Vectors in 3 dimensional space

- Distance formula

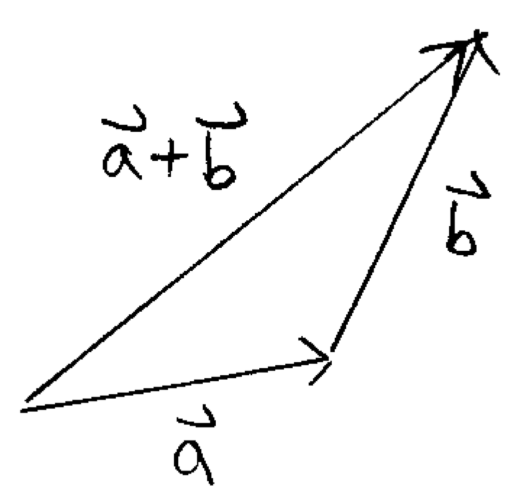
  $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2)$ be two points

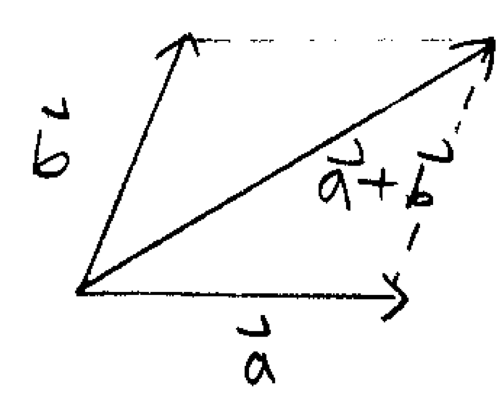
  $$d(P_1, P_2) = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$$

- Vectors

  $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$ be two vectors.

  $$\vec{a}+\vec{b} \triangleq (a_1+b_1,\ a_2+b_2,\ a_3+b_3)$$





For α real, $\alpha\vec{a} = (\alpha a_1, \alpha a_2, \alpha a_3)$

The norm of $\vec{a}$ (length)

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Properties

(1) $\|\vec{a}\| \geq 0$ and $\|\vec{a}\| = 0 \Leftrightarrow \vec{a} = 0$.

(2) $\|\alpha\vec{a}\| = |\alpha|\,\|\vec{a}\|$

(3) $\|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$

§1 Dot & Cross Product

let $\vec{i}=(1,0,0)$, $\vec{j}=(0,1,0)$, $\vec{k}=(0,0,1)$

$\vec{a}=(a_1,a_2,a_3)=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$

$\vec{b}=(b_1,b_2,b_3)=b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$

Definition 1.1

The dot product $\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+a_3b_3$

Properties

1. $\vec{a}\cdot\vec{a}=\|\vec{a}\|^2$

$\vec{a}\cdot\vec{a}=a_1a_1+a_2\cdot a_2+a_3\cdot a_3=a_1^2+a_2^2+a_3^2$

2. $\vec{a}\cdot\vec{0}=0=\vec{0}\cdot\vec{a}$

$a_1\cdot 0+a_2\cdot 0+a_3\cdot 0=0=0\cdot a_1+0\cdot a_2+0\cdot a_3$

3. $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$

$\vec{a}\cdot\vec{b}=a_1\cdot b_1+a_2\cdot b_2+a_3\cdot b_3=b_1\cdot a_1+b_2\cdot a_2+b_3\cdot a_3=\vec{b}\cdot\vec{a}$

4. $\alpha\vec{a}\cdot\beta\vec{b}=\alpha\beta(\vec{a}\cdot\vec{b})$

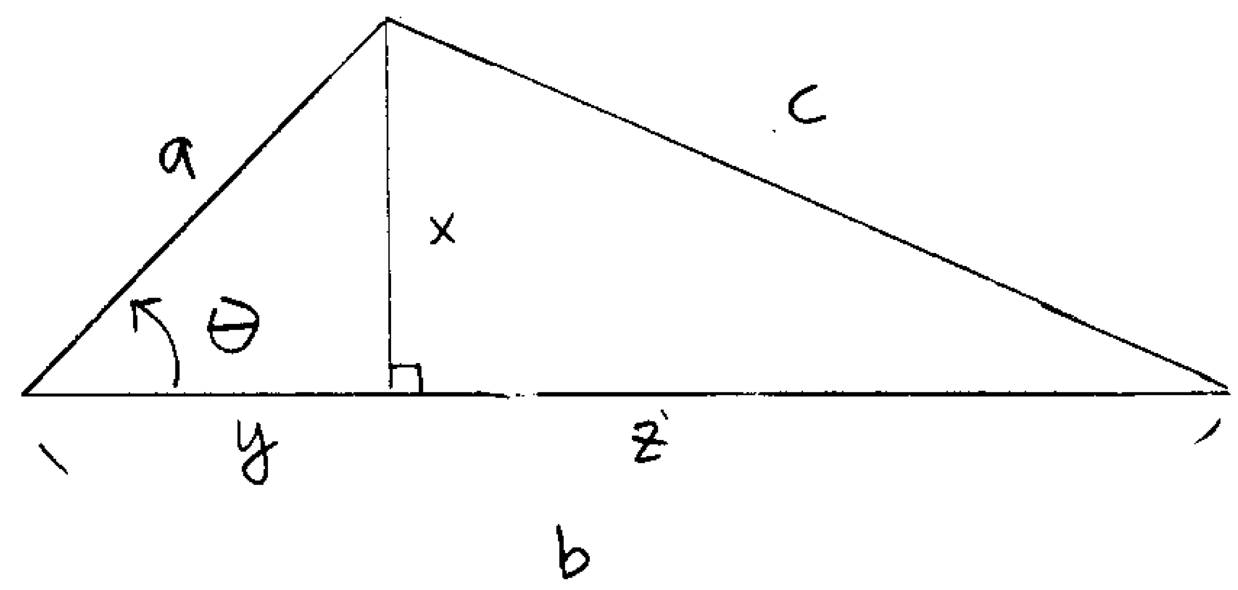
$$\alpha\vec{a}\cdot\beta\vec{b}=(\alpha a_1,\alpha a_2,\alpha a_3)\cdot(\beta b_1,\beta b_2,\beta b_3)$$
$$=\alpha a_1\beta b_1+\alpha a_2\beta b_2+\alpha a_3\beta b_3=\alpha\beta a_1b_1+\alpha\beta a_2b_2+\alpha\beta a_3b_3$$
$$=\alpha\beta(a_1b_1+a_2b_2+a_3b_3)=\alpha\beta(\vec{a}\cdot\vec{b})$$

5. $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$

$(\vec{a}+\vec{b})\cdot\vec{c}=\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}$.

exercise.

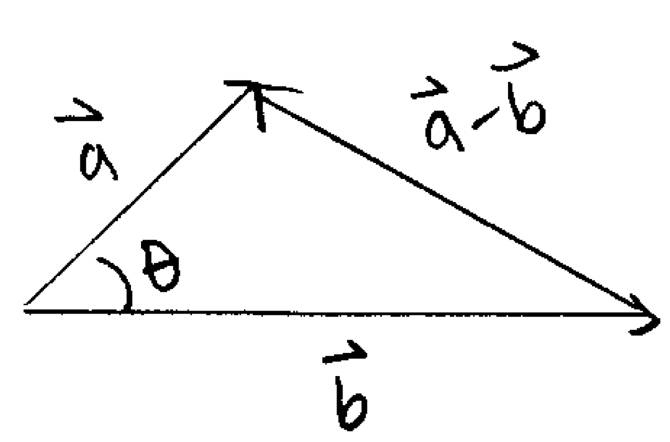
$$\boxed{\vec{a}\cdot\vec{b} = \|\vec{a}\|\cdot\|\vec{b}\|\cdot\cos\theta}$$



$$c^2 = z^2 + x^2 = (b-y)^2 + x^2 = b^2 - 2by + y^2 + x^2$$

Note $\quad x^2+y^2=a^2 \quad$ and $\quad y = a\cos\theta$

$$\Rightarrow \quad c^2 = b^2 - 2ba\cos\theta + a^2 = a^2+b^2-2ab\cos\theta$$



$$\Rightarrow \quad \|\vec{a}-\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta$$

$$\Rightarrow \quad 2\|\vec{a}\|\cdot\|\vec{b}\|\cos\theta = \|\vec{a}\|^2 + \|\vec{b}\|^2 - \|\vec{a}-\vec{b}\|^2$$

$$= \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} - (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$$

$$= \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} - (\vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b})$$

$$= \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} = 2\vec{a}\cdot\vec{b}$$

$$\Rightarrow \quad \vec{a}\cdot\vec{b} = \|\vec{a}\|\cdot\|\vec{b}\|\cos\theta$$

Remark $\quad \vec{a}\cdot\vec{b} = 0 \iff \vec{a}\perp\vec{b}$

exercise.

Example

$\vec{a} = 2i + 3j + 2k \quad , \quad \vec{b} = i + 2j - k$

$\vec{a} \cdot \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \cos\theta$

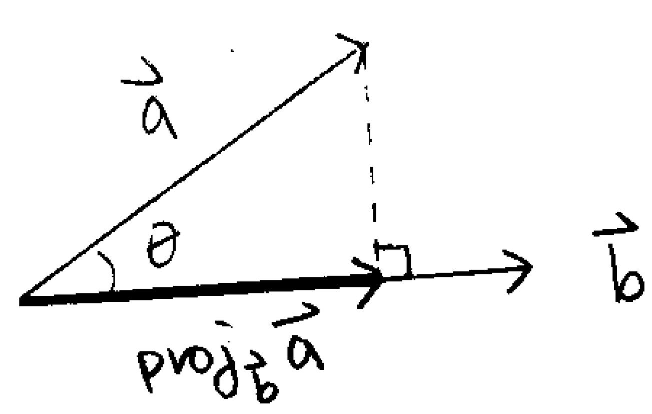
$$\Rightarrow \cos\theta = \frac{2+6-2}{\sqrt{2^2+3^2+2^2} \cdot \sqrt{1^2+2^2+(-1)^2}} = \frac{6}{\sqrt{102}}$$

Theorem 1.2

$\text{proj}_{\vec{b}} \vec{a}$ be the projection of $\vec{a}$ on $\vec{b}$

$\text{proj}_{\vec{b}} \vec{a} = (\vec{a} \cdot \vec{u}_b) \vec{u}_b$, where $\vec{u}_b$ is the unit vector of $\vec{b}$

<proof>



$$\text{proj}_{\vec{b}} \vec{a} = \|\vec{a}\| \cdot \cos\theta \cdot \vec{u}_b = \cancel{\|\vec{a}\|} \cdot \frac{\vec{a} \cdot \vec{b}}{\cancel{\|\vec{a}\|} \|\vec{b}\|} = (\vec{a} \cdot \vec{u}_b) \vec{u}_b \quad //$$

Theorem 1.3 (Schwarz's inequality)

$|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \cdot \|\vec{b}\|$.

<proof>

$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cdot \cos\theta$

$\Rightarrow |\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \cdot \|\vec{b}\|$

Corollary 1.4

$\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$.

<proof>

$$\|\vec{a}+\vec{b}\|^2 = (\vec{a}+\vec{b})(\vec{a}+\vec{b}) = \vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b} = \|\vec{a}\|^2 + 2\vec{a}\cdot\vec{b} + \|\vec{b}\|^2$$

$$\leq \|\vec{a}\|^2 + \|\vec{b}\|^2 + 2|\vec{a}\cdot\vec{b}| \leq \|\vec{a}\|^2 + \|\vec{b}\|^2 + 2\|\vec{a}\| \|\vec{b}\| = (\|\vec{a}\| + \|\vec{b}\|)^2$$

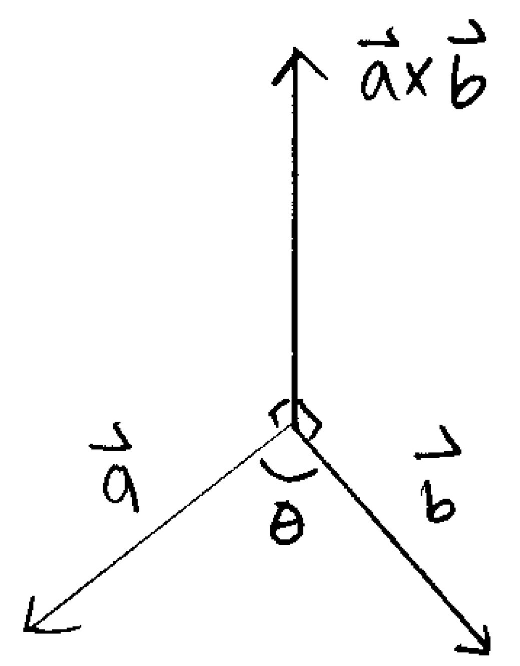
$\Rightarrow \|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$.

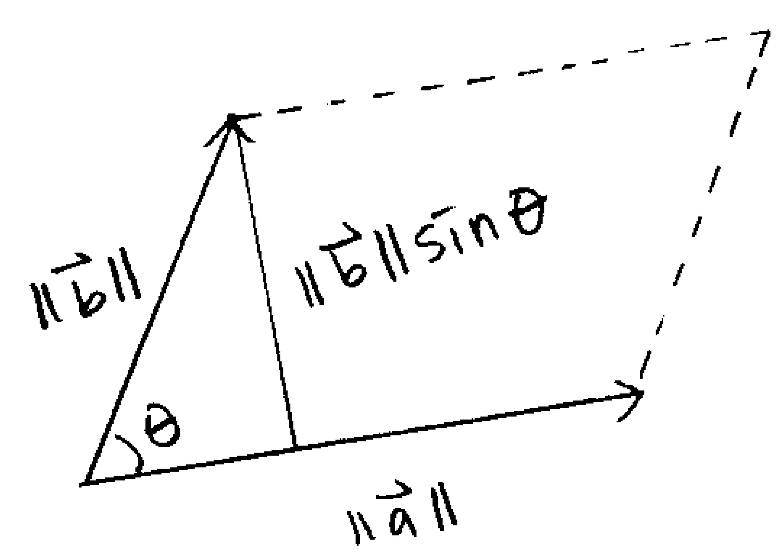
Definition 1.5

If $\vec{a}, \vec{b}$ are not parallel, then $\vec{a}\times\vec{b}$ is the vector with the following properties

1. $\vec{a}\times\vec{b} \perp$ the plane of $\vec{a}$ and $\vec{b}$
2. $\vec{a}, \vec{b}, \vec{a}\times\vec{b}$ form a right-handed triple.
3. $\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\|\vec{b}\| \sin\theta$

If $\vec{a}$ and $\vec{b}$ are parallel, then $\vec{a}\times\vec{b} = 0$.





Properties

1. $\vec{b}\times\vec{a} = -(\vec{a}\times\vec{b})$
2. $\alpha\vec{a}\times\beta\vec{b} = \alpha\beta(\vec{a}\times\vec{b})$
3. $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$, $(\vec{a}+\vec{b})\times\vec{c} = (\vec{a}\times\vec{c}) + (\vec{b}\times\vec{c})$
4. $V = |(\vec{a}\times\vec{b})\cdot\vec{c}|$, the volume of the parallelepiped.
5. $(\vec{a}\times\vec{b})\cdot\vec{c} = (\vec{c}\times\vec{a})\cdot\vec{b} = (\vec{b}\times\vec{c})\cdot\vec{a}$.

Theorem 1.6

$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

$$\vec{a}\times\vec{b} = (a_2b_3 - a_3b_2)\vec{i} - (a_1b_3 - a_3b_1)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}$$

Corollary 1.7

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

§2 Lines and Planes

Theorem 2.1

The line pass through (x_0, y_0, z_0) with direction vector $\vec{d} = d_1 i + d_2 j + d_3 k$ can be parametrized by

1. $r(t) = (x_0 + t d_1) i + (y_0 + t d_2) j + (z_0 + t d_3) k$

2. $x(t) = x_0 + t d_1$, $y(t) = y_0 + t d_2$, $z(t) = z_0 + t d_3$

3. $\dfrac{x - x_0}{d_1} = \dfrac{y - y_0}{d_2} = \dfrac{z - z_0}{d_3}$.

Example

1. $P = (-1, 4, 2)$, $\vec{d} = i + 2j + 3k$

   1. $r(t) = (-1 + t) i + (4 + 2t) j + (2 + 3t) k$

   2. $x(t) = -1 + t$, $y(t) = 4 + 2t$, $z(t) = 2 + 3t$.

   3. $\dfrac{x+1}{1} = \dfrac{y-4}{2} = \dfrac{z-2}{3}$.

2. Find the point at which the lines

   $l_1 : r(t) = (i - 6j + 2k) + t(i + 2j + k)$, $l_2 : R(u) = (4j + k) + u(2i + j + 2k)$

$(i - 6j + 2k) + t(i + 2j + k) = (4j + k) + u(2i + j + 2k)$

$\Rightarrow (1+t) i + (-6 + 2t) j + (2 + t) k = 2u i + (4 + u) j + (1 + 2u) k$

$\Rightarrow \left.\begin{aligned} 1 + t &= 2u \\ -6 + 2t &= 4 + u \end{aligned}\right\} t = 7,\ u = 4$

$\quad 2 + t = 1 + 2u$

$\Rightarrow (8, 8, 9)$.

3. What is the angle between ℓ_1 & ℓ_2 ?

$\vec{d} = i+2j+k \quad \vec{D} = 2i+j+2k$

$$\cos\theta = \frac{\vec{d}\cdot\vec{D}}{\|\vec{d}\|\cdot\|\vec{D}\|} = \frac{2+2+2}{\sqrt{6}\cdot 3} = \frac{1}{3}\sqrt{6}$$

4. Find the line pass through $(8,8,9)$ and perpendicular to ℓ_1 & ℓ_2.

the direction vector $= (i+2j+k)\times(2i+j+2k)$

$$= \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{vmatrix} = 3i-3k$$

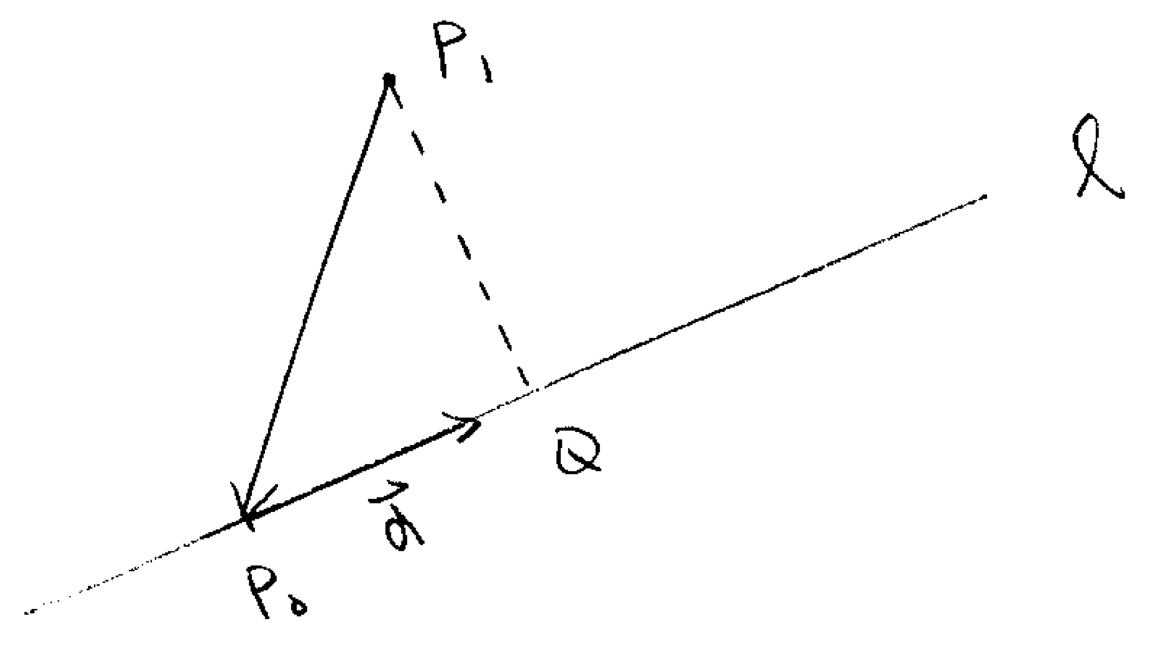
$\Rightarrow \ell_3 : S(t) = (8+3t)i + 8j + (9-3t)k$.

Theorem 2.2

Let P_1 be a point not on ℓ, P_0 be a point on ℓ

The distance $d(P_1,\ell) = \dfrac{\|\overrightarrow{P_0P_1}\times\vec{d}\|}{\|\vec{d}\|}$.

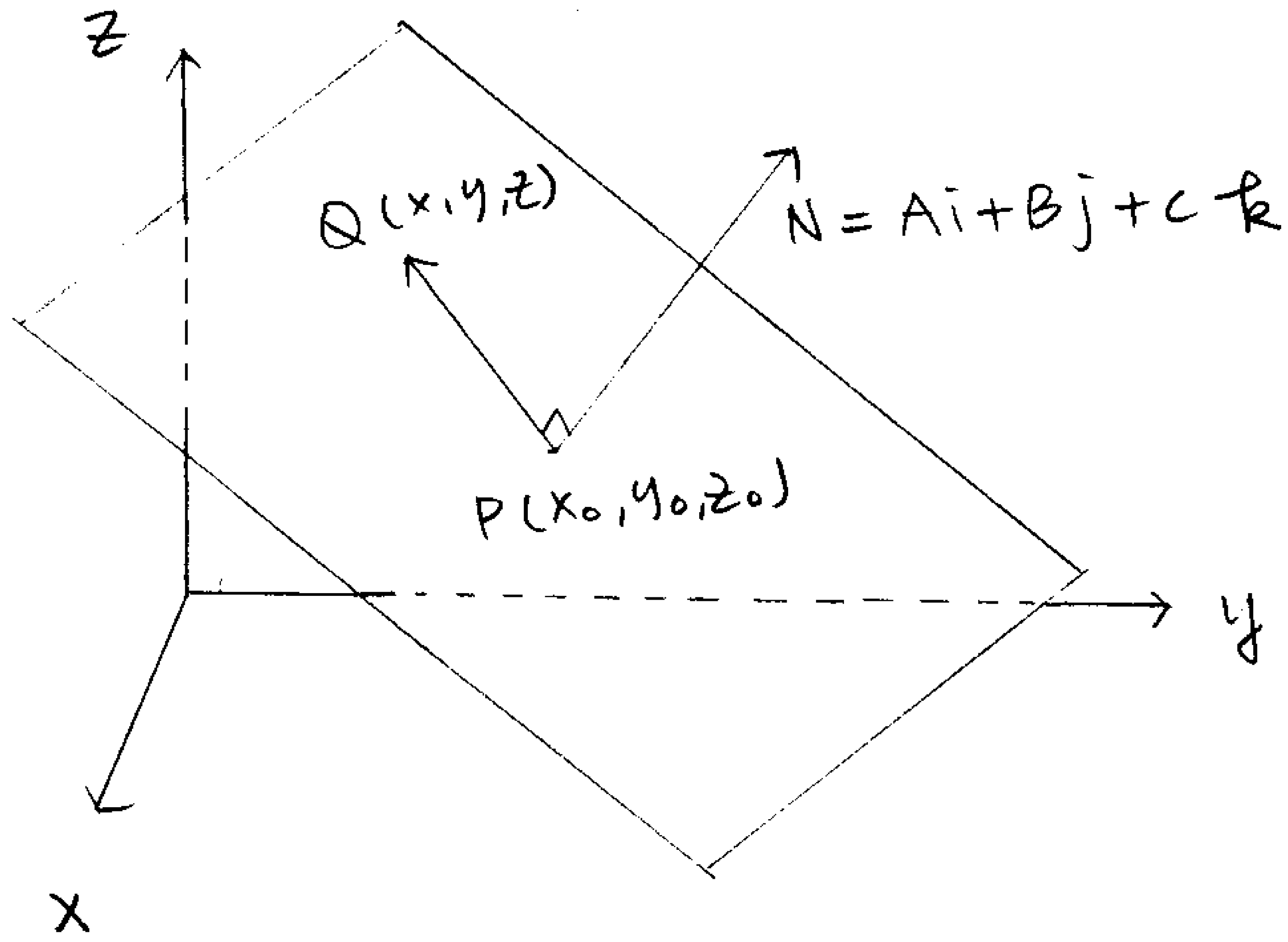
<proof>



$\Rightarrow d(P_1,\ell) = d(P_1,Q) = \|\overrightarrow{P_0P_1}\|\cdot\sin\theta$

and $\|\overrightarrow{P_0P_1}\times\vec{d}\| = \|\overrightarrow{P_0P_1}\|\|\vec{d}\|\sin\theta$

$$\Rightarrow d(P_1,\ell) = \frac{\|\overrightarrow{P_0P_1}\times\vec{d}\|}{\|\vec{d}\|}$$



The point Q will lie on the given plane $\Leftrightarrow N \cdot \overrightarrow{PQ} = 0$.

$$\Leftrightarrow A(x-x_0)+B(y-y_0)+C(z-z_0)=0$$

Example

1. Write an equation for the plane that passes through $P(1,0,2)$ and has normal vector $N = 3i-2j+k$

$3(x-1)-2(y-0)+1\cdot(z-2)=0 \Rightarrow 3x-2y+z=5$.

2. Find an equation for the plane that pass through $(-2,3,5)$ and perpendicular to $x=-2+t,\ y=1+2t,\ z=4$.

$N=(1,2,0)$

$\Rightarrow (x+2)+2(y-3)+0\cdot(z-5)=0$

$\Rightarrow x+2y-4=0$.

3. Show that every equation $ax+by+cz+d=0$ with $\sqrt{a^2+b^2+c^2} \neq 0$ represents a plane in space.

since $\sqrt{a^2+b^2+c^2} \neq 0 \Rightarrow a, b, c$ are not all zero

$\Rightarrow \exists\, x_0, y_0, z_0$ s.t. $ax_0+by_0+cz_0+d=0$

$\Rightarrow \quad (ax+by+cz+d)-(ax_0+by_0+cz_0+d)=0$

$\Rightarrow \quad a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

It represents the plane through the point (x_0, y_0, z_0) with normal $N = a\vec{i}+b\vec{j}+c\vec{k}$

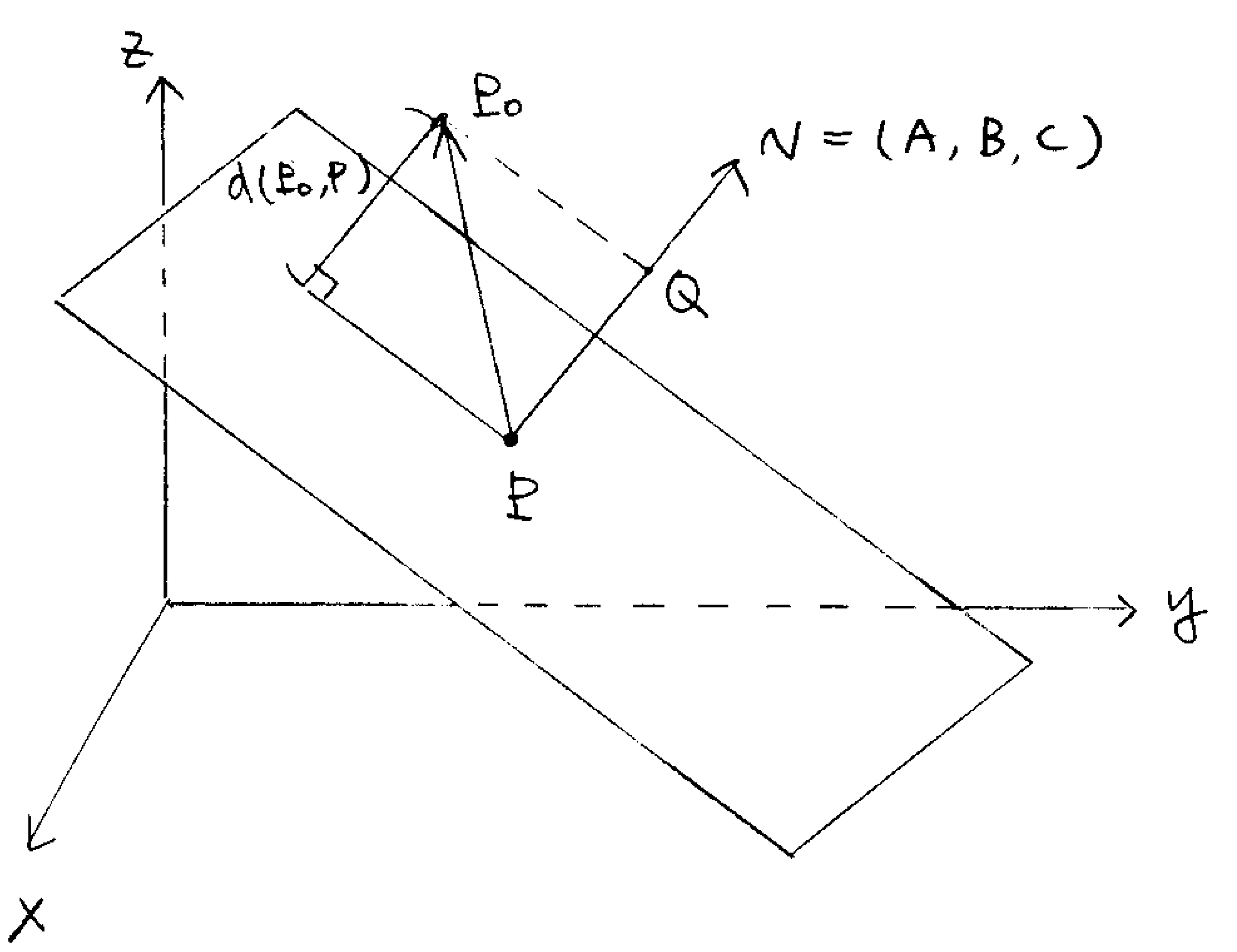
Theorem 2.3

$P: Ax+By+Cz+D=0$ be a plane and $\underset{P_0}{(x_0, y_0, z_0)}$ not on the plane.
The distance between the point and the plane is

$$d(P_0, P) = \frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}$$

<proof>

Pick any $P(x,y,z)$ in the plane



, $\overrightarrow{PP_0} = (x-x_0, y-y_0, z-z_0)$

$$d(P_0,P) = d(P,Q) = \left\| \left(\overrightarrow{PP_0} \cdot \vec{u}_N \right) \cdot \vec{u}_N \right\|$$

$$= \left\| \left((x-x_0, y-y_0, z-z_0) \cdot \frac{1}{\sqrt{A^2+B^2+C^2}} (A,B,C) \right) \cdot \frac{1}{\sqrt{A^2+B^2+C^2}} (A,B,C) \right\|$$

$$= \left\| \frac{1}{A^2+B^2+C^2} \left(A(x-x_0)+B(y-y_0)+C(z-z_0) \right) \cdot (A,B,C) \right\|$$

$$= \frac{1}{\sqrt{A^2+B^2+C^2}} \left| Ax+By+Cz-Ax_0-By_0-Cz_0 \right| = \frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}} \quad //$$