Line Integrals and Surface Integrals

§1 Line Integrals

Definition 1.1

Let $\underline{h}(x,y,z) = h_1(x,y,z)\,\hat{i} + h_2(x,y,z)\,\hat{j} + h_3(x,y,z)\,\hat{k}$ be a vector field that is continuous on a smooth curve $C: \vec{r}(u) = x(u)\,i + y(u)\,j + z(u)\,k$ for $u \in [a,b]$. The line integral of $\underline{h}$ over C is the number

$$\int_C (\underline{h})(\vec{r})\cdot d\vec{r} = \int_a^b (\underline{h}(\vec{r}(u))\cdot \vec{r}\,'(u))\,du.$$

Theorem 1.2

Let $\underline{h}$ be a vector field that is continuous on a smooth curve C. The line integral $\int_C \underline{h}(\vec{r})\cdot d\vec{r} = \int_a^b (\underline{h}(\vec{r}(u))\cdot \vec{r}\,'(u))\,du$ is left invariant by every direction-preserving change of parameter.

<proof>

Suppose $\phi : [c,d] \to [a,b]$ is an onto map and ϕ' is positive and continuous on $[c,d]$

Let $\vec{R}(w) = \vec{r}(\phi(w))$, $w \in [c,d]$

Show that $\int_C \underline{h}(\vec{R})\cdot d\vec{R} = \int_C \underline{h}(\vec{r})\cdot d\vec{r}$

$$\int_C \underline{h}(\vec{R})\cdot d\vec{R} = \int_c^d (\underline{h}(\vec{R}(w))\cdot \vec{R}\,'(w))\,dw = \int_c^d \underline{h}(\vec{r}(\phi(w)))\cdot \vec{r}\,'(\phi(w))\,\phi'(w)\,dw$$

$$= \int_a^b (\underline{h}(\vec{r}(u))\cdot \vec{r}\,'(u))\,du = \int_C \underline{h}(\vec{r})\cdot d\vec{r}.$$

Examples

1. Calculate $\int_C \underline{h}(\vec{r})\cdot d\vec{r}$ given that $\underline{h}(x,y) = xy\,\vec{i} + y^2\vec{j}$ and

$C: \vec{r}(u) = u\vec{i} + u^2\vec{j}, \quad u\in[0,1]$.

$$\int_C \underline{h}(\vec{r})\cdot d\vec{r} = \int_0^1 \underline{h}(\vec{r}(u))\cdot\vec{r}\,'(u)\,du = \int_0^1 (u^3\vec{i} + u^4\vec{j})\cdot(\vec{i}+2u\vec{j})\,du$$

$$= \int_0^1 u^3 + 2u^5\,du = \frac{1}{4}u^4 + \frac{1}{3}u^6\Big|_0^1 = \frac{7}{12}.$$

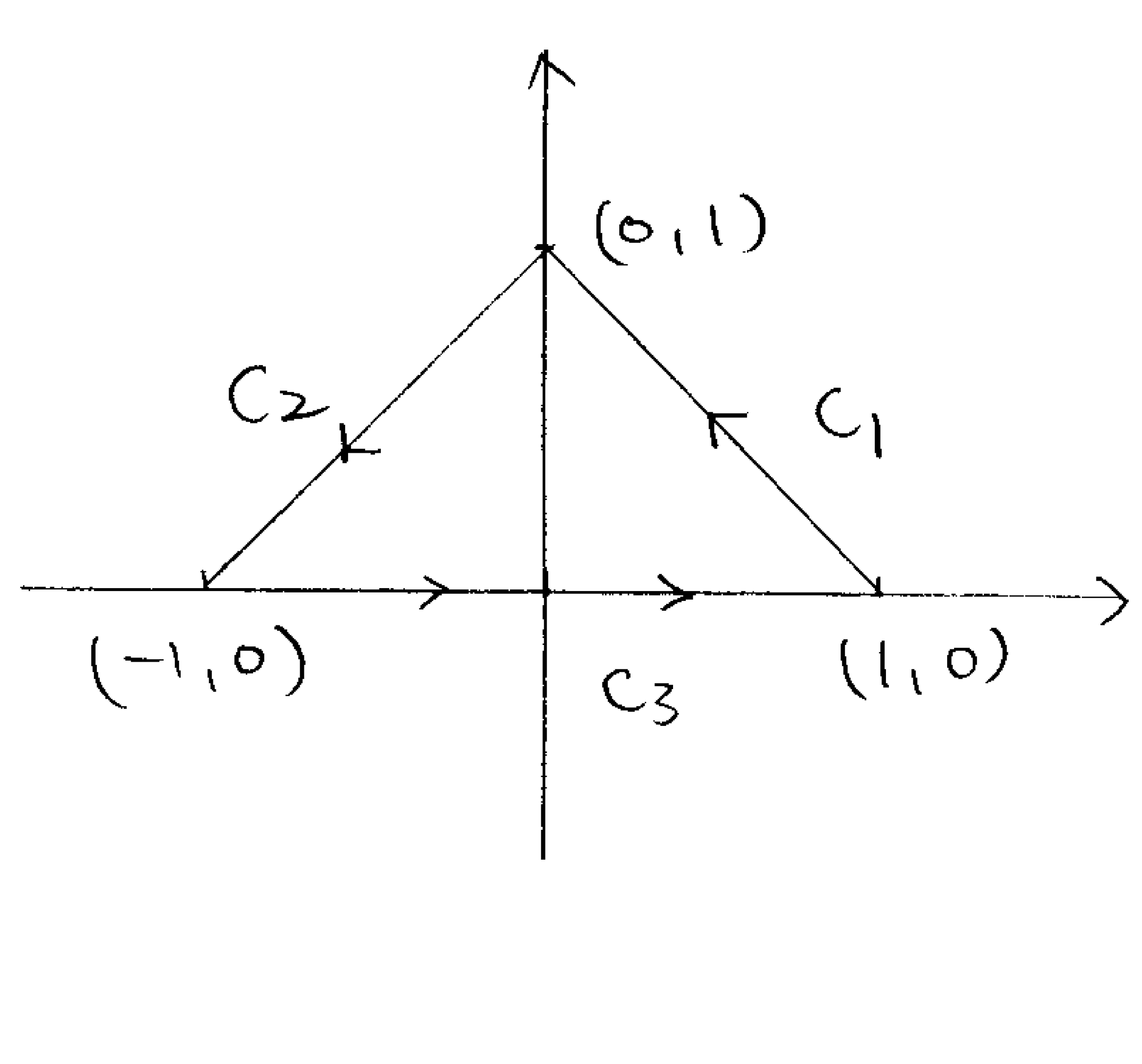
2. Integrate the vector field $\underline{h}(x,y,z) = xy\,\vec{i} + yz\,\vec{j} + xz\,\vec{k}$ over the twisted cubic $\vec{r}(u) = u\vec{i} + u^2\vec{j} + u^3\vec{k}$ from $(-1,1,-1)$ to $(1,1,1)$

$$\underline{h}(\vec{r}(u))\cdot\vec{r}\,'(u) = (u\cdot u^2\vec{i} + u^2\cdot u^3\vec{j} + u\cdot u^3\vec{k})\cdot(\vec{i}+2u\vec{j}+3u^2\vec{k})$$

$$= u^3 + 2u^6 + 3u^6 = u^3 + 5u^6.$$

$$\int_C \underline{h}(\vec{r})\cdot d\vec{r} = \int_{-1}^1 u^3 + 5u^6\,du = \frac{1}{4}u^4 + \frac{5}{7}u^7\Big|_{-1}^1 = \frac{10}{7}.$$

3. Evaluate $\int_C \underline{h}(\vec{r})\cdot d\vec{r}$ if $\underline{h}(x,y) = e^y\,\vec{i} - \sin\pi x\,\vec{j}$ and C is the triangle with vertices $(1,0)$, $(0,1)$, $(-1,0)$



$C_1: \vec{r}(u) = (1-u)\vec{i} + u\vec{j}, \quad u\in[0,1]$

$C_2: \vec{r}(u) = -u\vec{i} + (1-u)\vec{j}, \quad u\in[0,1]$

$C_3: \vec{r}(u) = (2u-1)\vec{i}, \quad u\in[0,1]$

$$\int_{C_1} \underline{h}(\vec{r})\cdot d\vec{r} = \int_0^1 (e^u\hat{i} - \sin(\pi(1-u))\hat{j})\cdot(-\hat{i}+\hat{j})\,du$$

$$= \int_0^1 -e^u - \sin(\pi(1-u))\,du = 1-e-\frac{2}{\pi}$$

$$\int_{C_2} \underline{h}(\vec{r})\cdot d\vec{r} = \int_0^1 (e^{1-u}\hat{i} - \sin(\pi(-u))\hat{j})(-\hat{i}-\hat{j})\,du$$

$$= \int_0^1 -e^{1-u} + \sin(-\pi u)\,du = 1-e-\frac{2}{\pi}.$$

$$\int_{C_3} \underline{h}(\vec{r})\cdot d\vec{r} = \int_0^1 (e^0\hat{i} - \sin(\pi(2u-1))\hat{j})\cdot(2\hat{i})\,du$$

$$= \int_0^1 2\,du = 2.$$

$$\int_C \underline{h}(\vec{r})\cdot d\vec{r} = 4-2e-\frac{4}{\pi}.$$

Theorem 1.3 (The fundamental Theorem for line integrals)

Let $C: \vec{r}=\vec{r}(u)$, $u\in[a,b]$, be a piecewise-smooth curve with $\underline{a}=\vec{r}(a)$ & $\underline{b}=\vec{r}(b)$. If f is continuously differentiable on an open set that contains the curve C, then

$$\int_C \nabla f(\vec{r})\cdot d\vec{r} = f(\underline{b}) - f(\underline{a}).$$

<proof>

If C is smooth

$$\Rightarrow \int_C \nabla f(\vec{r})\cdot d\vec{r} = \int_a^b (\nabla f(\vec{r}(u))\,\vec{r}'(u))\,du = \int_a^b \frac{d}{du}(f(\vec{r}(u)))\,du$$

$$= f(\vec{r}(b)) - f(\vec{r}(a)) = f(\underline{b}) - f(\underline{a})$$

If C is not smooth but only piecewise smooth,

let $C = C_1 \cup C_2 \cup \cdots \cup C_n$, where c_i are smooth curves.

$$\int_C \nabla f(\vec{r})\cdot d\vec{r} = \int_{C_1} \nabla f(\vec{r})\cdot d\vec{r} + \int_{C_2} \nabla f(\vec{r})\cdot d\vec{r} + \cdots + \int_{C_n} \nabla f(\vec{r})\cdot d\vec{r}$$

$$= (f(\underline{a}_1)-f(\underline{a}_0)) + (f(\underline{a}_2)-f(\underline{a}_1)) + \cdots + (f(\underline{a}_n)-f(\underline{a}_{n-1}))$$

$$= f(\underline{a}_n) - f(\underline{a}_0) = f(\underline{b}) - f(\underline{a}) \quad //$$

Corollary 1.4

If the curve C is closed. i.e. if $\underline{b} = \underline{a}$, then

$$\int_C \nabla f(\vec{r})\cdot d\vec{r} = 0.$$

Examples

1. Integrate $h(x,y) = y^2\,\hat{i} + (2xy - e^{2y})\,\hat{j}$ over $C: \vec{r}(u) = \cos u\,\hat{i} + \sin u\,\hat{j}$

$u \in [0, \frac{1}{2}\pi]$.

Step 1: Determine whether h is a gradient.

Let $P(x,y) = y^2$ & $Q(x,y) = 2xy - e^{2y}$

$\frac{\partial P}{\partial y} = 2y = \frac{\partial Q}{\partial x} \Rightarrow h$ is a gradient.

Step 2: Find f s.t. $\nabla f = h$.

$f(x,y) = \int P(x,y)\,dx = \int y^2\,dx = xy^2 + \phi(y)$

$\frac{\partial f}{\partial y} = Q(x,y) \Rightarrow 2xy + \phi'(y) = 2xy - e^{2y} \Rightarrow \phi'(y) = -e^{2y}$

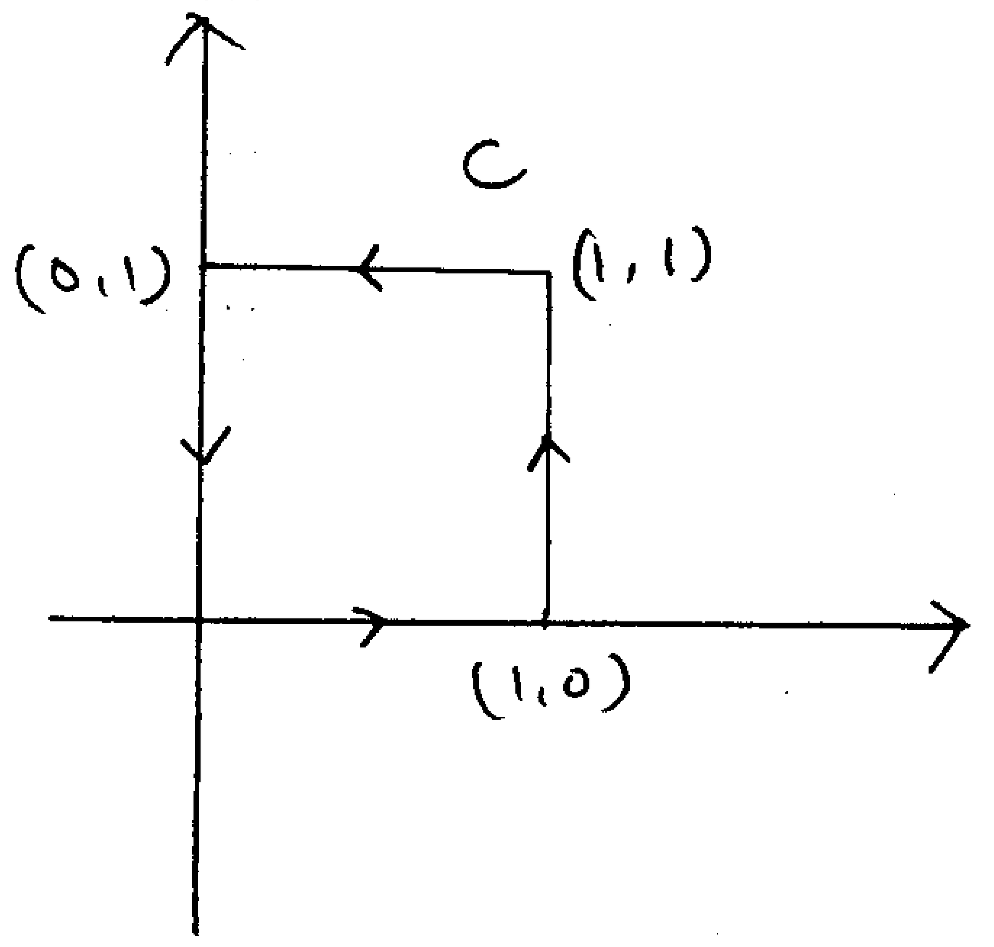
$\Rightarrow \phi(y) = -\frac{1}{2}e^{2y} + c$, c : constant

$\therefore f(x,y) = xy^2 - \frac{1}{2}e^{2y} + c$

step3 : $\int_C h(\vec{r})\cdot dr = \int_C \nabla f(\vec{r})\cdot d\vec{r} = f(b) - f(a)$

$= f(0,1) - f(1,0) = \frac{1}{2} - \frac{1}{2}e^2.$

2. $\int_C h(\vec{r})\cdot d\vec{r}$, $h(x,y) = (3x^2y + xy^2 - 1)\,\hat{i} + (x^3 + x^2y + 4y^3)\,\hat{j}$



Step 1 : Determine whether h is a gradient

$P(x,y) = 3x^2y + xy^2 - 1$, $Q(x,y) = x^3 + x^2y + 4y^3$

$\frac{\partial P}{\partial y} = 3x^2 + 2x = \frac{\partial Q}{\partial x} \Rightarrow h$ is a gradient.

Then we have f s.t. $\nabla f = h$ and C is a closed curve

$\Rightarrow \int_C h(\vec{r})\cdot d\vec{r} = \int_C \nabla f(\vec{r})\cdot d\vec{r} = 0$ (by Cor 1.4)

3. $\int_C h(\vec{r})\cdot d\vec{r}$, $h(x,y)=(y^2+y)\,i+(2xy-e^{2y})\,j$ and

$C: \vec{r}(u)=\cos u\, i+\sin u\, j$, $u\in[0,2\pi]$, the unit circle.

Step 1: Determine whether h is a gradient.

$P(x,y)=y^2+y$, $Q(x,y)=2xy-e^{2y}$

$\frac{\partial P}{\partial y}=2y+1$, $\frac{\partial Q}{\partial x}=2y$ $\Rightarrow$ $\frac{\partial P}{\partial y}\neq\frac{\partial Q}{\partial x}$

$\Rightarrow$ h is not a gradient

Step 2: $\int_C h(\vec{r})\cdot d\vec{r}=\int_0^{2\pi} h(\vec{r})\cdot\vec{r}'(u)\,du$

$=\int_0^{2\pi}\left((\sin^2u+\sin u)\,i+(2\cos u\sin u-e^{2\sin u})\,j\right)\cdot(-\sin u\, i+\cos u\, j)\,du$

$=\int_0^{2\pi} -\sin^3u-\sin^2u+2\cos^2u\sin u-\cos u\cdot e^{2\sin u}\,du$

$=\int_0^{2\pi} -\sin u+\sin u\cos^2u-\frac{1}{2}(1-\cos 2u)+2\cos^2u\sin u-\cos u\cdot e^{2\sin u}\,du$

$=\int_0^{2\pi} -\sin u-\frac{1}{2}+\frac{1}{2}\cos 2u+3\cos^2u\sin u-\cos u\cdot e^{2\sin u}\,du$

$=\cos u-\frac{1}{2}u+\frac{1}{4}\sin 2u-\cos^3u-\frac{1}{2}e^{2\sin u}\Big|_0^{2\pi}=-\pi.$

§2 Another notation for line integrals ; Line integrals with respect to arc length.

If $h(x,y,z) = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$

$C:\ \vec{r}(u) = x(u)\,\vec{i} + y(u)\,\vec{j} + z(u)\,\vec{k}\ ,\ u \in [a,b]$

$$\int_C h(\vec{r})\cdot d\vec{r} = \int_a^b h(\vec{r}(u))\cdot \vec{r}\,'(u)\,du$$

$$= \int_a^b \big(P(x(u),y(u),z(u))\,\vec{i} + Q(x(u),y(u),z(u))\,\vec{j} + R(x(u),y(u),z(u))\,\vec{k}\big)$$

$$\cdot\ (x'(u)\,\vec{i} + y'(u)\,\vec{j} + z'(u)\,\vec{k})\,du$$

$$= \int_a^b P(x(u),y(u),z(u))\cdot x'(u) + Q(x(u),y(u),z(u))\cdot y'(u) + R(x(u),y(u),z(u))\,z'(u)\,du$$

$$= \int_C P(x,y,z)\,dx + \int_C Q(x,y,z)\,dy + \int_C R(x,y,z)\,dz$$

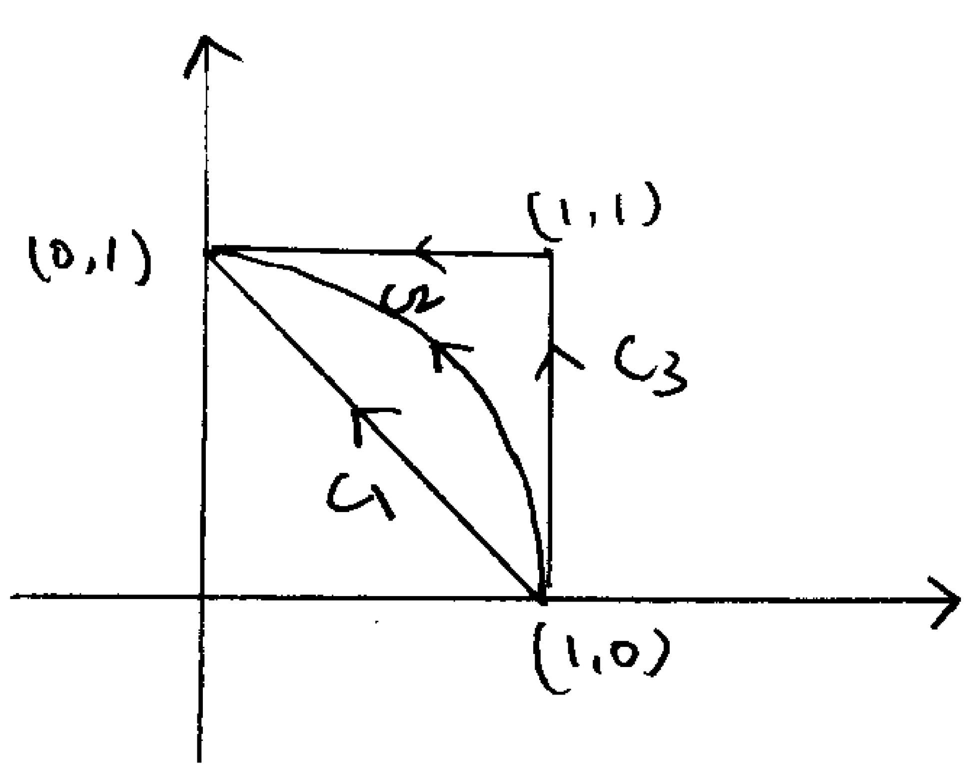
Example.

Evaluate $\int_C x^2y\,dx + xy\,dy$, where C is

(a) the straight-line path from $(1,0)$ to $(0,1)$

(b) the circular path $y = \sqrt{1-x^2}$ from $(1,0)$ to $(0,1)$

(c) the polygonal path from $(1,0)$, $(1,1)$, to $(0,1)$



(a) $C_1 : \vec{r}(u) = (1-u)\,\vec{i} + u\,\vec{j}\ , \quad 0 \le u \le 1$

$$\int_{C_1} x^2y\,dx + xy\,dy = \int_0^1 \left((1-u)^2 u(-1) + (1-u)\cdot u \cdot 1\right) du$$

$$= \int_0^1 u^2 - u^3\,du = \tfrac{1}{3}u^3 - \tfrac{1}{4}u^4 \Big|_0^1 = \tfrac{1}{12}$$

(b) $C_2 : \vec{r}(u) = \cos u\,\vec{i} + \sin u\,\vec{j}\ , \quad u \in [0, \tfrac{\pi}{2}]$

$$\int_{C_2} x^2y\,dx + xy\,dy = \int_0^{\frac{\pi}{2}} (\cos^2 u \cdot \sin u \cdot (-\sin u) + \cos u \sin u \cdot \cos u)\,du$$

$$= \int_0^{\frac{\pi}{2}} -\cos^2 u \sin^2 u + \cos^2 u \cdot \sin u\,du$$

$$= \int_0^{\frac{\pi}{2}} -\tfrac{1}{8}(1-\cos 4u) + \cos^2 u \sin u\,du$$

$$= -\tfrac{1}{8}u + \tfrac{1}{32}\sin 4u - \tfrac{1}{3}\cos^3 u \Big|_0^{\frac{\pi}{2}} = \tfrac{1}{3} - \tfrac{\pi}{16}$$

(c) $C_3 = C_4 \cup C_5\ , \quad C_4 : \vec{r}(u) = \vec{i} + u\,\vec{j}\ , \quad u \in [0,1]$

$C_5 : \vec{r}(u) = (1-u)\,\vec{i} + \vec{j}\ , \quad u \in [0,1]$

$$\int_{C_3} x^2y\,dx + xy\,dy = \int_{C_4} x^2y\,dx + xy\,dy + \int_{C_5} x^2y\,dx + xy\,dy$$

$$\int_{C_4} x^2y\,dx + xy\,dy = \int_0^1 u\cdot du = \tfrac{1}{2}$$

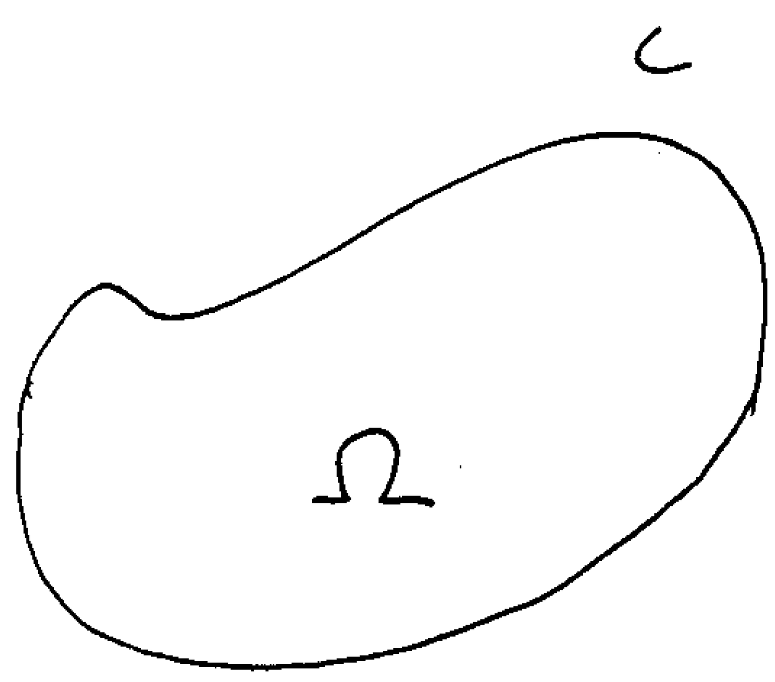
$$\int_{C_5} x^2y\,dx + xy\,dy = \int_0^1 (1-u)^2 \cdot 1 \cdot (-1)\,du = -\tfrac{1}{3}$$

$$\int_{C_3} x^2y\,dx + xy\,dy = \tfrac{1}{2} - \tfrac{1}{3} = \tfrac{1}{6}$$

§3 Green's Theorem

Definition 3.1

A Jordan curve is a plane curve which is both closed and simple.



C : Jordan curve

Ω : Jordan region.

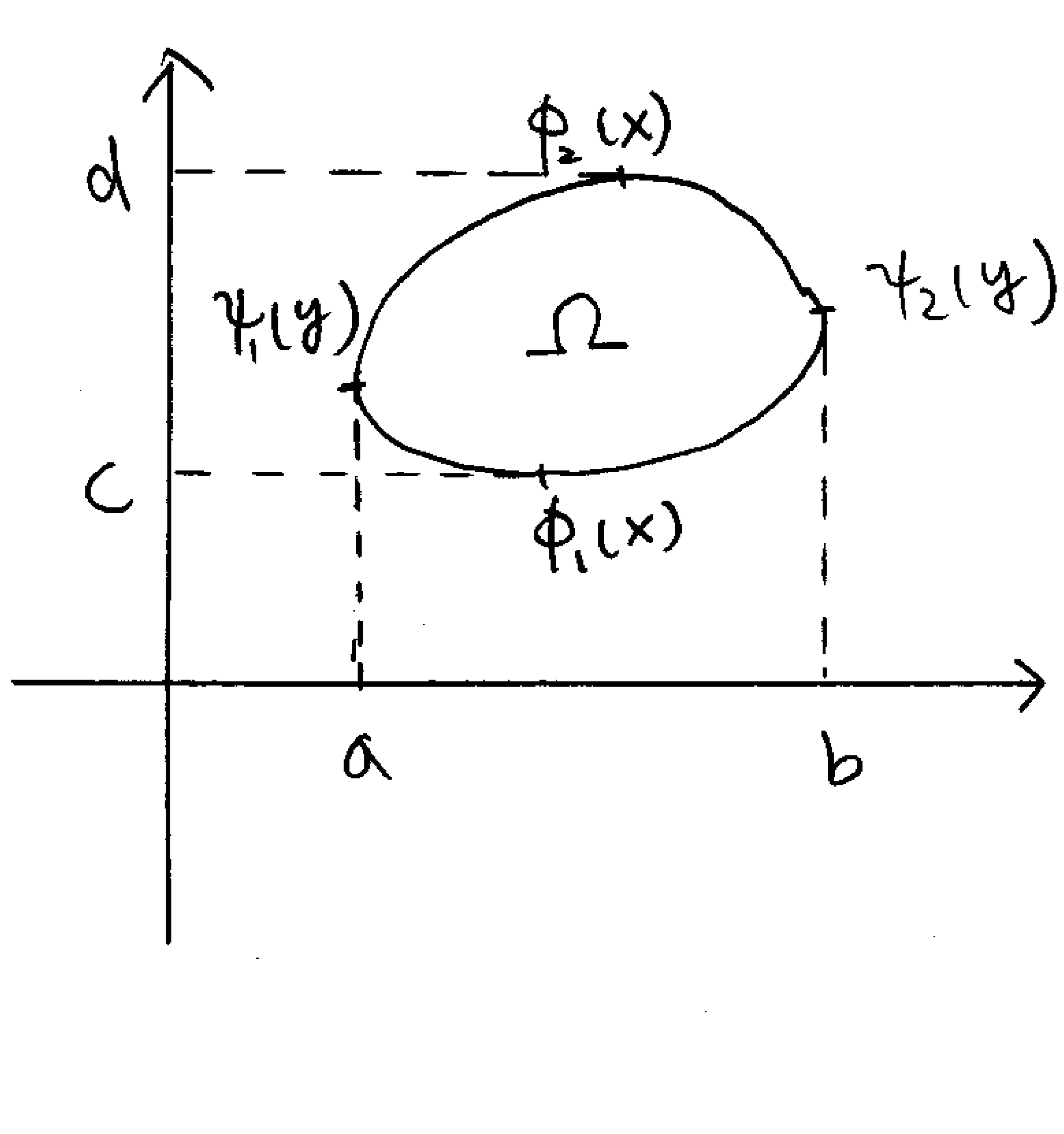
Theorem 3.2 Green's Theorem

Ω : Jordan region with a piecewise-smooth boundary C. If P & Q are scalar fields continuously differentiable on an open set that contains Ω, then

$$\iint_{\Omega} \left[\frac{\partial Q}{\partial x}(x,y) - \frac{\partial P}{\partial y}(x,y) \right] dx dy = \oint_C P(x,y)\,dx + Q(x,y)\,dy .$$

<proof>

We only prove the theorem in special case.



<u>Claim</u> $\oint_C P(x,y)\,dx = \iint_\Omega -\frac{\partial P}{\partial y}(x,y)\,dx\,dy$

$$\iint_\Omega -\frac{\partial P}{\partial y}(x,y)\,dx\,dy = -\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial P}{\partial y}\,dy\,dx$$

$$= -\int_a^b \left(P(x,\phi_2(x)) - P(x,\phi_1(x))\right) dx$$

$$= \int_a^b P(x,\phi_1(x))\,dx - \int_a^b P(x,\phi_2(x))\,dx$$

$C_1 : \vec{r}_1(u) = u\hat{i} + \phi_1(u)\hat{j}$, $C_2 : \vec{r}_2(u) = u\hat{i} + \phi_2(u)\hat{j}$, $u \in [a,b]$

$$\oint_C P(x,y)\,dx = \int_{C_1} P(x,y)\,dx + \int_{-C_2} P(x,y)\,dx$$

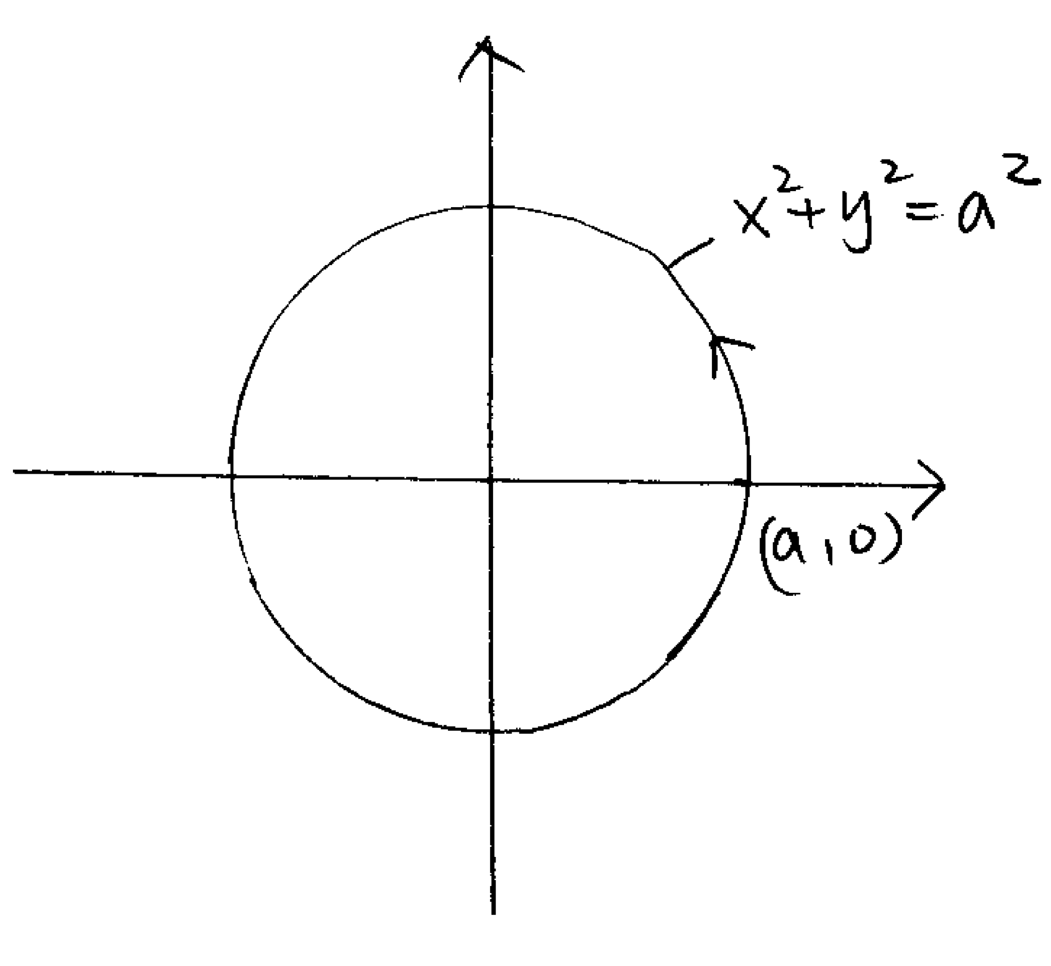
$$= \int_a^b P(u,\phi_1(u))\,du - \int_a^b P(u,\phi_2(u))\,du$$

$$\Rightarrow \oint_C P(x,y)\,dx = \iint_\Omega -\frac{\partial P}{\partial y}(x,y)\,dx\,dy$$

Similarly, $\oint_C Q(x,y)\,dy = \iint_\Omega \frac{\partial Q}{\partial x}(x,y)\,dx\,dy$ //.

Examples

1. Evaluate $\oint_C (3x^2+y)\,dx + (2x+y^3)\,dy$, $C : x^2+y^2=a^2$



Let Ω be the closed disc

$0 \le x^2+y^2 \le a^2$

$P(x,y) = 3x^2+y$, $Q(x,y) = 2x+y^3$

Use Green's Thm

$$\frac{\partial P}{\partial y} = 1 \quad , \quad \frac{\partial Q}{\partial x} = 2 \quad \text{and} \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

$$\oint_C (3x^2+y)\,dx + (2x+y^3)\,dy = \iint_\Omega 1\,dx\,dy = \text{area of } \Omega = \pi a^2.$$

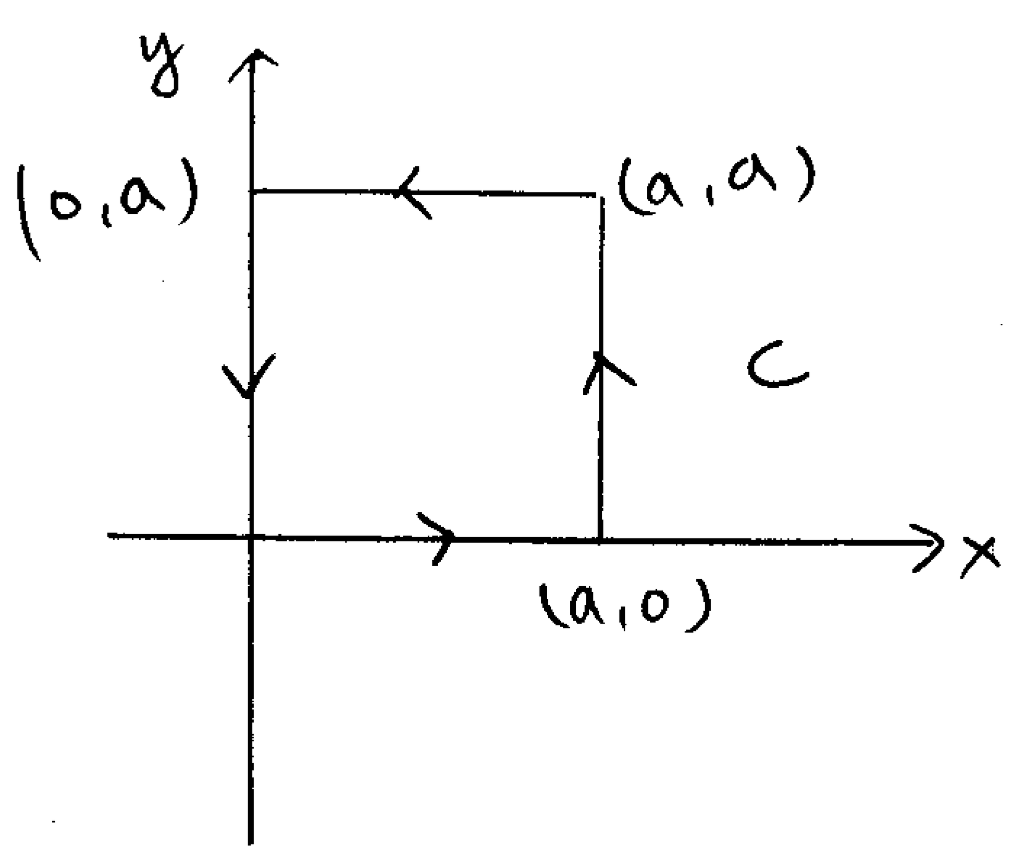
Remark: The circle $x^2+y^2=a^2$ can be parametrized counterclockwise by $x = a\cos u$, $y = a\sin u$, $0 \le u \le 2\pi$.

$$\oint_C (3x^2+y)\,dx + (2x+y^3)\,dy$$

$$= \int_0^{2\pi} \left[(3a^2\cos^2 u + a\sin u)(-a\sin u) + (2a\cos u + a^3\sin^3 u)\,a\cos u\right] du$$

$$= \int_0^{2\pi} -3a^3\cos^2 u\sin u - a^2\sin^2 u + 2a^2\cos^2 u + a^4\sin^3 u\cos u\,du$$

2. $\oint_C (1+10xy+y^2)\,dx + (6xy+5x^2)\,dy$



$P(x,y) = 1 + 10xy + y^2$

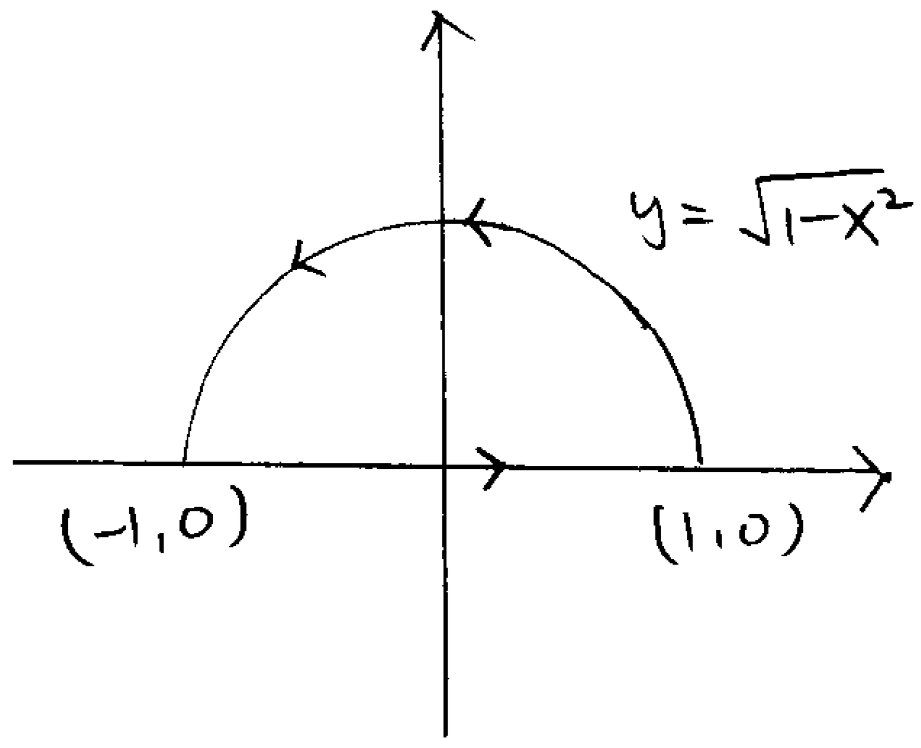
$Q(x,y) = 6xy + 5x^2$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 6y + 10x - 10x - 2y = 4y$$

$$\oint_C (1+10xy+y^2)\,dx + (6xy+5x^2)\,dy = \iint_\Omega 4y\,dx\,dy = \int_0^a\int_0^a 4y\,dx\,dy$$

$$= 2a^3.$$

3. $\oint_C e^x \sin y\,dx + e^x \cos y\,dy$



$P(x,y) = e^x \sin y \quad , \quad Q(x,y) = e^x \cos y$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = e^x \cos y - e^x \cos y = 0$$

$$\oint_C e^x \sin y\,dx + e^x \cos y\,dy = \iint_\Omega 0\,dxdy = 0.$$

Theorem 3.3

Ω: Jordan region with boundary C.

The area of Ω is given by each of the following integrals

$$\oint_C -y\,dx, \quad \oint_C x\,dy, \quad \frac{1}{2}\oint_C -y\,dx + x\,dy.$$

<proof>

1. $P(x,y) = -y, \quad Q(x,y) = 0$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 - (-1) = 1$$

$$\oint_C -y\,dx = \iint_\Omega 1 \cdot dxdy = \text{area of } \Omega.$$

2. $P(x,y) = 0 \quad , \quad Q(x,y) = x$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - 0 = 1$$

$$\oint_C x\,dy = \iint_\Omega 1 \cdot dxdy = \text{area of } \Omega$$

3. $P(x,y) = -y, \quad Q(x,y) = x$

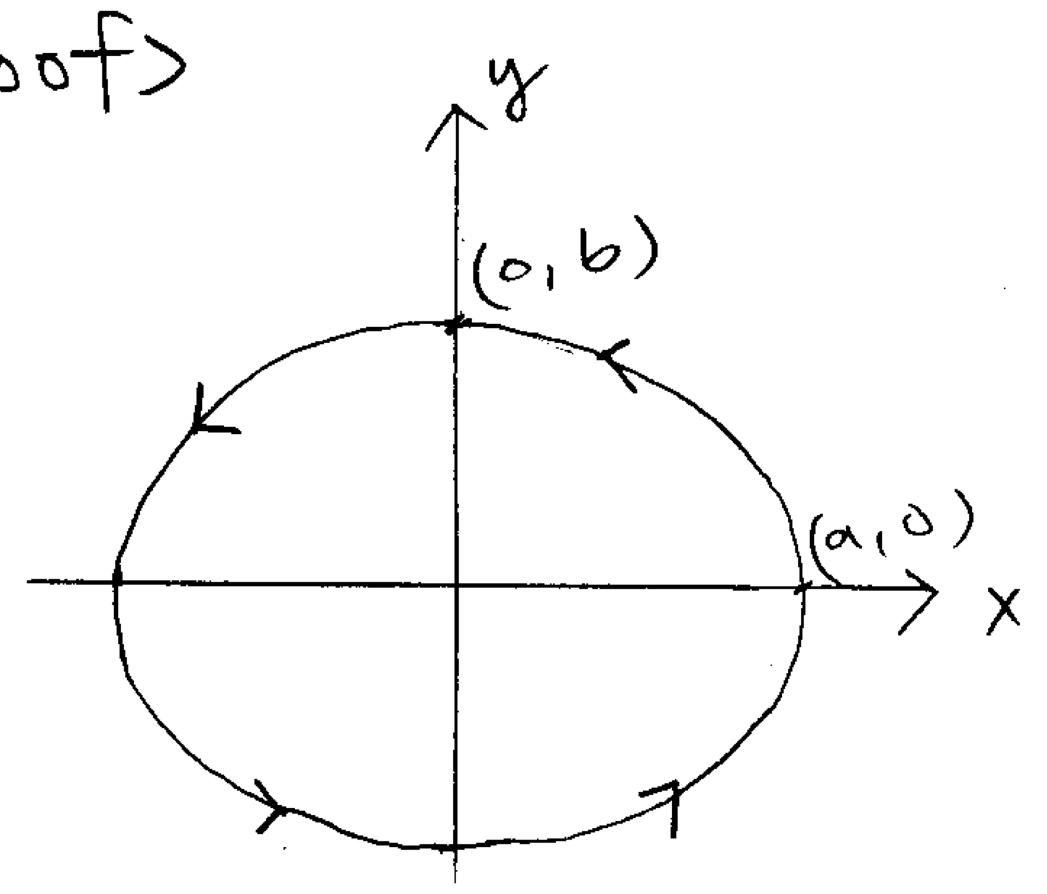
$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - (-1) = 2$$

$$\oint_C -y\,dx + x\,dy = \iint_\Omega 2\,dxdy = 2 \cdot \text{area of } \Omega$$

Example

Show that the area of Ω enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab.

<proof>



Can parametrize the ellipse counterclockwise by

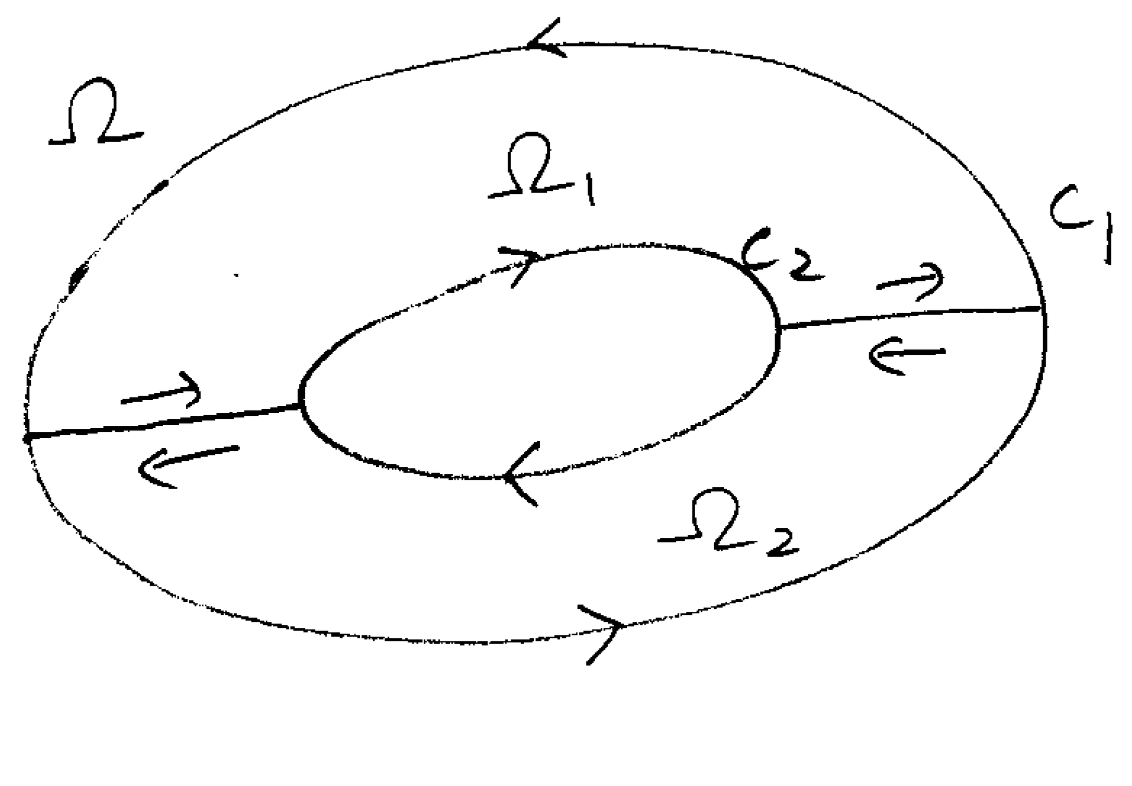
$x=a\cos u$, $y=b\sin u$, $u\in[0,2\pi]$

$$\text{area of } \Omega = \frac{1}{2}\oint_C -y\,dx+x\,dy$$

$$=\frac{1}{2}\int_0^{2\pi}(-b\sin u\cdot(-a\sin u)+a\cos u\cdot b\cos u)\,du$$

$$=\frac{1}{2}\int_0^{2\pi} ab\sin^2 u+ab\cos^2 u\,du=\frac{1}{2}\int_0^{2\pi} ab\,du=ab\pi.$$

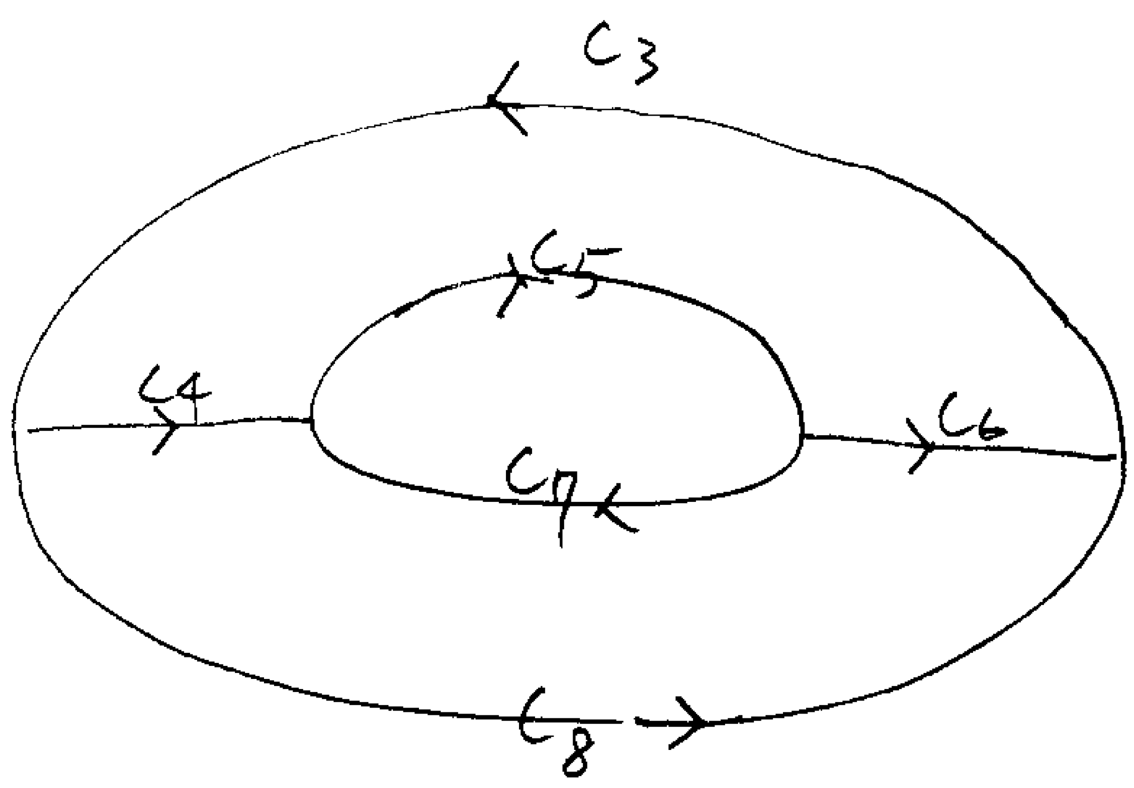
Regions bounded by 2 or more Jordan curves.



By Green's Thm

$$\iint_{\Omega_1}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy=\oint_{\text{bdy of }\Omega_1} P\,dx+Q\,dy$$

$$\iint_{\Omega_2}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy=\oint_{\text{bdy of }\Omega_2} P\,dx+Q\,dy$$



$$\oint_{\text{bdy of } \Omega_1} P\,dx + Q\,dy$$

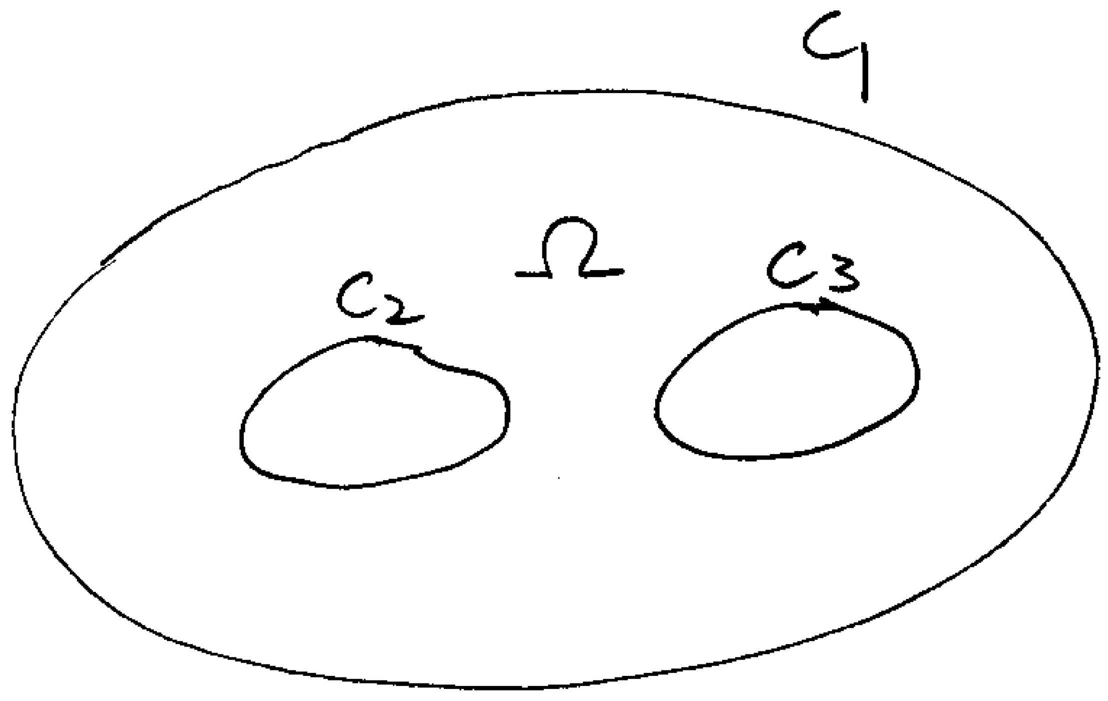
$$= \left(\oint_{c_3} + \oint_{c_4} + \oint_{c_5} + \oint_{c_6}\right) P\,dx + Q\,dy$$

$$\oint_{\text{bdy of } \Omega_2} P\,dx + Q\,dy$$

$$= \left(\oint_{-c_6} + \oint_{c_7} + \oint_{-c_4} + \oint_{c_8}\right) P\,dx + Q\,dy$$

$$\iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \oint_{c_1} P\,dx + Q\,dy + \oint_{c_2} P\,dx + Q\,dy$$

Moreover,



$$\iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \oint_{c_1} P\,dx + Q\,dy + \oint_{c_2} P\,dx + Q\,dy + \oint_{c_3} P\,dx + Q\,dy$$

Example

Let c_1 be a Jordan curve that does not pass through the origin $(0,0)$. Show that

$$\oint_{c_1} -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = \begin{cases} 0, & \text{if } c_1 \text{ doesn't enclose the origin} \\ 2\pi, & \text{if } c_1 \text{ does enclose the origin} \end{cases}$$

<proof>

Let $P(x,y) = -\dfrac{y}{x^2+y^2}$, $Q(x,y) = \dfrac{x}{x^2+y^2}$

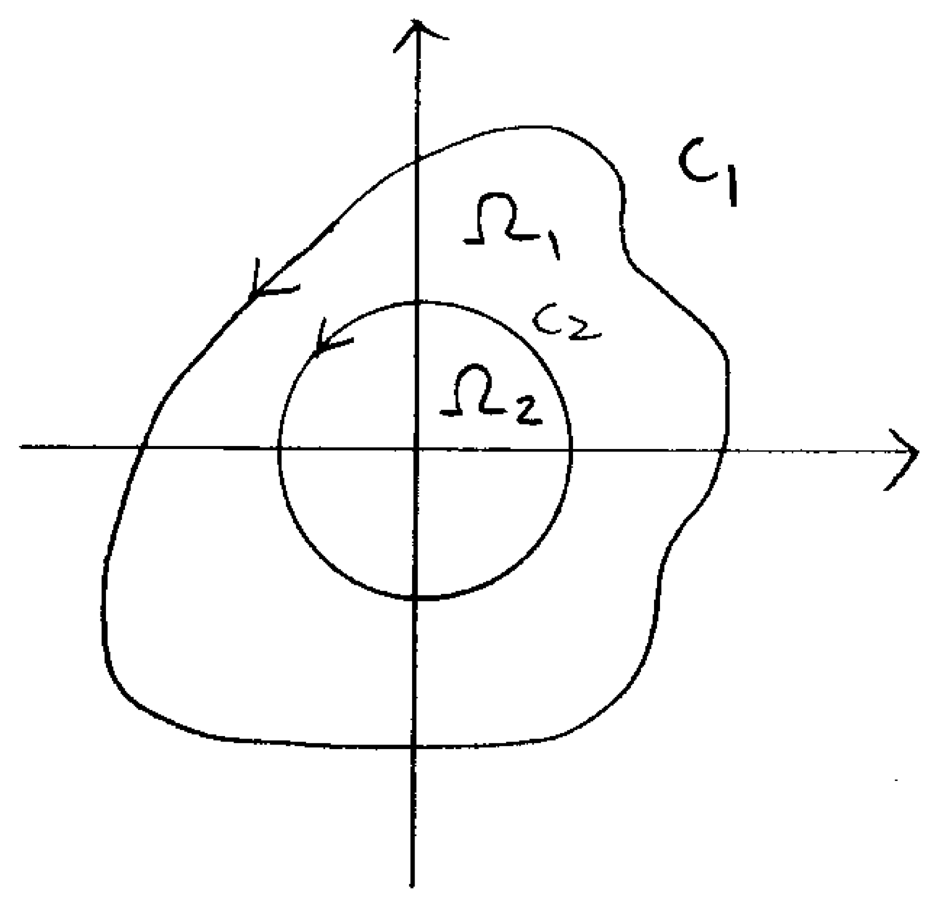
$$\frac{\partial Q}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}, \quad \frac{\partial P}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

Thus $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ except the origin.

If C_1 doesn't enclosed the origin.

$$\Rightarrow \oint_{C_1} -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = \iint_{\Omega} 0\,dx\,dy = 0$$

If C_1 does enclose the origin.



$\Omega = \Omega_1 \cup \Omega_2$

Note that

$$\iint_{\Omega_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint_{C_1} P\,dx + Q\,dy - \oint_{C_2} P\,dx + Q\,dy$$

$$\Rightarrow \oint_{C_1} P\,dx + Q\,dy = \oint_{C_2} P\,dx + Q\,dy$$

Parametrize the circle by $x = a\cos u$, $y = a\sin u$, $u \in [0, 2\pi]$

$$\oint_{C_2} P\,dx + Q\,dy = \int_0^{2\pi} (\sin^2 u + \cos^2 u)\,du = \int_0^{2\pi} du = 2\pi. \quad //$$

Exercise

C: piecewise-smooth Jordan curve that doesn't pass through the origin. Evaluate $\oint_C -\frac{y^3}{(x^2+y^2)^2}\,dx + \frac{xy^2}{(x^2+y^2)^2}\,dy$

(a) If C doesn't enclose the origin. 0

(b) If C does enclose the origin. π.