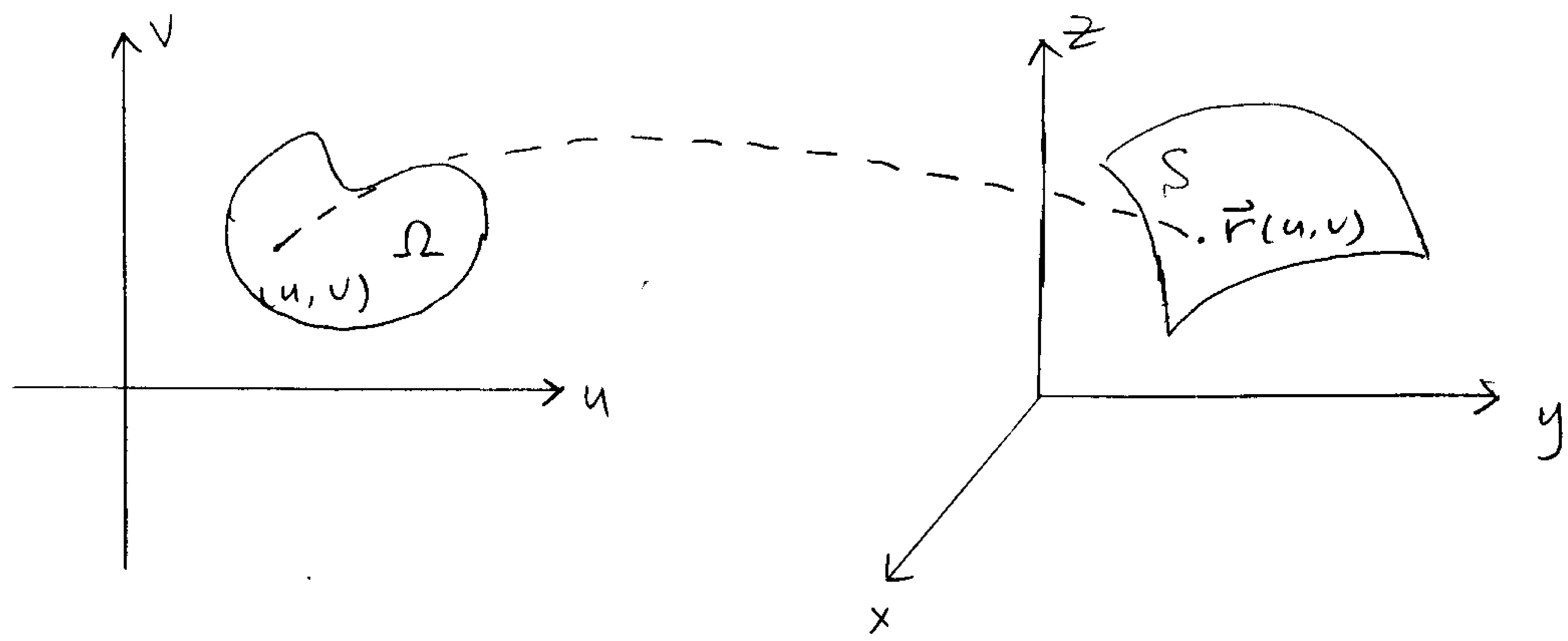
§4 Parametrized Surfaces ; Surface Area

S : surface.

We can parametrize S by a vector function $\vec{r} = \vec{r}(u,v)$ where (u,v) ranges over some region Ω of the uv-plane



Examples

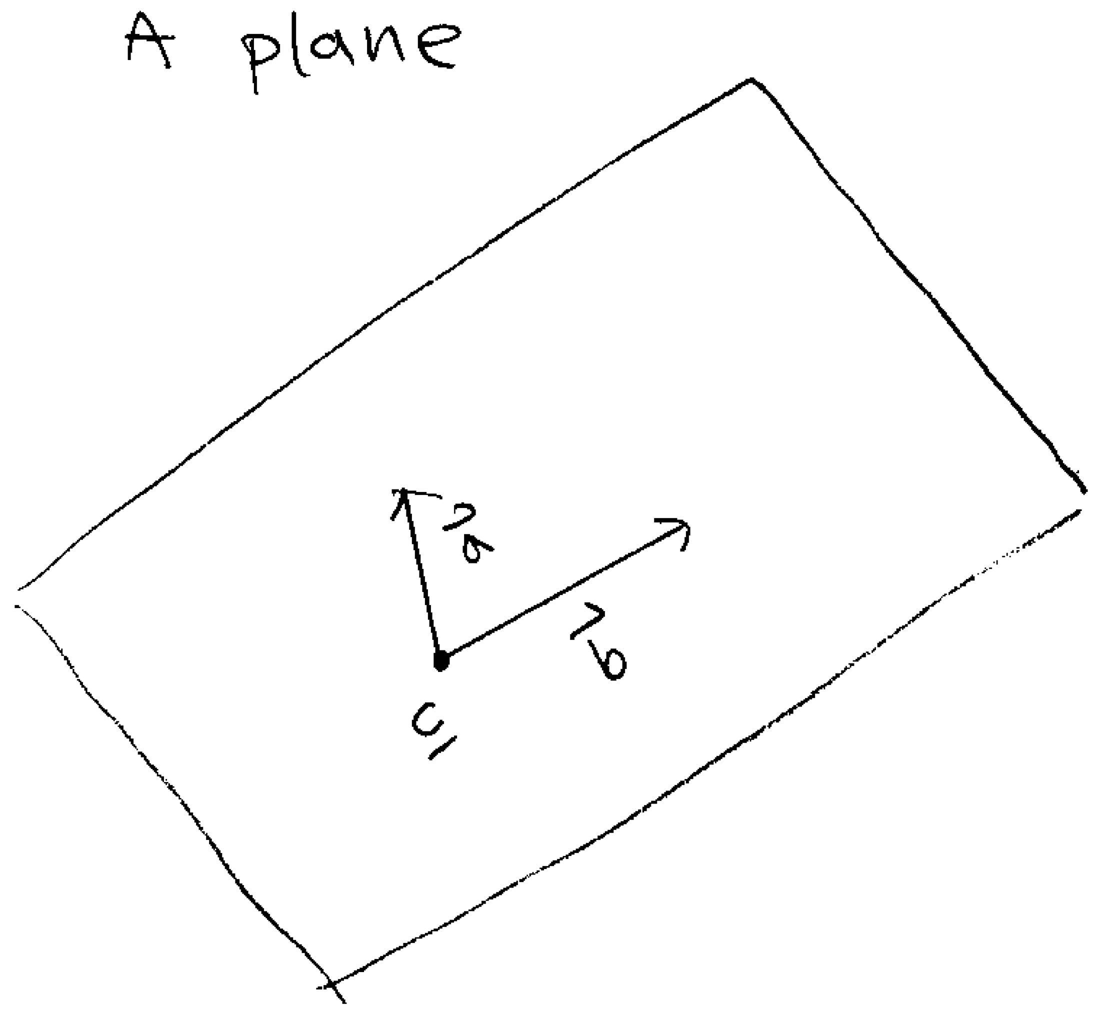
1. $y = f(x)$, $x \in [a,b]$ can be parametrized by

setting $\vec{r}(u) = ui + f(u)j$, $u \in [a,b]$

$z = f(x,y)$, $(x,y) \in \Omega$ can be parametrized by

setting $\vec{r}(u,v) = ui + vj + f(u,v)k$, $(u,v) \in \Omega$

2. A plane



The plane can be parametrized by setting $\vec{r}(u,v) = u\vec{a} + v\vec{b} + \underline{c}$, u, v : real.

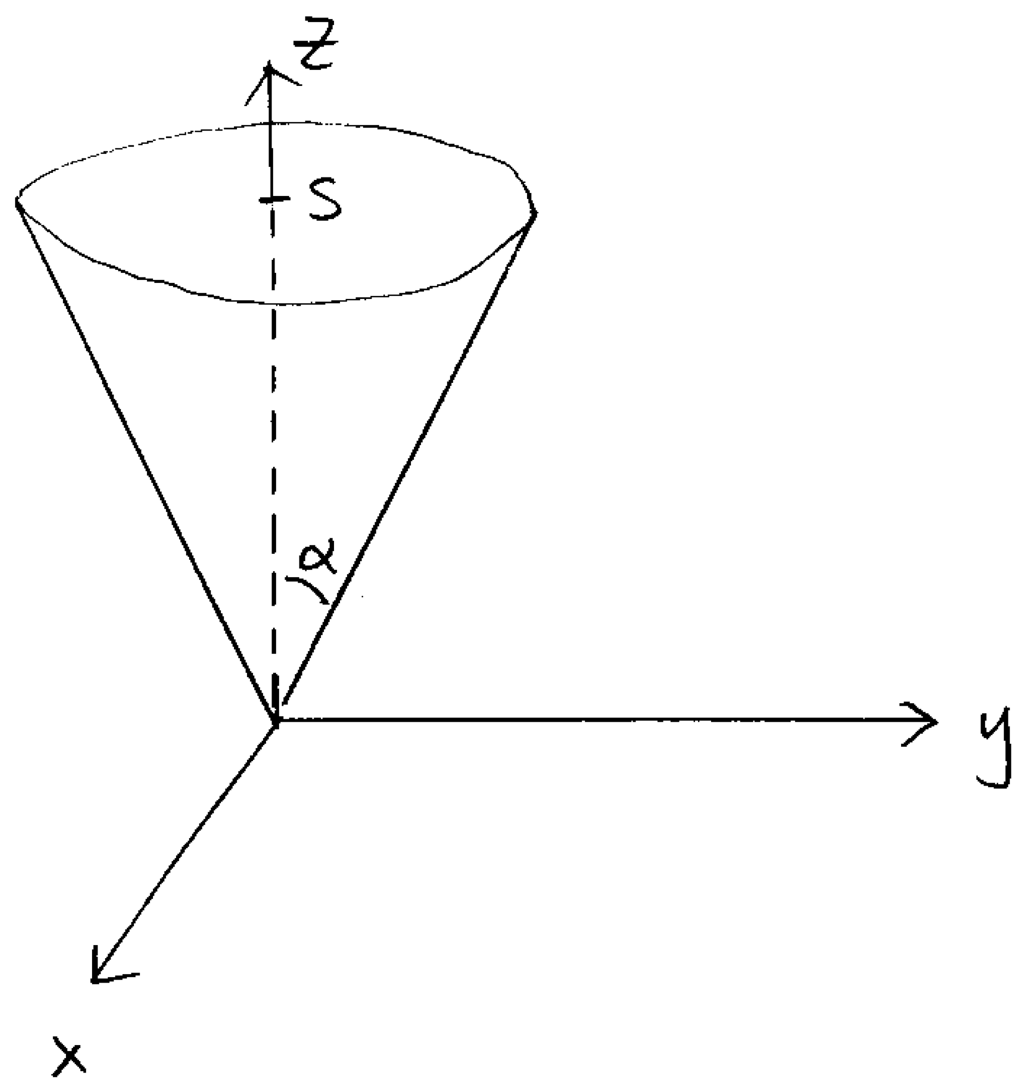
3. A sphere

The sphere of radius a centered at the origin can be parametrized by setting

$$\vec{r}(u,v) = a\cos u\cos v\,\vec{i} + a\cos u\sin v\,\vec{j} + a\sin u\,\vec{k}$$

$0 \le u \le 2\pi,\ -\frac{1}{2}\pi \le v \le \frac{1}{2}\pi$.

4. A Cone

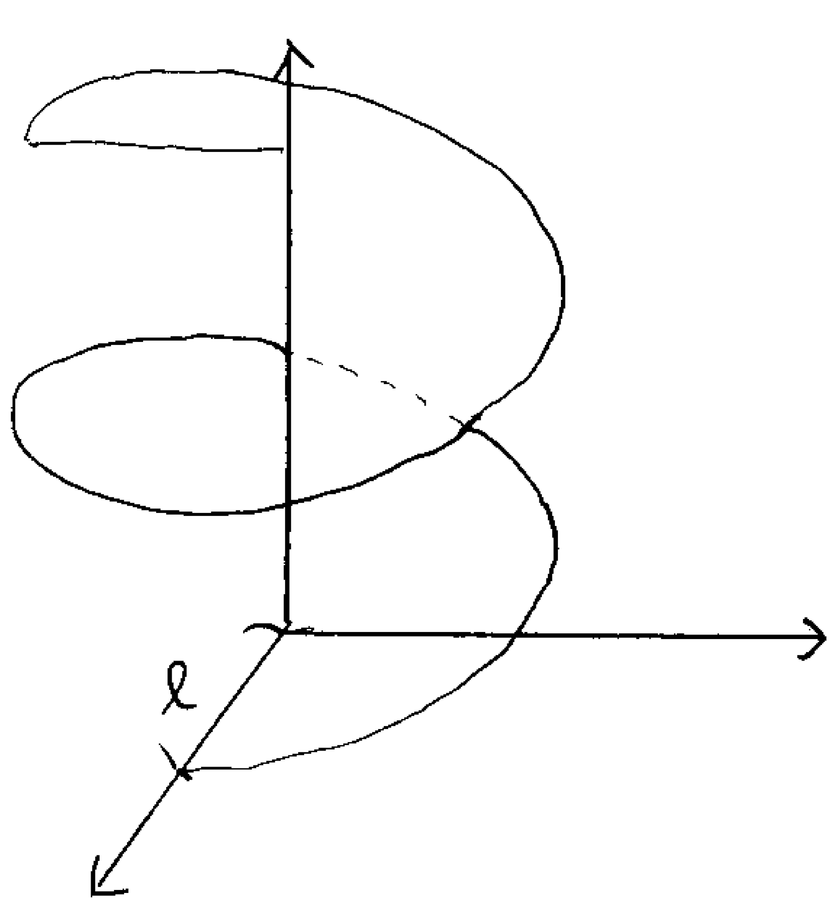


The cone is parametrized by setting

$$\vec{r}(u,v) = v\cos u\cos\alpha\,\vec{i} + v\sin u\sin\alpha\,\vec{j} + v\cos\alpha\,\vec{k}$$

$0 \le u \le 2\pi,\ 0 \le v \le s$.

5. A spiral ramp



$$\vec{r}(u,v) = u\cos\omega v\,\vec{i} + u\sin\omega v\,\vec{j} + bv\,\vec{k}$$

$0 \le u \le l,\ v \ge 0$.

The Fundamental Vector Product

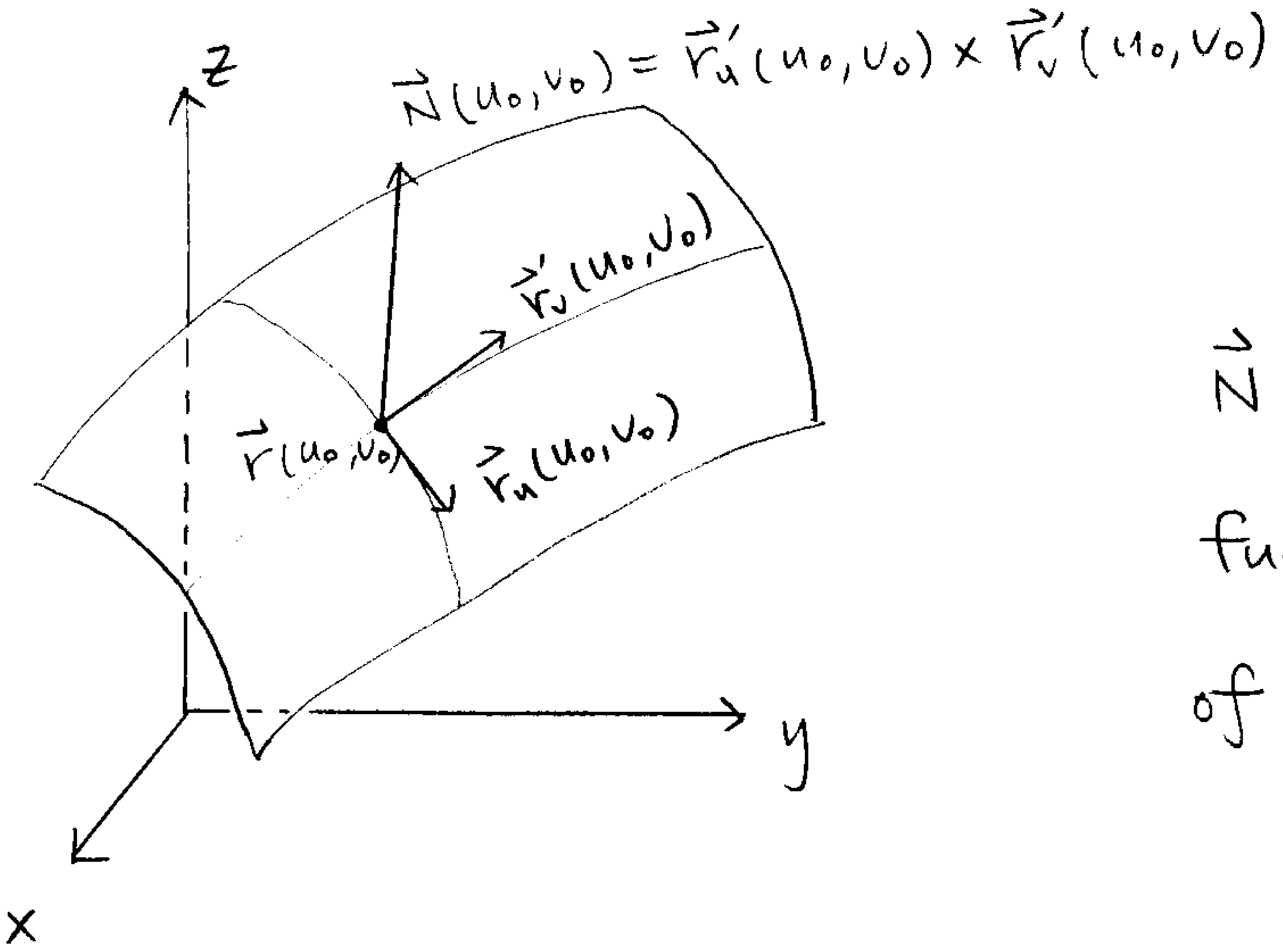
S : surface parametrized by a differentiable vector function

$$\vec{r} = \vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$$

For simplicity, let R : $a<u<b$, $c<v<d$

$$\vec{r}_u' = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial y}\vec{j} + \frac{\partial z}{\partial u}\vec{k} \quad , \quad \vec{r}_v' = \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k}$$

let $(u_0, v_0) \in R$ s.t. $\vec{r}_u'(u_0,v_0) \times \vec{r}_v'(u_0,v_0) \neq \vec{0}$



$\vec{N}(u_0,v_0) \triangleq \vec{r}_u' \times \vec{r}_v'$, the fundamental vector product of the surface

Examples

1. Plane $\vec{r}(u,v) = u\cdot\vec{a} + v\cdot\vec{b} + \underline{c}$

$\vec{r}_u'(u,v) = \vec{a}$, $\vec{r}_v'(u,v) = \vec{b}$

$\vec{N}(u,v) = \vec{a} \times \vec{b}$

2. The sphere $x^2+y^2+z^2=a^2$ can be parametrized by setting

$$\vec{r}(u,v) = a\cos u\cos v\,\vec{i} + a\cos u\sin v\,\vec{j} + a\sin u\,\vec{k}$$

$$0 \le u \le 2\pi,\quad -\tfrac{1}{2}\pi \le v \le \tfrac{1}{2}\pi .$$

$$\vec{r}_u'(u,v) = -a\sin u\cos v\,\vec{i} + a\cos u\cos v\,\vec{j}$$

$$\vec{r}_v'(u,v) = -a\cos u\sin v\,\vec{i} - a\sin u\sin v\,\vec{j} + a\cos v\,\vec{k}$$

$$\vec{N}(u,v) = \begin{vmatrix} i & j & k \\ -a\sin u\cos v & a\cos u\cos v & 0 \\ -a\cos u\sin u & -a\sin u\sin v & a\cos v \end{vmatrix}$$

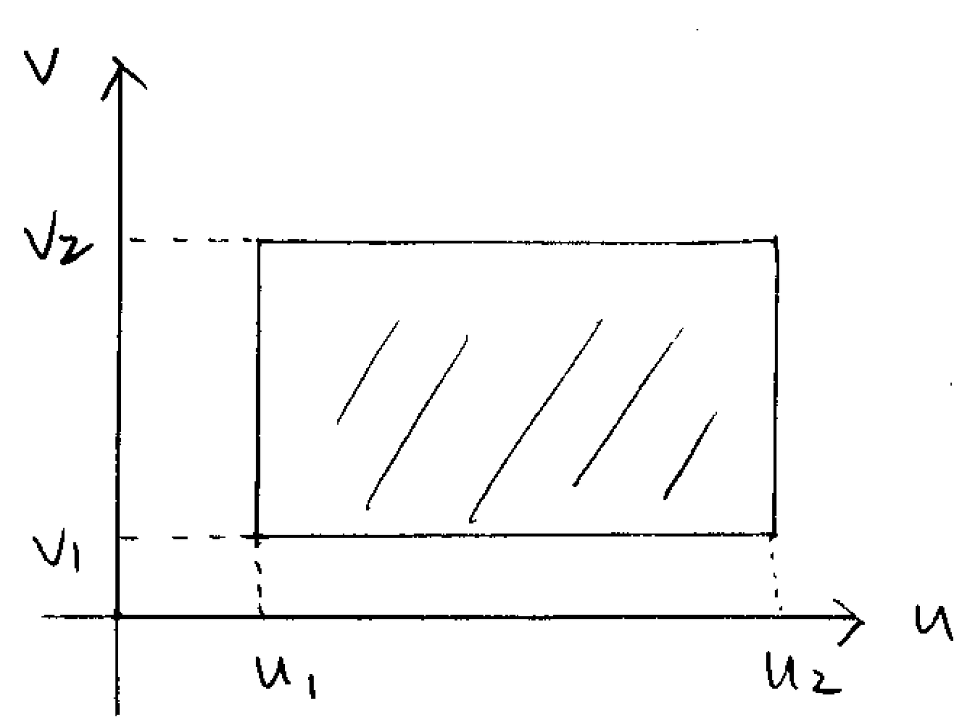
$$= a\cos v\,(a\cos u\cos v\,\vec{i} + a\sin u\cos v\,\vec{j} + a\sin v\,\vec{k})$$

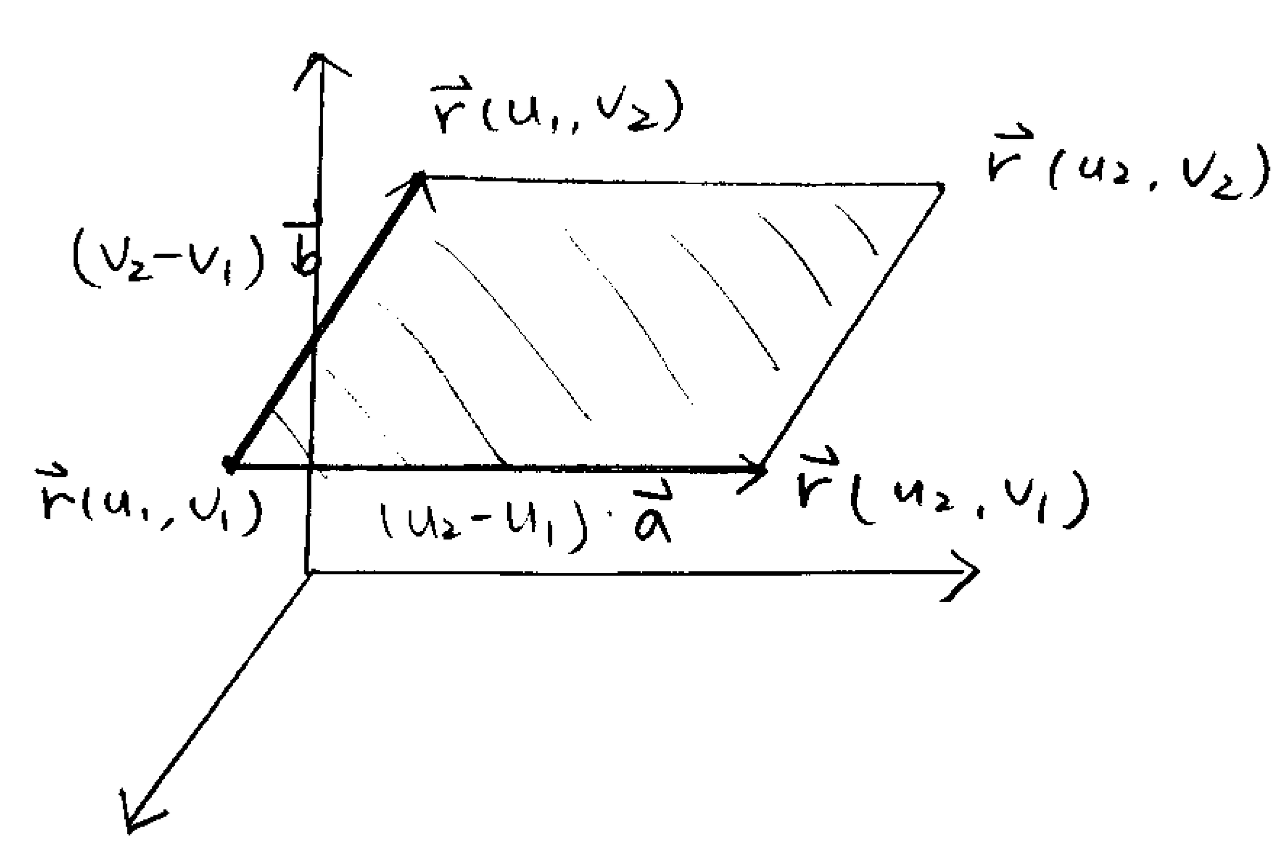
$$= a\cos v \cdot \vec{r}(u,v)$$

The Area of a Parametrized surface.

A plane can be parametrized by a linear function

$$\vec{r}(u,v) = u\vec{a} + v\vec{b} + \underline{c}$$





$$\vec{r}(u_2,v_1) - \vec{r}(u_1,v_1) = (u_2-u_1)\,\vec{a} \quad , \quad \vec{r}(u_1,v_2) - \vec{r}(u_1,v_1) = (v_2-v_1)\cdot\vec{b}$$

The area of the parallelogram is

$$\|(u_2-u_1)\cdot\vec{a} \times (v_2-v_1)\cdot\vec{b}\| = \|\vec{a}\times\vec{b}\|\,(u_2-u_1)(v_2-v_1) = \|\vec{a}\times\vec{b}\|\cdot\text{area of } R$$

Theorem 4.1

Ω : basic region in uv-plane

Suppose $N = \vec{r}_u' \times \vec{r}_v'$ is never zero on the interior of Ω

Then area of $S = \iint_\Omega \| \vec{N}(u,v) \| \, du dv$.

Examples

1. The surface area of a sphere

$\vec{r}(u,v) = a\cos u \cos v \, \vec{i} + a \cos u \sin v \, \vec{j} + a \sin u \, \vec{k}$

Ω : $0 \le u \le 2\pi$, $-\frac{1}{2}\pi \le v \le \frac{1}{2}\pi$

$\vec{N}(u,v) = a\cos v \cdot \vec{r}(u,v)$ and $\| \vec{N}(u,v) \| = a^2 |\cos v| = a^2 \cos v$

$(\because -\frac{1}{2}\pi \le v \le \frac{1}{2}\pi)$

$$\text{area of the sphere} = \iint_\Omega \| \vec{N}(u,v) \| \, du dv$$

$$= \iint_\Omega a^2 \cos v \, du dv$$

$$= \int_0^{2\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} a^2 \cos v \, dv du = \int_0^{2\pi} a^2 \sin v \Big|_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} du$$

$$= \int_0^{2\pi} 2a^2 \, du = 4\pi a^2.$$

2. The area of a plane region

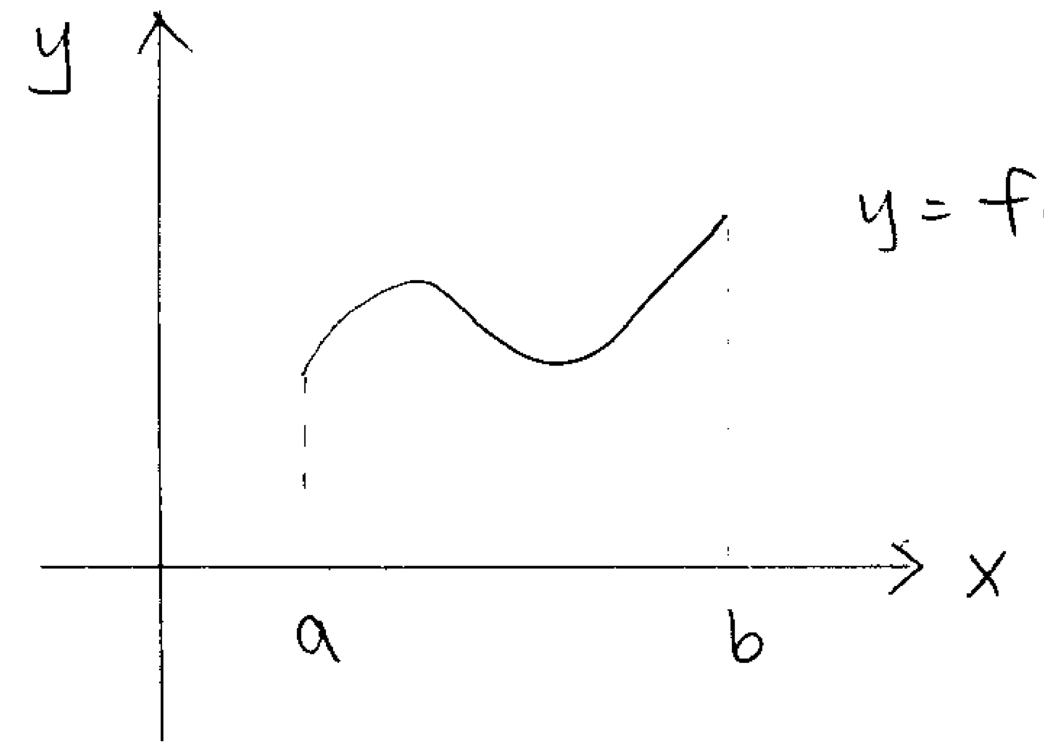
S : a plane region Ω can be parametrized by setting

$\vec{r}(u,v) = u\vec{i} + v\vec{j}$, $(u,v) \in \Omega$

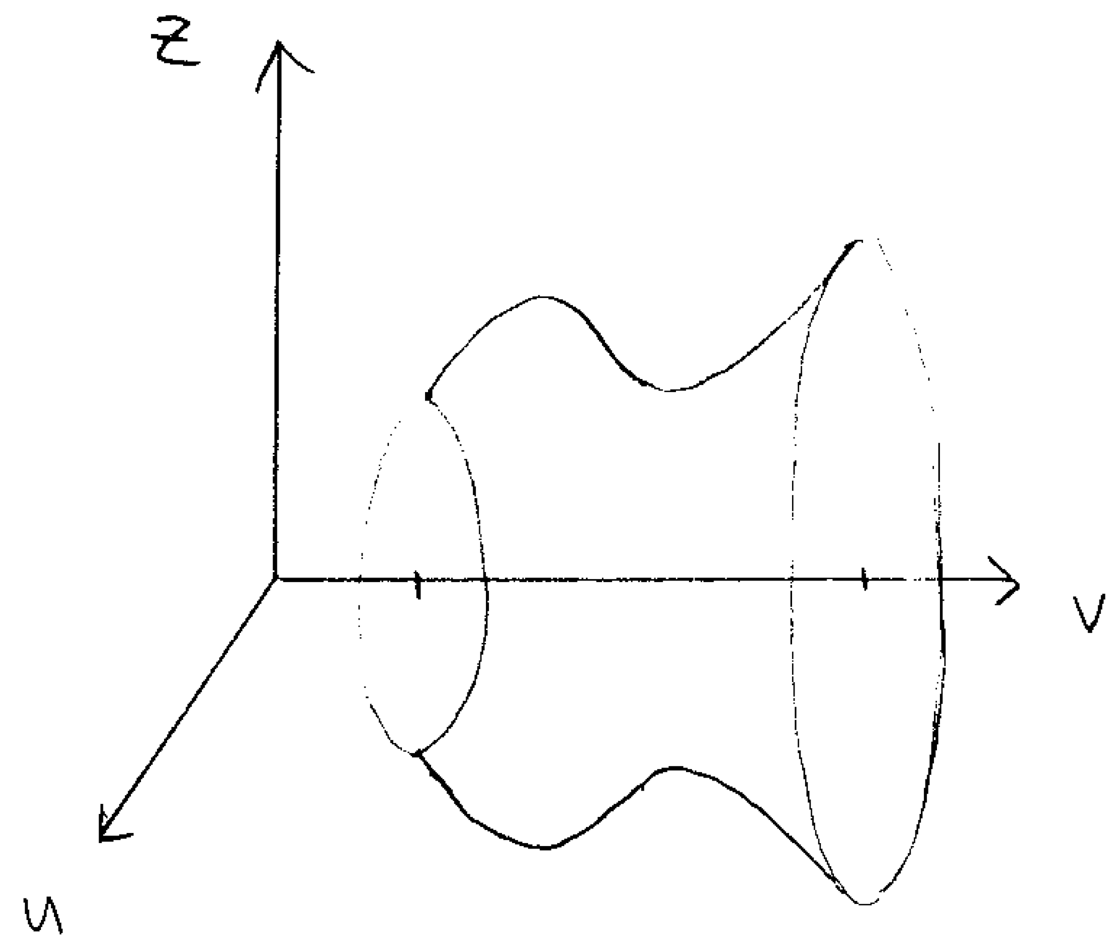
$\vec{N}(u,v) = \vec{r}_u'(u,v) \times \vec{r}_v'(u,v) = \vec{i} \times \vec{j} = \vec{k} \Rightarrow \| \vec{N}(u,v) \| = 1$

$\Rightarrow$ area of $S = \iint_\Omega du dv$.

3. The area of a surface of revolution



$y = f(x)$, $x \in [a, b]$



If f is positive and continuously differentiable

we can parametrize S by setting

$\vec{r}(u,v) = v\,\vec{i} + f(v)\cos u\,\vec{j} + f(v)\sin u\,\vec{k}$

Ω : $0 \le u \le 2\pi$, $a \le v \le b$.

$$\vec{N}(u,v) = \vec{r}_u{}'(u,v) \times \vec{r}_v{}'(u,v) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -f(v)\sin u & f(v)\cos u \\ 1 & f'(v)\cos u & f'(v)\sin u \end{vmatrix}$$

$$= -f(v)f'(v)\,\vec{i} + f(v)\cos u\,\vec{j} + f(v)\sin u\,\vec{k}$$

$$\|\vec{N}(u,v)\| = f(v)\sqrt{f'(v)^2 + 1}$$

$$\text{area of } S = \iint_{\Omega} f(v)\sqrt{f'(v)^2+1}\,du\,dv = \int_0^{2\pi}\int_a^b f(v)\sqrt{f'(v)+1}\,dv\,du$$

$$= 2\pi\int_a^b f(v)\sqrt{f'(v)+1}\,dv.$$

4. Spiral ramp

$S : \vec{r}(u,v) = u\cos wv\,\vec{i} + u\sin wv\,\vec{j} + bv\,\vec{k}$, $\Omega : 0 \le u \le \ell$, $0 \le v \le \frac{2\pi}{w}$

$\vec{r}_u(u,v) = \cos wv\,\vec{i} + \sin wv\,\vec{j}$, $\vec{r}_v(u,v) = -wu\sin wv\,\vec{i} + wu\cos wu\,\vec{j} + b\vec{k}$

$$\vec{N}(u,v) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos wv & \sin wv & 0 \\ -wu\sin wv & wu\cos wu & b \end{vmatrix} = b\sin wv\,\vec{i} - b\cos wv\,\vec{j} + wu\,\vec{k}$$

$$\|\vec{N}(u,v)\| = \sqrt{b^2 + w^2u^2}$$

$$\iint_\Omega \sqrt{b^2+w^2u^2}\,du\,dv = \int_0^{\frac{2\pi}{w}} \int_0^{\ell} \sqrt{b^2+w^2u^2}\,du\,dv = \frac{2\pi}{w}\int_0^{\ell} \sqrt{b^2+w^2u^2}\,du$$

The area of a surface $z = f(x,y)$

We can parametrize S by setting $\vec{r}(u,v) = u\vec{i} + v\vec{j} + f(u,v)\vec{k}$, $(u,v) \in \Omega$

We may just as well use x & y and write

$\vec{r}(x,y) = x\vec{i} + y\vec{j} + f(x,y)\vec{k}$, $(x,y) \in \Omega$

$\vec{r}_x(x,y) = \vec{i} + f_x(x,y)\vec{k}$, $\vec{r}_y(x,y) = \vec{j} + f_y(x,y)\vec{k}$, $(x,y) \in \Omega$

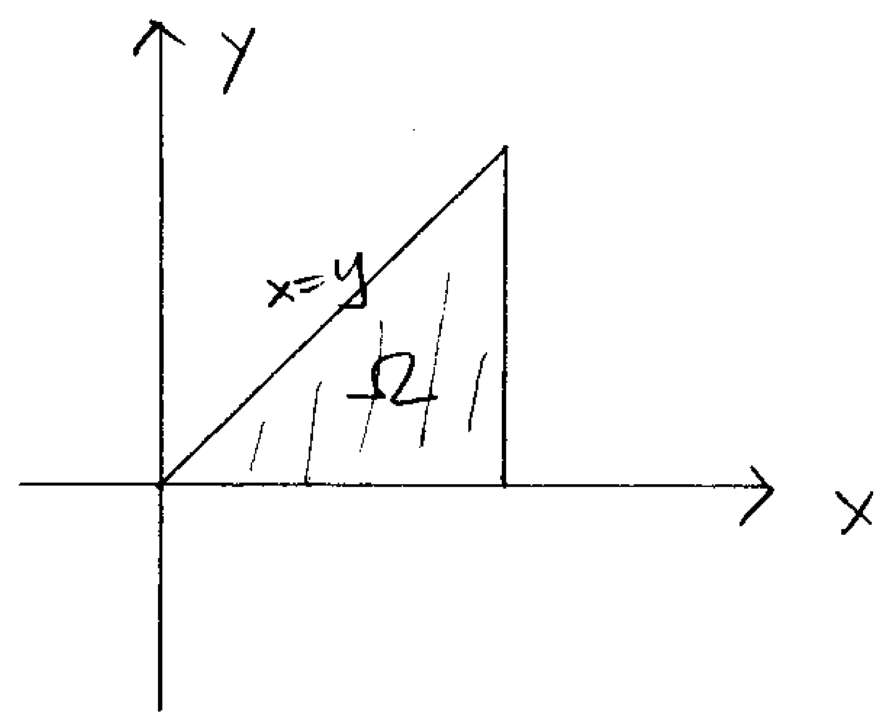
$$\vec{N}(x,y) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x(x,y) \\ 0 & 1 & f_y(x,y) \end{vmatrix} = -f_x(x,y)\vec{i} - f_y(x,y))\vec{j} + \vec{k}$$

$$\|\vec{N}(x,y)\| = \sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1}$$

$$\text{area of } S = \iint_\Omega \sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1}\,dx\,dy$$

Examples

1. Find the surface area of that part of the parabolic cylinder $z=y^2$ that lies over the triangle with vertices $(0,0)$, $(0,1)$ $(1,1)$ in the xy-plane



$f_x(x,y)=0$, $f_y(x,y)=2y$

Ω : $0\le y\le 1$, $0\le x\le y$

$$A=\iint_\Omega \sqrt{f_x(x,y)^2+f_y(x,y)^2+1} = \int_0^1\int_0^y \sqrt{0^2+(2y)^2+1}\,dx\,dy$$

$$=\int_0^1\int_0^y \sqrt{4y^2+1}\,dx\,dy = \int_0^1 y\sqrt{4y^2+1}\,dy = \frac{1}{12}(5\sqrt{5}-1)$$

2. Find the surface area of that part of the hyperbolic paraboloid $z=xy$ that lies inside the cylinder $x^2+y^2=a^2$

$f_x(x,y)=y$, $f_y(x,y)=x$

$$A=\iint_\Omega \sqrt{y^2+x^2+1}\,dx\,dy = \int_0^{2\pi}\int_0^a \sqrt{r^2+1}\,r\,dr\,d\theta = \frac{2}{3}\pi\left[(a^2+1)^{\frac{3}{2}}-1\right]$$