{2}\theta'(t)x(t)y(t)\cos\theta(t) = 0$

At $x = 4,\ y = 5,\ \theta = \pi/3,\ \dfrac{dx}{dt} = \dfrac{dy}{dt} = 2,$ we have

$$5\frac{d\theta}{dt} + 2\sqrt{3} + \frac{5\sqrt{3}}{2} = 0 \quad \Longrightarrow \quad \frac{d\theta}{dt} = -\frac{9\sqrt{3}}{10}.$$

28. $V = \pi r^2 h; \quad \dfrac{dV}{dt} = 2\pi r h\,\dfrac{dr}{dt} + \pi r^2 \dfrac{dh}{dt}$

Measure in centimeters: at $r = 12,\ h = 1000,\ \dfrac{dr}{dt} = 4,\ \dfrac{dh}{dt} = 150,$

$$\frac{dV}{dt} = 2\pi(12)(1000)(4) + \pi(144)(150) = 117{,}600\pi \text{ cu.cm/yr} \cong 0.37 \text{ cu m/yr.}$$

29. $\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y}; \qquad \dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y}$

$\dfrac{\partial u}{\partial s}\dfrac{\partial u}{\partial t} = \left(\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y}\right)\left(\dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y}\right) = \left(\dfrac{\partial u}{\partial x}\right)^2 - \left(\dfrac{\partial u}{\partial y}\right)^2$