§3 Taylor polynomials in x ; Taylor series.

f continuous at 0. let $P_0(x) = f(0)$, a constant function

If f is differentiable at 0, the linear function that best approximates f at points 0 is $P_1(x) = f(0) + f'(0)x$

If f is twice diff. at 0, then we get a better approximate by the polynomial $P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$

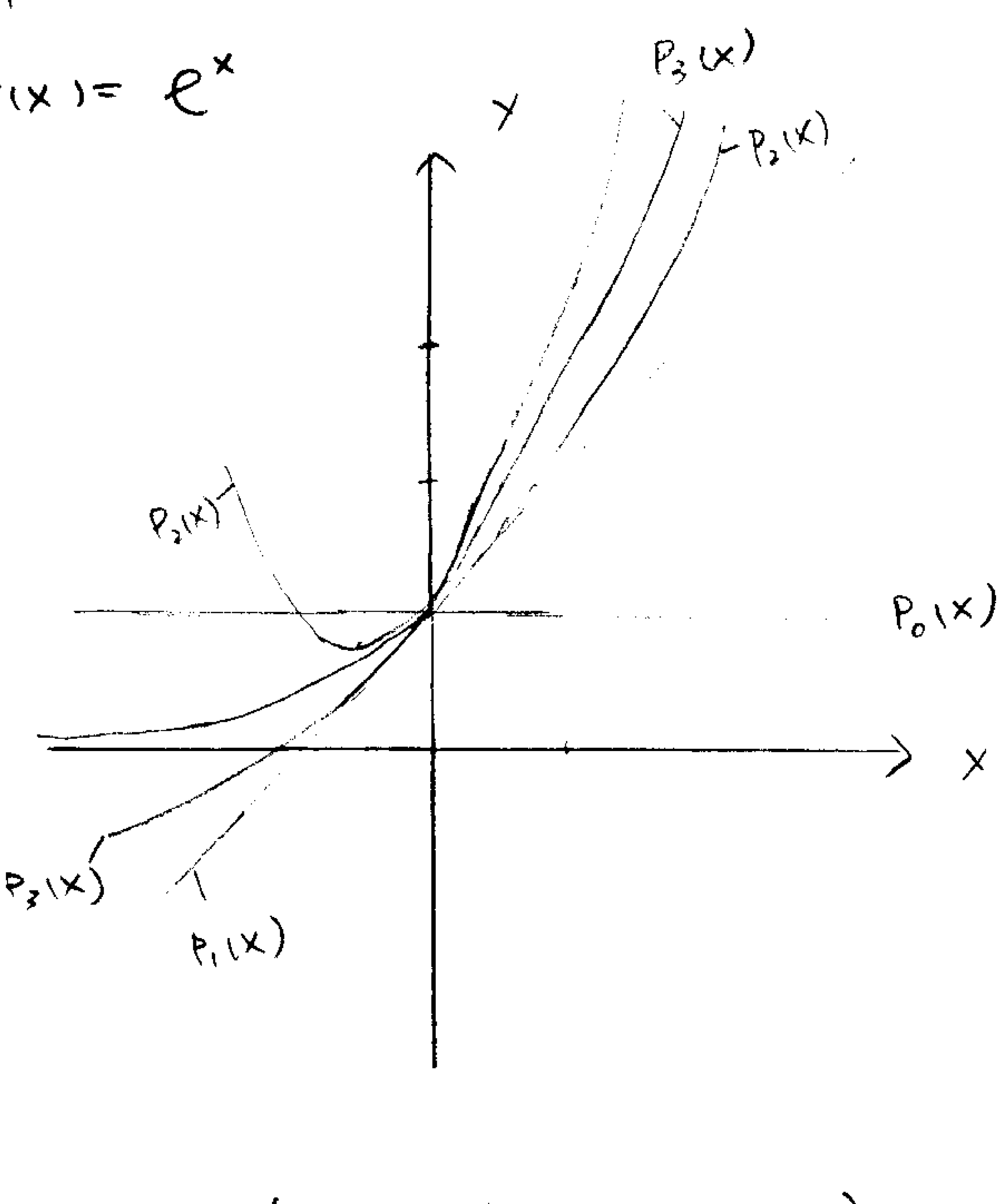
$\vdots$

In general, if f has n derivatives at 0, we can form the polynomial

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

Example

$f(x) = e^x$



$f'(0) = 1 \Rightarrow P_1(x) = 1 + x$

$f''(0) = 1 \Rightarrow P_2(x) = 1 + x + \frac{1}{2}x^2$

$f'''(0) = 1 \Rightarrow P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$

Theorem 3.1 (Taylor's Theorem)

If f has $n+1$ continuous derivatives on the open interval I that contains 0, then for each $x \in I$,

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + R_n(x)$$

with $R_n(x) = \frac{1}{n!}\int_0^x f^{(n+1)}(t)(x-t)^n\,dt$. We call $R_n(x)$ the remainder.

<proof>

Fix $x \in I$

Since $\int_0^x f'(t)\,dt = f(x) - f(0)$

We have $f(x) = f(0) + \int_0^x f'(t)\,dt$

let $u = f'(t)$, $v = -(x-t)$

$\Rightarrow du = f''(t)dt$, $dv = dt$

Use integrate by parts $\Rightarrow \int_0^x f'(t)dt = -f'(t)(x-t)\Big|_0^x + \int_0^x f''(t)(x-t)\,dt$

$$= f(0) + f'(0)\,x + \int_0^x f''(t)(x-t)\,dt$$

let $u = f''(t)$, $v = -\frac{1}{2}(x-t)^2$ $\Rightarrow du = f'''(t)dt$, $dv = (x-t)\,dt$

Use integrate by parts $\Rightarrow \int_0^x f''(t)(x-t)\,dt = -\frac{1}{2}f''(t)(x-t)^2\Big|_0^x + \frac{1}{2!}\int_0^x f'''(t)(x-t)^2 dt$

$$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{1}{2!}\int_0^x f'''(t)(x-t)^2\,dt$$

$\vdots$ continuing to integrate by parts

$$= f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \frac{1}{n!}\int_0^x f^{(n+1)}(t)(x-t)^n\,dt$$ //

Corollary 3.2 Lagrange formula for the Remainder.

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$$ for some c between 0 & x

Recall the Mean Value Theorem

If f is differentiable on (a,b) and continuous on $[a,b]$,

then $\exists c \in (a,b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$

Observe that $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!} + R_n(x)$

$$= f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!} + \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$$

is an extension result of the MVT.

Corollary 3.3

$$|R_n(x)| \le \max_{t\in J} |f^{(n+1)}(t)| \frac{|x|^{n+1}}{(n+1)!}$$ where J is the closed interval joins 0 to x.

Examples

1. The Taylor polynomials for the exponential function $f(x) = e^x$ take the form $P_n(x) = 1 + x + \frac{1}{2!}x^2 + \dots + \frac{1}{n!}x^n$

Show that the remainder estimate that for each real x

$$R_n(x) \longrightarrow 0 \quad \text{as} \quad n \to \infty$$

Fix x and let $M = \max\{e^y \mid y \in J\}$ (J is the closed interval joins 0 to x)

⊙ Note $M = e^x$ if $x < 0$ & $M = e^0 = 1$, if $x > 0$

Since $f^{(n+1)}(t) = e^t$, $\forall n$

we have $\max_{t\in J} |f^{(n+1)}(t)| = M$, $\forall n$ & $|R_n(x)| < \max_{t\in J} |f^{(n+1)}(t)| \frac{|x|^{n+1}}{(n+1)!} = M \cdot \frac{|x|^{n+1}}{(n+1)!}$

$\Rightarrow |R_n(x)| \longrightarrow 0$ as $n \to \infty$ $\left(\because \frac{|x|^{n+1}}{(n+1)!} \longrightarrow 0 \text{ as } n \to \infty \right)$

2. $f(x) = \sin x$

$f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$

$f^{(4)}(x) = \sin x$, $f^{(5)}(x) = \cos x$, $f^{(6)}(x) = -\sin x$, $f^{(7)}(x) = -\cos x$

$f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1$

$\Rightarrow P_0(x) = 0$, $P_1(x) = P_2(x) = x$, $P_3(x) = P_4(x) = x - \frac{1}{3!}x^3$, $P_5(x) = P_6(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$

$P_7(x) = P_8(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7$, ...

Hence $f^{(4k)}(x) = \sin x$, $f^{(4k+1)}(x) = \cos x$, $f^{(4k+2)}(x) = -\sin x$, $f^{(4k+3)}(x) = -\cos x$

Thus $|f^{(n+1)}(t)| \leq 1 \quad , \forall n \ , \forall x$

$$\Rightarrow |R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!} \longrightarrow 0 \quad \text{as} \quad n \to \infty$$

We can write Taylor polynomials $P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k.$$

* If f is infinitely differentiable on an open interval I that contains 0,

then we have $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + R_n(x)$

Suppose $R_n(x) \to 0$ as $n \to \infty$, $\forall x \in I$, we say $f(x)$ can be

expanded as a Taylor series in x and write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k.$$

Remark

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad \text{for all real } x.$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots \quad \text{for all real } x.$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots \quad \text{for all real } x.$$

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \quad \text{for } -1 < x < 1.$$

<proof>

Let $f(x) = \ln(1+x)$ is defined on $(-1, \infty)$ and infinitely differentiable.

$$f'(x) = \frac{1}{1+x}, \quad f''(x) = -\frac{1}{(1+x)^2}, \quad f'''(x) = \frac{2}{(1+x)^3}, \quad f^{(4)}(x) = -\frac{3!}{(1+x)^4}$$

$$f^{(5)}(x) = \frac{4!}{(1+x)^5} \quad \dots$$

Observe that $f^{(k)}(x) = (-1)^{k+1} \frac{(k-1)!}{(1+x)^k} \Rightarrow f^{(k)}(0) = (-1)^{k+1}(k-1)!$

$$\Rightarrow P_n(x) = f(0) + \sum_{k=1}^{n} \frac{(-1)^{k+1}(k-1)!}{k!} x^k = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} x^k$$

Corollary 3.3

$$|R_n(x)| \leq \max_{t\in J} |f^{(n+1)}(t)| \frac{|x|^{n+1}}{(n+1)!}$$ where J is the closed interval joins 0 to x.

Examples

1. The Taylor polynomials for the exponential function $f(x) = e^x$

take the form $P_n(x) = 1 + x + \frac{1}{2!}x^2 + \dots + \frac{1}{n!}x^n$

Show that the remainder estimate that for each real x

$$R_n(x) \longrightarrow 0 \quad \text{as} \quad n \to \infty$$

Fix x and let $M = \max\{e^y \mid y \in J\}$ (J is the closed interval joins 0 to x)

⊙ Note $M = e^x$ if $x < 0$ & $M = e^0 = 1$ if $x > 0$

Since $f^{(n+1)}(t) = e^t$, $\forall n$

we have $\max_{t\in J} |f^{(n+1)}(t)| = M$, $\forall n$ & $|R_n(x)| < \max_{t\in J} |f^{(n+1)}(t)| \frac{|x|^{n+1}}{(n+1)!} = M \cdot \frac{|x|^{n+1}}{(n+1)!}$

$\Rightarrow |R_n(x)| \longrightarrow 0$ as $n \to \infty$ ($\because \frac{|x|^{n+1}}{(n+1)!} \to 0$ as $n \to \infty$)

2. $f(x) = \sin x$

$f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$

$f^{(4)}(x) = \sin x$, $f^{(5)}(x) = \cos x$, $f^{(6)}(x) = -\sin x$, $f^{(7)}(x) = -\cos x$

$f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1$

$\Rightarrow P_0(x) = 0$, $P_1(x) = P_2(x) = x$, $P_3(x) = P_4(x) = x - \frac{1}{3!}x^3$, $P_5(x) = P_6(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$

$P_7(x) = P_8(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7$, ...

Hence $f^{(4k)}(x) = \sin x$, $f^{(4k+1)}(x) = \cos x$, $f^{(4k+2)}(x) = -\sin x$, $f^{(4k+3)}(x) = -\cos x$

Thus $|f^{(n+1)}(t)| \le 1 \quad , \forall n \ , \forall x$

$$\Rightarrow |R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!} \longrightarrow 0 \quad \text{as} \quad n \to \infty$$

We can write Taylor polynomials $P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k .$$

* If f is infinitely differentiable on an open interval I that contains 0,

then we have $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + R_n(x)$

Suppose $R_n(x) \to 0$ as $n \to \infty$, $\forall x \in I$, we say $f(x)$ can be

expanded as a Taylor series in x and write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

Remark

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad \text{for all real } x$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots \quad \text{for all real } x$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots \quad \text{for all real } x.$$

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \quad \text{for } -1 < x < 1$$

<proof>

Let $f(x) = \ln(1+x)$ is defined on $(-1, \infty)$ and infinitely differentiable

$f'(x) = \frac{1}{1+x}$, $f''(x) = -\frac{1}{(1+x)^2}$, $f'''(x) = \frac{2}{(1+x)^3}$, $f^{(4)}(x) = -\frac{3!}{(1+x)^4}$

$f^{(5)}(x) = \frac{4!}{(1+x)^5}$ ...

Observe that $f^{(k)}(x) = (-1)^{k+1} \frac{(k-1)!}{(1+x)^k} \Rightarrow f^{(k)}(0) = (-1)^{k+1}(k-1)!$

$$\Rightarrow P_n(x) = f(0) + \sum_{k=1}^{n} \frac{(-1)^{k+1}(k-1)!}{k!} x^k = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} x^k$$

Show that $R_n(x) \to 0$ as $-1 < x < 1$

$$R_n(x) = \frac{1}{n!}\int_0^x f^{(n+1)}(t)(x-t)^n\,dt$$

$$= \frac{1}{n!}\int_0^x (-1)^{n+2}\frac{n!}{(1+t)^{n+1}}(x-t)^n\,dt = (-1)^n \int_0^x \frac{(x-t)^n}{(1+t)^{n+1}}\,dt$$

For $0 < x < 1$,

$$|R_n(x)| = \int_0^x \frac{(x-t)^n}{(1+t)^{n+1}}\,dt \le \int_0^x (x-t)^n\,dt = \frac{-1}{n+1}(x-t)^{n+1}\Big|_0^x$$

$$= \frac{x^{n+1}}{n+1} \to 0 \quad \text{as } n \to \infty.$$

For $-1 \le x < 0$,

$$|R_n(x)| = \left|\int_0^x (-1)^n \frac{(x-t)^n}{(1+t)^{n+1}}\,dt\right| = \int_x^0 \frac{(t-x)^n}{(1+t)^{n+1}}\,dt = \int_0^x \left(\frac{t-x}{1+t}\right)^n \cdot \frac{1}{1+t}\,dt$$

(By the first mean value theorem for integrals,
$\exists x_n \in (x, 0)$ s.t. $\int_x^0 \left(\frac{t-x}{1+t}\right)^n \cdot \frac{1}{1+t}\,dt = \left(\frac{x_n - x}{1+x_n}\right)^n \cdot \frac{1}{1+x_n} \cdot (-x)$)

$$= \left(\frac{x_n - x}{1+x_n}\right)^n \cdot \frac{1}{1+x_n} \cdot (-x)$$

$$= \left(\frac{x_n + |x|}{1+x_n}\right)^n \cdot \frac{1}{1+x_n} \cdot |x| \qquad (\because x < 0 \Rightarrow |x| = -x)$$

$$< \left(\frac{|x|\cdot x_n + |x|}{1+x_n}\right)^n \cdot \frac{1}{1+x_n} \cdot |x| \qquad (\because |x| < 1)$$

$$= \left(\frac{|x|(x_n+1)}{1+x_n}\right)^n \cdot \frac{1}{1+x_n} \cdot |x| = |x|^n \frac{|x|}{1+x_n} \to 0 \quad \text{as } n \to \infty \quad //$$

Remark

For any function f with derivatives of all orders at $x=0$, we can form a Taylor series $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$

The Taylor series converges to $f(x)$ $\Leftrightarrow$ the remainder term $R_n(x) \to 0$.

Examples

1. Determine the maximum possible error we incur by using P_6 to estimate $f(x) = e^x$ for $x \in (0,1)$

By Taylor's Thm & Lagrange formula, we have

$$e^x = P_6(x) + R_6(x)$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{1}{6!}\int_0^x f^{(7)}(t)(x-t)^6\,dt.$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{f^{(7)}(c)}{7!}x^7 \quad \text{for some } c \text{ between } 0 \text{ and } x$$

$$|R_6(x)| \le \max_{0\le t\le x} |f^{(7)}(t)| \frac{|x|^7}{7!} = \max_{0\le t\le x} e^t \cdot \frac{1}{7!} \le e^x \cdot \frac{1}{7!} \le \frac{e}{7!} < \frac{3}{7!} = \frac{1}{1680} < 0.0006.$$

2. Given an estimate for $e^{0.2}$ correct to 3 decimal places.

We know $P_n(0.2) = 1 + 0.2 + \frac{(0.2)^2}{2!} + \cdots + \frac{(0.2)^n}{n!}$

Find n s.t. $|R_n(0.2)| < 0.0005$

$$|R_n(0.2)| \le \max_{0\le t\le 0.2} |f^{(n+1)}(t)| \cdot \frac{(0.2)^{n+1}}{(n+1)!} \le e^{0.2} \cdot \frac{(0.2)^{n+1}}{(n+1)!} < e \cdot \frac{(0.2)^{n+1}}{(n+1)!} < 3 \cdot \frac{(0.2)^{n+1}}{(n+1)!}$$

$$= 3 \cdot \frac{1}{5^{n+1}(n+1)!} < 0.0005$$

$\Rightarrow n \ge 3$

3. Use the sine series to estimate $\sin 0.5$ within 0.001.

Find n s.t. $|R_n(0.5)| < 0.001$.

$$|R_n(0.5)| \le \max_{0\le t\le 0.5} |f^{(n+1)}(t)| \frac{(0.5)^{n+1}}{(n+1)!} \le \frac{(0.5)^{n+1}}{(n+1)!} = \frac{1}{2^{n+1}(n+1)!} < 0.001$$

$\Rightarrow n \ge 4$

Note that $\sin x = P_4(x) + R_4(x) \Rightarrow \sin 0.5 = P_4(0.5) + R_4(0.5)$

and $P_4(0.5) = P_3(0.5) = 0.5 - \frac{(0.5)^3}{3!} = \frac{23}{48}$

4. Use the series for $\ln(1+x)$ to estimate $\ln 1.4$ within 0.01.

$$\ln 1.4 = 0.4 - \frac{(0.4)^2}{2} + \frac{(0.4)^3}{3} - \frac{(0.4)^4}{4} + \cdots$$

The first term less than 0.01 is $\frac{1}{4}(0.4)^4 = 0.0064$.

$$\Rightarrow 0.4 - \tfrac{1}{2}(0.4)^2 + \tfrac{1}{3}(0.4)^3 - \tfrac{1}{4}(0.4)^4 < \ln 1.4 < 0.4 - \tfrac{1}{2}(0.4)^2 + \tfrac{1}{3}(0.4)^3$$

$$\Rightarrow 0.335 < \ln 1.4 < 0.341$$

We can take $\ln 1.4 \approx 0.34$.

Theorem 3.3 (Taylor's Theorem in $x-a$)

If g has $n+1$ continuous derivatives on an open interval I that contains a, then for each $x \in I$,

$$g(x) = g(a) + g'(a)(x-a) + \frac{g''(a)}{2!}(x-a)^2 + \cdots + \frac{g^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

with $R_n(x) = \frac{1}{n!}\int_a^x g^{(n+1)}(t)(x-t)^n\,dt$.

Corollary 3.4 (Lagrange Formula)

$$R_n(x) = \frac{g^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } c \text{ between } a \text{ and } x.$$

Moreover, let J be the closed interval that joins a to x, then

$$|R_n(x)| \le \max_{x\in J}|g^{(n+1)}(t)| \cdot \frac{|x-a|^{n+1}}{(n+1)!}$$

If $R_n(x) \to 0$, then we have $g(x) = g(a) + g'(a)(x-a) + \cdots + \frac{g^{(n)}(a)}{n!}(x-a)^n + \cdots$

$$= \sum_{k=0}^{\infty} \frac{g^{(k)}(a)}{k!}(x-a)^k.$$

Examples

1. Expand $g(x) = 4x^3 - 3x^2 + 5x - 1$ in powers of $x-2$.

$g'(x) = 12x^2 - 6x + 5$, $g'(2) = 41$

$g''(x) = 24x - 6$, $g''(2) = 42$

$g'''(x) = 24$, $g'''(2) = 24$

$g^{(4)}(x) = 0$, $g^{(4)}(2) = 0$

$g(x) = 29 + 41(x-2) + 21(x-2)^2 + 4(x-3)^3$

2. Expand $g(x) = x^2 \ln x$ in powers of $x-1$.

$g(x) = x^2 \ln x$

$g'(x) = 2x \ln x + x$

$g''(x) = 2\ln x + 2 + 1 = 2\ln x + 3$

$g'''(x) = \frac{2}{x}$

$g^{(4)}(x) = -2 \cdot x^{-2}$

$g^{(5)}(x) = -2 \cdot (-2) x^{-3}$

$g^{(6)}(x) = -2 \cdot (-2)(-3) x^{-4}$

$g^{(7)}(x) = -2 \cdot (-2) \cdot (-3) \cdot (-4) x^{-5}$ ... $g^{(k)}(x) = (-1)^{k+1} 2(k-3)!\, x^{-k+2}$ for $k \geq 3$

$\Rightarrow g^{(k)}(x) = (-1)^{k+1} \cdot 2 \cdot (k-3)!$ for $k \geq 3$.

$g(x) = (x-1) + \frac{3}{2!}(x-1)^2 + \sum_{k=3}^{\infty} \frac{(-1)^{k+1} \cdot 2 \cdot (k-3)!}{k!} (x-1)^k$

$= (x-1) + \frac{3}{2!}(x-1)^2 + \sum_{k=3}^{\infty} \frac{2 \cdot (-1)^{k+1}}{k(k-1)(k-2)} (x-1)^k$

Proposition 3.5

The series expansion $\ln x = \ln a + \frac{1}{a}(x-a) - \frac{1}{2a^2}(x-a)^2 + \frac{1}{3a^3}(x-a)^3 - \cdots$ is valid for $0 < x \leq 2a$.

<proof>

We expand $\ln(a+t)$ in powers of t and let $t = x-a$.

Note that $\ln(a+t) = \ln(a(1+\frac{t}{a})) = \ln a + \ln(1+\frac{t}{a})$

We know ~~that~~

$\ln(1+\frac{t}{a}) = \frac{t}{a} - \frac{1}{2}(\frac{t}{a})^2 + \frac{1}{3}(\frac{t}{a})^3 - \cdots$ holds for $-1 < \frac{t}{a} < 1$ $(-a < t < a)$.

$\Rightarrow \ln(a+t) = \ln a + \ln(1+\frac{t}{a})$

$= \ln a + \frac{t}{a} - \frac{1}{2}(\frac{t}{a})^2 + \frac{1}{3}(\frac{t}{a})^3 - \cdots$ for $-a < t < a$.

let $t = x-a$,

$\ln(a+t) = \ln(a+x-a) = \ln x$

$= \ln a + \frac{1}{a}(x-a) - \frac{1}{2a^2}(x-a)^2 + \frac{1}{3a^3}(x-a)^3 - \cdots$ for $-a < x-a < a$

$(0 < x < 2a)$ //

§4 Power Series

Definition 4.1

A power series $\sum a_k x^k$ is said to converge

(i) at c if $\sum a_k c^k$ converges

(ii) on the set S if $\sum a_k x^k$ converges at each $x \in S$.

Examples

1. $\sum \frac{(-1)^k}{k} x^k$

Consider $\sum \left| \frac{(-1)^k}{k} x^k \right| = \sum \frac{1}{k} |x|^k$

Applying the ratio test,

$$\lim_{k\to\infty} \frac{\frac{1}{k+1}|x|^{k+1}}{\frac{1}{k}|x|^k} = \lim_{k\to\infty} \frac{k}{k+1}|x| = |x|$$

By the ratio test, $\sum \frac{1}{k}|x|^k$ converges for $|x|<1$ and diverges for $|x|>1$

$\Rightarrow \sum \frac{(-1)^k}{k}|x|^k$ converges absolutely for $|x|<1$.

Now we test the endpoints $x=-1$ & $x=1$

$\sum \frac{(-1)^k}{k}(-1)^k = \sum \frac{1}{k}$ div.

$\sum \frac{(-1)^k}{k}(1)^k = \sum \frac{(-1)^k}{k}$ conv.

The interval of convergence is $(-1, 1]$.

2. $\sum \frac{1}{k^2} x^k$

Consider $\sum \left|\frac{1}{k^2} x^k\right| = \sum \frac{1}{k^2}|x|^k$

$$\lim_{k\to\infty} \frac{\frac{1}{(k+1)^2}|x|^{k+1}}{\frac{1}{k^2}|x|^k} = \lim_{k\to\infty} \frac{k^2}{(k+1)^2}|x| = |x|$$

By ratio test, $\sum \frac{1}{k^2}x^k$ converges absolutely for $|x|<1$.

check $x=-1$ & $x=1$

$x=-1$ $\sum \frac{1}{k^2}(-1)^k = \sum \frac{(-1)^k}{k^2}$ is an alternating series & $\frac{1}{k^2} \to 0$ as $k \to \infty$

$\Rightarrow \sum \frac{1}{k^2}(-1)^k$ converges.

$x=1$, $\sum \frac{1}{k^2}(1)^k = \sum \frac{1}{k^2}$ converges

The interval of convergence $[-1,1]$.

3. $\sum \frac{k}{6^k} x^k$

Consider $\sum \left|\frac{k}{6^k} x^k\right| = \sum \frac{k}{6^k}|x|^k$

$\lim_{k\to\infty}\left(\frac{k}{6^k}|x|^k\right)^{\frac{1}{k}} = \lim_{k\to\infty} \frac{1}{6}\cdot k^{\frac{1}{k}}\cdot|x| = \frac{1}{6}|x| < 1 \Rightarrow |x|<6$

$\sum \frac{k}{6^k}x^k$ converges absolutely for $|x|<6$

Check $x=-6$ & $x=6$

$x=-6$, $\sum \frac{k}{6^k}(-6)^k = \sum \frac{k}{\cancel{6^k}}(-1)^k \cancel{6^k} = \sum (-1)^k k$ div.

$x=6$, $\sum \frac{k}{\cancel{6^k}}\cancel{(6)^k} = \sum k$ div.

The interval of convergence $(-6,6)$

4. $\sum \frac{k!}{(3k)!}x^k$

Consider $\sum \left|\frac{k!}{(3k)!}x^k\right| = \sum \frac{k!}{(3k)!}|x|^k$

$\lim_{k\to\infty} \dfrac{\frac{(k+1)!}{(3(k+1))!}|x|^{k+1}}{\frac{k!}{(3k)!}|x|^k} = \lim_{k\to\infty} \frac{\cancel{k!}(k+1)\,|x|^{k+1}}{\cancel{(3k)!}(3k+1)(3k+2)(3k+3)} \cdot \frac{\cancel{(3k)!}}{\cancel{k!}|x|^k} = \lim_{k\to\infty}\frac{(k+1)\,|x|}{(3k+1)(3k+2)(3k+3)}$

$$= \lim_{k\to\infty}\frac{\cancel{(k+1)}|x|}{(3k+1)(3k+2)3\cancel{(k+1)}} = 0$$

Hence $\sum \frac{k!}{(3k)!}x^k$ converges absolutely for all x.

5. $\sum \frac{k^k}{2^k} x^k$

consider $\sum \left| \frac{k^k}{2^k} x^k \right| = \sum \frac{k^k}{2^k} |x|^k$

$\lim_{k\to\infty} \left(\frac{k^k}{2^k} |x|^k \right)^{\frac{1}{k}} = \lim_{k\to\infty} \frac{k}{2}|x| = \infty$ for $x \neq 0$

The series converges only at $x=0$.

6. $\sum \frac{(-1)^k}{k^2 3^k} (x+2)^k$

consider $\sum \left| \frac{(-1)^k}{k^2 3^k} (x+2)^k \right| = \sum \frac{1}{k^2 3^k} |x+2|^k$

$\lim_{k\to\infty} \left| \frac{\frac{1}{(k+1)^2 3^{k+1}} (x+2)^{k+1}}{\frac{1}{k^2 3^k}(x+2)^k} \right| = \lim_{k\to\infty} \frac{k^2}{3(k+1)^2} |x+2| = \frac{1}{3}|x+2|$

The series converges absolutely for $\frac{1}{3}|x+2| < 1 \Rightarrow -1 < \frac{1}{3}(x+2) < 1$

$\Rightarrow -5 < x < 1$.

check $x=-5$

$\sum \frac{(-1)^k}{k^2 3^k} (-3)^k = \sum \frac{(-1)^k (-1)^k 3^k}{k^2 \cdot 3^k} = \sum \frac{1}{k^2}$ conv.

$x=1$

$\sum \frac{(-1)^k}{k^2 3^k} \cdot 3^k = \sum \frac{(-1)^k}{k^2}$ conv.

The interval of convergent $[-5, 1]$.

Theorem 4.1 (The differentiability Theorem)

If $f(x) = \sum_{k=0}^{\infty} a_k x^k$ for all x in $(-c,c)$, then f is differentiable on $(-c,c)$ and $f'(x) = \sum_{k=0}^{\infty} \frac{d}{dx}(a_k x^k) = \sum_{k=1}^{\infty} k a_k x^{k-1}$ for all $x \in (-c,c)$.

Examples

1. $f(x) = e^x$

$$\frac{d}{dx}(e^x) = \frac{d}{dx}\left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right) = \sum_{k=0}^{\infty} \frac{d}{dx}\left(\frac{x^k}{k!}\right) = \sum_{k=1}^{\infty} \frac{k \cdot x^{k-1}}{k!} = \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

2. $f(x) = \sin x$, $g(x) = \cos x$

$$\frac{d}{dx}(\sin x) = \frac{d}{dx}\left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}\right) = \sum_{k=0}^{\infty} \frac{d}{dx}\left(\frac{(-1)^k}{(2k+1)!} x^{2k+1}\right)$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k (2k+1)}{(2k+1)!} x^{2k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = \cos x$$

$$\frac{d}{dx}(\cos x) = \frac{d}{dx}\left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}\right) = \sum_{k=0}^{\infty} \frac{d}{dx}\left(\frac{(-1)^k}{(2k)!} x^{2k}\right) = \sum_{k=1}^{\infty} \frac{(-1)^k 2k}{(2k)!} x^{2k-1}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)!} x^{2k-1} = -\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = -\sin x.$$

3. $g(x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$ for $x \in (-1,1)$

$$\Rightarrow g'(x) = \sum_{k=1}^{\infty} \frac{k \cdot x^{k-1}}{k} = \sum_{k=1}^{\infty} x^{k-1} = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

Note that $g(0) = 0$

$$\Rightarrow g(x) = -\ln(1-x) = \ln\left(\frac{1}{1-x}\right), \quad \forall x \in (-1,1).$$

Theorem 4.2

If $f(x) = \sum_{k=0}^{\infty} a_k x^k$ converges on $(-c,c)$, then $F(x) = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}$ converges on $(-c,c)$ and $\int f(x)dx = F(x) + C$

That is, $\int \left(\sum_{k=0}^{\infty} a_k x^k\right) dx = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1} + C$.

Examples

1. We know that $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{k=0}^{\infty} (-1)^k x^k$ for $x \in (-1,1)$

$\Rightarrow \ln(1+x) = \int \left(\sum_{k=0}^{\infty} (-1)^k x^k\right) dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1} + C \quad , \forall x \in (-1,1)$

Take $x=0$

$\ln 1 = 0 + C \Rightarrow C = 0$

$\Rightarrow \ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1} \quad , \forall x \in (-1,1)$.