Sequences ; Indeterminate Forms ; Improper Integrals.

§1 The least upper bound axiom

S : nonempty set of real numbers

If there exists M such that $x \le M$ for all $x \in S$, say M is a <u>upper bound</u> of S

Axiom 1.1

S : nonempty set of real numbers and has an upper bound

Then S has a least upper bound.

Theorem 1.2

If M is the least upper bound of the set S and ε is a positive number, then there is at least one number $r \in S$, such that $M-\varepsilon < r \le M$

<proof>

Since M is an upper bound of $S \Rightarrow r \le M \ , \forall r \in S$

Suppose $\nexists\, r \in S$ s.t. $M-\varepsilon < r \le M$

$\Rightarrow r \le M-\varepsilon \ , \forall r \in S$

$\Rightarrow M-\varepsilon$ is also an upper bound of S $\rightarrow\!\leftarrow$ //.

Example

$S = \{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots \}$, let $\varepsilon = 0.0001$.

Note 1 is the least upper bound of S

Find $r \in S$ s.t. $1-0.0001 < r \le 1$

Take $r = \dfrac{99999}{100000}$

Theorem 1.3

Every nonempty set of real numbers that has a lower bound has a greatest lower bound.

<proof>

S : nonempty set with a lower bound k.

$\Rightarrow k \leq x, \forall x \in S$

$\Rightarrow -x \leq -k, \forall x \in S$

Let $S' = \{-x \mid x \in S\}$, then S' has an upper bound $-k$.

$\Rightarrow S'$ has a least upper bound, say m.

$\Rightarrow -x \leq m, \forall x \in S \Rightarrow -m \leq x, \forall x \in S$

Then $-m$ is a lower bound of S.

If $\exists m_1$ s.t. $-m < m_1 \leq x, \forall x \in S \Rightarrow -x \leq m_1 < m, \forall x \in S$ $\times$

Hence $-m$ is the greatest lower bound. //

Theorem 1.4

If m is the greast lower bound of the set S and $\varepsilon > 0$, then there is at least one number $x \in S$ s.t. $m \leq x < m + \varepsilon$.

<proof> exercise.

§2 Sequences

$\{a_n\}_{n=1}^{\infty}$ be a sequence

Definition 2.1

$\{a_n\}$ is said to be

increasing if $a_n < a_{n+1}$, $\forall n$

nondecreasing if $a_n \leq a_{n+1}$, $\forall n$

decreasing if $a_n > a_{n+1}$, $\forall n$

nonincreasing if $a_n \geq a_{n+1}$, $\forall n$.

Examples

1. $\{a_n = \frac{n}{n+1}\}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{n+2}}{\frac{n}{n+1}} = \frac{n+1}{n+2} \cdot \frac{n+1}{n} = \frac{n^2+2n+1}{n^2+2n} > 1$$

$\Rightarrow a_{n+1} > a_n \quad \Rightarrow \{a_n\}$ is increasing.

$\frac{1}{2}$ is the greastest lower bound & 1 is the least upper bound

2. $\{c^n\}$, $c > 1$

$$\frac{c^{n+1}}{c^n} = c > 1 \quad \Rightarrow c^{n+1} > c^n$$

$\Rightarrow \{c^n\}$ is increasing

Let M be arbitrary positive number.

let K s.t. $\frac{\ln M}{\ln c} \leq K \quad \Rightarrow \ln M \leq K \cdot \ln c \Rightarrow M \leq c^k$

$\{c^n\}$ has no upper bound.

3. $\{n^{\frac{1}{n}}\}$

consider $f(x) = x^{\frac{1}{x}} = e^{\frac{1}{x}\ln x}$

$\Rightarrow f'(x) = x^{\frac{1}{x}}\left(\frac{1-\ln x}{x^2}\right) < 0$, for $x > e$

$\Rightarrow f$ decreases on $[e, \infty)$

Note $3 > e$ $\Rightarrow$ $\{n^{\frac{1}{n}}\}$ is decreasing on $[3, \infty)$

Definition 2.2 Limit of a sequence

$$\lim_{n\to\infty} a_n = L$$

if $\forall \varepsilon > 0$, $\exists K$, positive integers s.t. $|a_n - L| < \varepsilon$, $\forall n \geq K$

Example

1. $a_n = \dfrac{4n-1}{n}$ $\quad\left(\lim_{n\to\infty} \dfrac{4n-1}{n} = 4\right)$

For each $\varepsilon > 0$, take K large enough s.t $\frac{1}{K} < \varepsilon$

$$\left|\frac{4n-1}{n} - 4\right| = \left|4 - \frac{1}{n} - 4\right| = \frac{1}{n} \leq \frac{1}{K} < \varepsilon \quad, \forall n \geq K$$

2. $a_n = \dfrac{2\sqrt{n}}{\sqrt{n}+1}$ $\quad\left(\lim_{n\to\infty} a_n = 2\right)$

$\forall \varepsilon > 0$, take K large enough s.t. $\frac{2}{\sqrt{K}} < \varepsilon$

$$\left|\frac{2\sqrt{n}}{\sqrt{n}+1} - 2\right| = \left|\frac{-2}{\sqrt{n}+1}\right| = \frac{2}{\sqrt{n}+1} < \frac{2}{\sqrt{n}} \leq \frac{2}{\sqrt{K}} < \varepsilon \quad, \forall n \geq K$$

Theorem 2.3 (Uniqueness of Limit)

If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = M$, then $L = M$

<proof> exercise

Definition 2.4

A sequence that has a limit is said to be convergent.

A sequence that has no limit is said to be divergent.

Theorem 2.5

Every convergent sequence is bounded.

<proof>

Suppose $a_n \to L$ as $n \to \infty$.

Let $\varepsilon = 1$, $\exists K$ large enough s.t. $|a_n - L| < \varepsilon$, $\forall n \geq K$

$\Rightarrow |a_n| - |L| \leq \big||a_n| - |L|\big| \leq |a_n - L| < \varepsilon = 1$ for all $n \geq K$

$\Rightarrow |a_n| < 1 + |L|$, $\forall n \geq K$

Take $M = \max\{|a_1|, |a_2|, \ldots, |a_K|, 1+|L|\} \Rightarrow |a_n| \leq M$, $\forall n$ //

Remark

Every unbounded sequence is divergent.

Theorem 2.6

A nondecreasing sequence which is bounded above converges to the least upper bound. A nonincreasing sequence which is bounded below converges to the greastest lower bound.

<proof>

Let $\{a_n\}$ be the sequence which is bounded above and nondecreasing.

let L be the least upper bound, then $a_n \leq L$, $\forall n$.

$\forall \varepsilon > 0$, $\exists K$ s.t. $L - \varepsilon < a_K$

Since $\{a_n\}$ is nondecreasing $\Rightarrow a_K \leq a_n$, $\forall n \geq K$

$\Rightarrow L - \varepsilon < a_K < a_n < L$, $\forall n \geq K$ $\Rightarrow |a_n - L| < \varepsilon$, $\forall n \geq K$

$\Rightarrow a_n \to L$. //

Theorem 2.7

α be a real number. If $a_n \to L$ and $b_n \to M$, then

(i) $a_n + b_n \longrightarrow L+M$,

(ii) $\alpha a_n \longrightarrow \alpha L$

(iii) $a_n b_n \longrightarrow L \cdot M$

Moreover, $M \neq 0$ & $b_n \neq 0, \forall n$, then

(iv) $\frac{1}{b_n} \to \frac{1}{M}$

(v) $\frac{a_n}{b_n} \to \frac{L}{M}$.

<proof>

(i) $\forall \varepsilon > 0$, $\exists K_1$ s.t. $|a_n - L| < \frac{\varepsilon}{2}$ for $n \geq K_1$, $\exists K_2$ s.t. $|b_n - M| < \frac{\varepsilon}{2}$ for $n \geq K_2$

let $K = \max\{K_1, K_2\}$

$|a_n + b_n - (L+M)| \leq |a_n - L| + |b_n - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for all $n \geq K$.

(ii) $\forall \varepsilon > 0$, $\exists K$ s.t. $|a_n - L| < \frac{\varepsilon}{|\alpha|}$ for all $n \geq K$

$|\alpha a_n - \alpha L| = |\alpha||a_n - L| < |\alpha| \cdot \frac{\varepsilon}{|\alpha|} = \varepsilon$ for all $n \geq K$.

(iii) – (v) exercises. //

Examples.

1. $$\frac{3n^4 - 2n^2 + 1}{n^5 + 3n^3} = \frac{\frac{3}{n} - \frac{2}{n^3} + \frac{1}{n^5}}{1 + \frac{3}{n^2}} \longrightarrow 0 \text{ as } n \to \infty$$

2. $$\frac{n^4 - 3n^2 + n + 2}{n^3 + 7n} = \frac{n - \frac{3}{n} + \frac{1}{n^2} + \frac{2}{n^3}}{1 + \frac{7}{n^2}} \longrightarrow \infty \text{ as } n \to \infty$$ diverges.

Remark

$$a_n \longrightarrow L \iff a_n - L \longrightarrow 0 \iff |a_n - L| \longrightarrow 0.$$

Theorem 2.8 The Pinching Theorem for sequences

Suppose $a_n \le b_n \le c_n$ for n large enough

If $a_n \to L$ and $c_n \to L$, then $b_n \to L$

Example

1. $\dfrac{\cos n}{n}$

$\because -1 \le \cos n \le 1 \Rightarrow \underset{\downarrow \atop 0}{-\dfrac{1}{n}} \le \dfrac{\cos n}{n} \le \underset{\downarrow \atop 0}{\dfrac{1}{n}} \Rightarrow \dfrac{\cos n}{n} \to 0$ as $n \to \infty$

2. $\sqrt{4+\left(\frac{1}{n}\right)^2}$

$\underset{\downarrow \atop 2}{2} \le \sqrt{4+\left(\frac{1}{n}\right)^2} \le \sqrt{4+4\cdot\frac{1}{n}+\left(\frac{1}{n}\right)^2} = \sqrt{\left(2+\frac{1}{n}\right)^2} = \underset{\downarrow \atop 2}{2+\dfrac{1}{n}}$

$\Rightarrow \sqrt{4+\left(\frac{1}{n}\right)^2} \to 2$ as $n \to \infty$.

3. $\left(1+\frac{1}{n}\right)^n$

observe that $\left(1+\frac{1}{n}\right)^n \le e \le \left(1+\frac{1}{n}\right)^{n+1}$, $\forall n$

$\Rightarrow \underset{\downarrow \atop e}{\dfrac{e}{1+\frac{1}{n}}} \le \left(1+\frac{1}{n}\right)^n \le \underset{\downarrow \atop e}{e} \Rightarrow \left(1+\frac{1}{n}\right)^n \to e$ as $n \to \infty$.

◎ Some Important Limits

1. If $x>0$, then $x^{\frac{1}{n}} \to 1$ as $n \to \infty$.

<pf>
For any $x>0$, $\ln x^{\frac{1}{n}} = \frac{1}{n}\ln x \to 0$ as $n \to \infty$

$\Rightarrow x^{\frac{1}{n}} = e^{\ln x^{\frac{1}{n}}} \to e^0 = 1$ as $n \to \infty$.

2. If $|x|<1$, then $x^n \to 0$ as $n \to \infty$.

<pf>

Let $\varepsilon > 0$, $\exists K$ large s.t. $|x|^K < \varepsilon$ ($\because |x|<1$)

Note that $|x|^{n+1} = |x| \cdot |x|^n < |x|^n$

$\Rightarrow x^n$ is decreasing & $|x|^n < \varepsilon$, $\forall n \geq K$

$\Rightarrow x^n \to 0$ as $n \to \infty$.

3. For each $\alpha > 0$, $\frac{1}{n^\alpha} \to 0$ as $n \to \infty$.

<pf>

For $\alpha > 0$, $\exists$ odd positive integer P s.t. $\frac{1}{P} < \alpha$

$$\Rightarrow \underset{\substack{\downarrow \\ 0}}{0} < \frac{1}{n^\alpha} = \left(\frac{1}{n}\right)^\alpha \leq \underset{\substack{\downarrow \\ 0}}{\left(\frac{1}{n}\right)^{\frac{1}{P}}} \qquad (f(x) = x^{\frac{1}{P}} \text{ is continuous at } 0)$$

$\Rightarrow \frac{1}{n^\alpha} \to 0$ as $n \to \infty$.

4. For each real x, $\frac{x^n}{n!} \to 0$ as $n \to \infty$

<pf> choose an integer K s.t. $K > |x|$.

For $n > K+1$,

$$\frac{K^n}{n!} = \left(\frac{K^K}{K!}\right)\left(\frac{K}{K+1} \cdot \frac{K}{K+2} \cdots \frac{K}{n-1}\right)\left(\frac{K}{n}\right) < \frac{K^{K+1}}{K!} \cdot \frac{1}{n}$$

$\because K > |x|$

$$\therefore \underset{\substack{\downarrow \\ 0}}{0} < \frac{|x|^n}{n!} < \frac{K^n}{n!} < \underset{\substack{\downarrow \\ 0}}{\frac{K^{K+1}}{K!} \cdot \frac{1}{n}} \quad \Rightarrow \frac{|x|^n}{n!} \to 0 \text{ as } n \to \infty$$

$\Rightarrow \frac{x^n}{n!} \to 0$ as $n \to \infty$.

5. $\frac{\ln n}{n} \to 0$ as $n \to \infty$.

<pf>

$$0 \le \frac{\ln n}{n} = \frac{1}{n}\int_1^n \frac{1}{t}\,dt \le \frac{1}{n}\int_1^n \frac{1}{\sqrt{t}}\,dt = \frac{2}{n}\cdot(\sqrt{n}-1) = 2\left(\frac{1}{\sqrt{n}}-\frac{1}{n}\right) \to 0 \text{ as } n \to \infty$$

Hence $\frac{\ln n}{n} \to 0$ as $n \to \infty$.

6. $n^{\frac{1}{n}} \to 1$ as $n \to \infty$.

<pf>

$\because \ln n^{\frac{1}{n}} = \frac{\ln n}{n} \to 0$ as $n \to \infty$

$\Rightarrow n^{\frac{1}{n}} = e^{\ln n^{\frac{1}{n}}} = e^{\frac{1}{n}\cdot \ln n} \to e^0 = 1$ as $n \to \infty$.

7. For each real x, $\left(1+\frac{x}{n}\right)^n \to e^x$ as $n \to \infty$.

<pf>

For $x=0$, it is obviously.

For $x \neq 0$, $\ln\left(1+\frac{x}{n}\right)^n = n\ln\left(1+\frac{x}{n}\right) = x\cdot\left(\frac{\ln\left(1+\frac{x}{n}\right)-\ln 1}{\frac{x}{n}}\right)$

Let $h = \frac{x}{n}$ $\Rightarrow h \to 0$

$$\Rightarrow \lim_{n\to\infty} \ln\left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty} x\cdot\left(\frac{\ln\left(1+\frac{x}{n}\right)-\ln 1}{\frac{x}{n}}\right) = \lim_{h\to 0} x\cdot\left(\frac{\ln(1+h)-\ln 1}{h}\right) = x$$

$\Rightarrow \left(1+\frac{x}{n}\right)^n = e^{\ln\left(1+\frac{x}{n}\right)^n} \to e^x$ as $n \to \infty$.

L'Hospital's Rule ($\frac{\infty}{\infty}$)

Suppose that $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ and in approach $g'(x) \neq 0$

If $\dfrac{f'(x)}{g'(x)} \to \gamma$, then $\dfrac{f(x)}{g(x)} \to \gamma$.

Examples

1. α be any positive number. Show that $\dfrac{\ln x}{x^\alpha} \to 0$ as $x \to \infty$.

<pf>

$$\lim_{x\to\infty} \frac{\ln x}{x^\alpha} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\alpha \cdot x^{\alpha-1}} = \lim_{x\to\infty} \frac{1}{\alpha \cdot x^\alpha} = 0$$

2. α be any positive number. $\dfrac{e^x}{x^\alpha} \to \infty$ as $x \to \infty$

<pf>

Choose integer $k > \alpha$, then $\dfrac{e^x}{x^k} < \dfrac{e^x}{x^\alpha}$

$$\lim_{x\to\infty} \frac{e^x}{x^k} = \lim_{x\to\infty} \frac{e^x}{k \cdot x^{k-1}} = \lim_{x\to\infty} \frac{e^x}{k(k-1)x^{k-2}} = \cdots = \lim_{x\to\infty} \frac{e^x}{k!}$$

$\Rightarrow \dfrac{e^x}{k!} \to \infty \quad \Rightarrow \dfrac{e^x}{x^k} \to \infty$ as $x \to \infty$.

Hence $\dfrac{e^x}{x^\alpha} \to \infty$ as $x \to \infty$.

Other indeterminate form

$(0 \cdot \infty)$

Find $\lim\limits_{x\to 0^+} \sqrt{x} \cdot \ln x$.

<pf>

$$\lim_{x\to 0^+} \sqrt{x}\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\left(-\frac{1}{2}\cdot x^{-\frac{3}{2}}\right)} = \lim_{x\to 0^+} -2\sqrt{x} = 0$$

$(\infty-\infty)$

Find $\lim_{x\to\frac{\pi}{2}^-}(\tan x-\sec x)$.

<pf>

$$\tan x-\sec x=\frac{\sin x}{\cos x}-\frac{1}{\cos x}=\frac{\sin x-1}{\cos x}$$

$$\lim_{x\to\frac{\pi}{2}^-}\tan x-\sec x=\lim_{x\to\frac{\pi}{2}^-}\frac{\sin x-1}{\cos x}=\lim_{x\to\frac{\pi}{2}^-}\frac{\cos x}{-\sin x}=\frac{0}{-1}=0.$$

(0^0)

show that $\lim_{x\to0^+}x^x=1$

<pf>

$$\lim_{x\to0^+}\ln x^x=\lim_{x\to0^+}x\ln x=\lim_{x\to0^+}\frac{\ln x}{1/x}=\lim_{x\to0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=\lim_{x\to0^+}-x=0$$

$$\Rightarrow\lim_{x\to0^+}e^{\ln x^x}=e^0=1$$

(1^∞)

Find $\lim_{x\to0^+}(1+x)^{\frac{1}{x}}$

<pf>

$$\lim_{x\to0^+}\ln(1+x)^{\frac{1}{x}}=\lim_{x\to0^+}\frac{1}{x}\ln(1+x)=\lim_{x\to0^+}\frac{\ln(1+x)}{x}=\lim_{x\to0^+}\frac{1}{1+x}=1$$

$$\Rightarrow\lim_{x\to0^+}(1+x)^{\frac{1}{x}}=\lim_{x\to0^+}e^{\ln(1+x)^{\frac{1}{x}}}=e^1=e$$

(∞^0)

Show that if $1<a<b$, then $\lim_{x\to\infty}(a^x+b^x)^{\frac{1}{x}}=b$

<pf>

$$\lim_{x\to\infty}\ln(a^x+b^x)^{\frac{1}{x}}=\lim_{x\to\infty}\frac{1}{x}\cdot\ln(a^x+b^x)=\lim_{x\to\infty}\frac{\ln(a^x+b^x)}{x}$$

$$=\lim_{x\to\infty}\frac{a^x\ln a+b^x\ln b}{a^x+b^x}=\lim_{x\to\infty}\frac{(\frac{a}{b})^x\ln a+\ln b}{(\frac{a}{b})^x+1}=\ln b$$

$$\lim_{x\to\infty}(a^x+b^x)^{\frac{1}{x}}=\lim_{x\to\infty}e^{\ln(a^x+b^x)^{\frac{1}{x}}}=\lim_{x\to\infty}e^{\ln b}=b.$$

§4 Improper Integrals

$f(x)$ continuous on $[a, \infty)$

If $\lim_{b\to\infty} \int_a^b f(x)\,dx = L$, then we write $\int_a^\infty f(x)\,dx = L$.

Say that the improper integral $\int_a^\infty f(x)\,dx$ converges to L.

In the similar manner,

$\int_{-\infty}^b f(x)\,dx$ arise as limits of the form $\lim_{a\to -\infty} \int_a^b f(x)\,dx$.

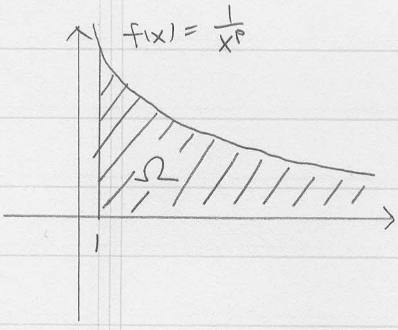
Examples

1. $\int_0^\infty e^{-2x}\,dx = \lim_{b\to\infty} \int_0^b e^{-2x}\,dx = \lim_{b\to\infty} \left. -\frac{1}{2}e^{-2x}\right|_0^b = \lim_{b\to\infty} \frac{1}{2} - \frac{1}{2e^b} = \frac{1}{2}$

$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln x\Big|_1^b = \lim_{b\to\infty} \ln b = \infty$.

2. Fix $P>0$, Ω be the region below the graph of $f(x) = \frac{1}{x^P}$

Show that area of $\Omega = \begin{cases} \frac{1}{P-1} & \text{, if } P>1 \\ \infty & \text{, if } P\le 1 \end{cases}$



area of $\Omega = \int_1^\infty f(x)\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^P}\,dx$

For $P\neq 1$

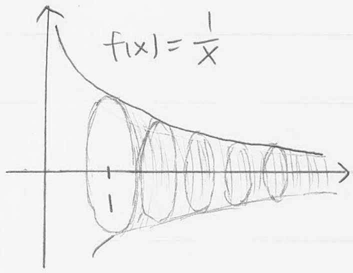
$\lim_{b\to\infty} \int_1^b \frac{1}{x^P}\,dx = \lim_{b\to\infty} \frac{1}{1-P}\left(b^{1-P}-1\right) = \begin{cases} \frac{1}{1-P} & \text{, if } P>1 \\ \infty & \text{, if } P<1 \end{cases}$

For $P=1$

$\lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln b = \infty$

$\boxed{\int_1^\infty \frac{1}{x^P}\,dx \text{ converges if } P>1 \text{ and diverges if } 0<P\le 1}$

3. Let $f(x) = \frac{1}{x}$ with $x \geq 1$



What is the volume of the resulting configuration?

$$V = \int_1^\infty \pi (f(x))^2 dx = \int_1^\infty \frac{\pi}{x^2} dx$$

$$= \lim_{b\to\infty} \pi \int_1^b \frac{1}{x^2} dx = \lim_{b\to\infty} \pi \cdot \left(-\frac{1}{x}\right)\Big|_1^b = \lim_{b\to\infty} -\frac{1}{b}\pi + \pi$$

$$= \pi.$$

A comparison Test

Suppose that f & g are continuous and $0 \leq f(x) \leq g(x)$, $\forall x \in [a, \infty)$

(i) If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.

(ii) If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges.

Examples

4. $\int_1^\infty \frac{1}{\sqrt{1+x^3}} dx$

Note that $\frac{1}{\sqrt{1+x^3}} < \frac{1}{x^{\frac{3}{2}}}$ for $x \in [1, \infty)$ and $\int_1^\infty \frac{1}{x^{\frac{3}{2}}} dx$ converges

$\Rightarrow \int_1^\infty \frac{1}{\sqrt{1+x^3}} dx$ converges.

5. $\int_1^\infty \frac{1}{\sqrt{1+x^2}} dx$

$$\frac{1}{\sqrt{1+x^2}} > \frac{1}{\sqrt{1+2x+x^2}} = \frac{1}{1+x} \quad \text{for } x \in [1, \infty)$$

and $\int_1^\infty \frac{1}{1+x} dx$ diverges.

$\Rightarrow \int_1^\infty \frac{1}{\sqrt{1+x^2}} dx$ diverges

Suppose f is continuous on $(-\infty, \infty)$. The improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ is said to be converge if $\int_{-\infty}^{0} f(x)\,dx$ & $\int_{0}^{\infty} f(x)\,dx$ are both converge.

Let $\int_{-\infty}^{\infty} f(x)\,dx = L + M$ where $\int_{-\infty}^{0} f(x)\,dx = L$ & $\int_{0}^{\infty} f(x)\,dx = M$.

Example

Let $r > 0$, Determine whether the improper integral $\int_{-\infty}^{\infty} \frac{r}{r^2+x^2}\,dx$ converges or diverges.

<pf>

Consider $\int_{-\infty}^{0} \frac{r}{r^2+x^2}\,dx = \lim\limits_{a\to-\infty} \int_{a}^{0} \frac{r}{r^2+x^2}\,dx = \lim\limits_{a\to-\infty} \int_{a}^{0} \frac{\frac{1}{r}}{1+(\frac{x}{r})^2}\,dx$

$$= \lim_{a\to-\infty} \arctan\frac{x}{r}\Big|_{a}^{0} = \lim_{a\to\infty} -\arctan\frac{x}{a} = -(-\frac{\pi}{2}) = \frac{\pi}{2}$$

$$\int_{0}^{\infty} \frac{r}{r^2+x^2}\,dx = \lim_{b\to\infty} \int_{0}^{b} \frac{r}{r^2+x^2}\,dx = \lim_{b\to\infty} \arctan\frac{x}{r}\Big|_{0}^{b}$$

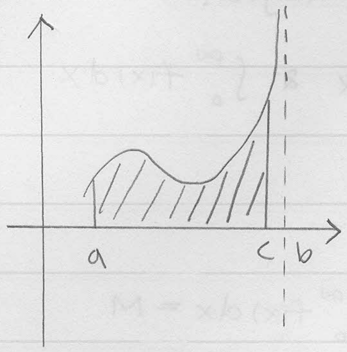
$$= \lim_{b\to\infty} \arctan\frac{b}{r} = \frac{\pi}{2}.$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{r}{r^2+x^2}\,dx = -\frac{\pi}{2} + \frac{\pi}{2} = 0 \quad \text{converges.}$$

Remark

(1) $\int_{-\infty}^{\infty} f(x)\,dx \neq \lim\limits_{b\to\infty} \int_{-b}^{b} f(x)\,dx$

suppose f is continuous on $[a, b)$ and unbounded there



If $\lim_{c \to b^-} \int_a^c f(x)\,dx = L$, then we write $\int_a^b f(x)\,dx = L$ and say that the improper integral $\int_a^b f(x)\,dx$ converges to L.

Similarly, suppose f is continuous on $(a, b]$

If $\lim_{c \to a^+} \int_c^b f(x)\,dx = L$, then we write $\int_a^b f(x)\,dx = L$ and say that the improper integral $\int_a^b f(x)\,dx$ converges to L.

Examples

1. $\int_0^1 (1-x)^{-\frac{2}{3}}\,dx = \lim_{c \to 1^-} \int_0^c (1-x)^{-\frac{2}{3}}\,dx = \lim_{c \to 1^-} -3(1-x)^{\frac{1}{3}}\Big|_0^c$

$$= \lim_{c \to 1^-} \left(-3(1-c)^{\frac{1}{3}} + 3\right) = 3$$

2. $\int_0^2 \frac{1}{x}\,dx = \lim_{c \to 0^+} \int_c^2 \frac{1}{x}\,dx = \lim_{c \to 0^+} \ln x\Big|_c^2 = \lim_{c \to 0^+} \ln 2 - \ln c = \infty$

Suppose f is continuous on $[a, b]$ except at some $c \in (a, b)$

We say $\int_a^b f(x)\,dx$ converges if $\int_a^c f(x)\,dx$ & $\int_c^b f(x)\,dx$ both converge.

If $\int_a^c f(x)\,dx = L$ & $\int_c^b f(x)\,dx = M$, $\int_a^b f(x)\,dx = L + M$.

Examples

1. $\int_1^4 \frac{1}{(x-2)^2}\,dx$

consider $\int_1^2 \frac{1}{(x-2)^2}dx$ & $\int_2^4 \frac{1}{(x-2)^2}dx$

$$\lim_{c\to 2^-}\int_1^c \frac{1}{(x-2)^2}\,dx = \lim_{c\to 2^-} \left.\frac{-1}{x-2}\right|_1^c = \lim_{c\to 2^-} \frac{1}{c-2} + 1 = \infty.$$

Hence $\int_1^4 \frac{1}{(x-2)^2}dx$ diverges.

* If we overlook the infinite discontinuity at $x=2$, we can be led to the incorrect conclusion that

$$\int_1^4 \frac{1}{(x-2)^2}\,dx = -\left.\frac{1}{x-2}\right|_1^4 = -\frac{3}{2}$$

2. $\int_{-2}^1 \frac{1}{x^{4/5}}\,dx$

$$\int_{-2}^0 \frac{1}{x^{4/5}}dx = \lim_{c\to 0^-}\int_{-2}^c \frac{1}{x^{4/5}}dx = \lim_{c\to 0^-} \left. 5x^{\frac{1}{5}}\right|_{-2}^c = \lim_{c\to 0^-} 5c^{\frac{1}{5}} - 5(-2) = 5\cdot 2^{\frac{1}{5}}$$

$$\int_0^1 \frac{1}{x^{\frac{4}{5}}}dx = \lim_{c\to 0^+}\int_c^1 \frac{1}{x^{\frac{4}{5}}}dx = \lim_{c\to 0^+} \left. 5x^{\frac{1}{5}}\right|_c^1 = \lim_{c\to 0^+} 5 - c^{\frac{1}{5}} = 5$$

$$\Rightarrow \int_{-2}^1 \frac{1}{x^{\frac{4}{5}}}dx = 5\cdot 2^{\frac{1}{5}} + 5 \quad \text{converges.}$$