Infinite Series

§1 Inifinite series

$a_0, a_1, a_2, \ldots$ an infinite sequence of real numbers

We can't form the sum of all the a_k (there are an infinite number of them), but we can form the partial sums

$$S_0 = a_0 = \sum_{k=0}^{0} a_k$$

$$S_1 = a_0 + a_1 = \sum_{k=0}^{1} a_k$$

$$\vdots$$

$$S_n = a_0 + a_1 + \ldots + a_n = \sum_{k=0}^{n} a_k$$

<u>Definition 1.1</u>

If $S_n = \sum_{k=0}^{n} a_k \longrightarrow L$ as $n \to \infty$ where L is a finite number, we write $\sum_{k=0}^{\infty} a_k = L$ and say that the series $\sum_{k=0}^{\infty} a_k$ converges to L.

We call L the sum of the series

If the sequence of partial sum diverges, we say that the series $\sum_{k=0}^{\infty} a_k$ diverges.

Examples

1. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} = \sum_{k=0}^{\infty} \frac{1}{k+1} - \frac{1}{k+2}$

$$S_n = \sum_{k=0}^{n} \frac{1}{k+1} - \frac{1}{k+2} = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \ldots + \left(\cancel{\frac{1}{n+1}} - \frac{1}{n+2}\right) = 1 - \frac{1}{n+2}$$

$\Rightarrow S_n \to 1$ as $n \to \infty$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} = 1$$

2. $\sum_{k=0}^{\infty} (-1)^k$

$$S_n = \sum_{k=0}^{n} (-1)^k = \begin{cases} 1 & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases} \Rightarrow \{S_n\} \text{ diverges} \Rightarrow \sum_{k=0}^{\infty} (-1)^k \text{ diverges.}$$

(bounded).

3. $\sum_{k=0}^{\infty} 2^k$

$S_n = \sum_{k=0}^{n} 2^k = 1+2+2^2+\dots+2^n > 2^n \longrightarrow \infty$ as $n \to \infty$

$\Rightarrow \sum_{k=0}^{\infty} 2^k$ diverges (unbounded)

4. Geometry series

If $|x|<1$, then $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$

If $|x| \geq 1$, then $\sum_{k=0}^{\infty} x^k$ diverges.

<proof> exercise

Theorem 1.2

1. If $\sum_{k=0}^{\infty} a_k$ & $\sum_{k=0}^{\infty} b_k$ both converge, then $\sum_{k=0}^{\infty} (a_k+b_k)$ converges.
2. If $\sum_{k=0}^{\infty} a_k$ converges, then $\sum_{k=0}^{\infty} \alpha a_k$ converges for each real number α.

   Moreover, if $\sum_{k=0}^{\infty} a_k = L$, then $\sum_{k=0}^{\infty} \alpha a_k = \alpha L$

<proof>

Let $s_n = \sum_{k=0}^{n} a_k$, $t_n = \sum_{k=0}^{n} b_k$, $u_n = \sum_{k=0}^{n} (a_k+b_k)$. $v_n = \sum_{k=0}^{n} \alpha a_k$

$\Rightarrow u_n = s_n + t_n$ and $v_n = \alpha s_n$

If $s_n \longrightarrow L$ and $t_n \longrightarrow M$ as $n \to \infty$

$\Rightarrow u_n \longrightarrow L+M$ and $v_n \longrightarrow \alpha L$ as $n \to \infty$. //

Theorem 1.3

If $\sum_{k=0}^{\infty} a_k$ converges, then $a_k \to 0$ as $k \to \infty$.

<proof>

Let $S_n = \sum_{k=0}^{n} a_k$ and $S_n \longrightarrow L$ as $n \to \infty$

Note that $S_{n-1} \longrightarrow L$ as $n \to \infty$

$\Rightarrow S_n - S_{n-1} \longrightarrow L - L = 0$ as $n \to \infty$ (by thm 1.2)

$\Rightarrow a_n \to 0$ as $n \to \infty$.

Remark

If $a_k \not\to 0$, then $\sum_{k=0}^{\infty} a_k$ diverges

"NOTE" Thm 1.3 "DOES NOT" say that, if $a_k \to 0$, then $\sum_{k=0}^{\infty} a_k$ converges.

Examples

1. $\sum_{k=0}^{\infty} \frac{k}{k+1}$

$\frac{k}{k+1} \longrightarrow \underset{0}{\overset{1}{\neq}}$ as $k \to \infty$ $\Rightarrow \sum_{k=0}^{\infty} \frac{k}{k+1}$ diverges.

2. $\sum_{k=0}^{\infty} \sin k$

$\sin k \not\to 0$ as $k \to \infty$, then $\sum_{k=0}^{\infty} \sin k$ diverges

3. $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$

$a_k = \frac{1}{\sqrt{k}} \longrightarrow 0$ as $k \to \infty$

$S_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} \geq \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \dots + \frac{1}{\sqrt{n}} = \frac{n}{\sqrt{n}} + \sqrt{n} \to \infty$ as $n \to \infty$

$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges

§2 The integral test, Basic comparison, Limit comparison

The root test, The ratio test

<u>Theorem 2.1</u> (The Integral Test)

If f is continuous, positive, and decreasing on $[1,\infty)$, then

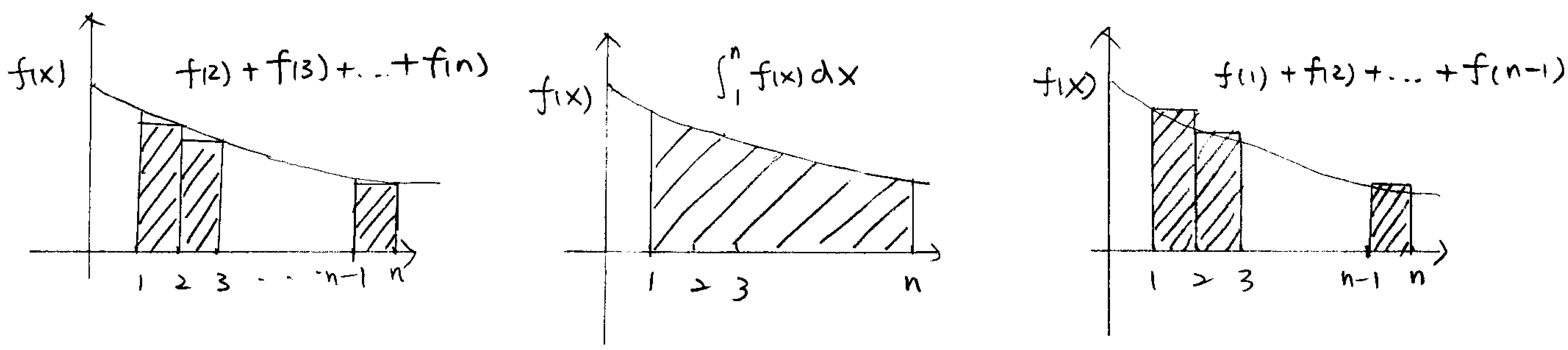
$$\sum_{k=1}^{\infty} f(k) \text{ converges} \iff \int_1^{\infty} f(x)\,dx \text{ converges}$$

<proof>

Fact

f is continuous, positive, and decreasing on $[1,\infty)$, then

$\int_1^{\infty} f(x)\,dx$ converges $\iff$ the sequence $a_n = \int_1^n f(x)\,dx$ converges



$(\Rightarrow)$ $\sum_{k=1}^{\infty} f(k) = L$

$$\int_1^n f(x)\,dx \le f(1) + f(2) + \dots + f(n-1) = \sum_{k=1}^{n-1} f(k)$$

$\Rightarrow \int_1^{\infty} f(x)\,dx \le \sum_{k=1}^{\infty} f(k) = L \;\rightarrow\; \int_1^{\infty} f(x)\,dx$ converges

$(\Leftarrow)$ $\int_1^{\infty} f(x)\,dx = L$

$$\sum_{k=1}^{n} f(k) = f(1) + \sum_{k=2}^{n} f(k) \le f(1) + \int_1^n f(x)\,dx$$

$\Rightarrow \sum_{k=1}^{\infty} f(k) \le f(1) + \int_1^{\infty} f(x)\,dx = f(1) + L \;\Rightarrow\; \sum_{k=1}^{\infty} f(k)$ converges

Examples

1. $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges

Let $f(x) = \frac{1}{x}$ is continuous, positive, and decreasing on $[1, \infty)$.

$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{x}\,dx$ diverges

By Integral test, $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges

2. $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges $\Leftrightarrow$ $p > 1$

If $p \le 0$, $\frac{1}{k^p} \not\to 0$ as $k \to \infty$ $\Rightarrow$ $\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges.

Enough to consider $p > 0$.

Let $f(x) = \frac{1}{x^p}$, continuous, positive, and descreasing on $[1, \infty)$

By integral test,

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \text{ converges} \Leftrightarrow \int_1^{\infty} \frac{1}{x^p}\,dx \text{ converges} \Leftrightarrow p > 1$$

3. $\sum_{k=1}^{\infty} \frac{1}{k \cdot \ln(k+1)}$ diverges.

Let $f(x) = \frac{1}{x \ln(x+1)}$, continuous, positive, and decreasing on $[1, \infty)$

We can use the integral test.

First, we observe that

$$\int_1^b \frac{1}{x \ln(x+1)}\,dx > \int_1^b \frac{1}{(x+1)\ln(x+1)}\,dx$$

$$= \ln(\ln(x+1))\Big|_1^b = \ln(\ln(b+1)) - \ln(\ln 2) \longrightarrow \infty \text{ as } b \to \infty$$

Hence $\int_1^{\infty} \frac{1}{x \ln(x+1)}\,dx$ diverges.

$\Rightarrow$ $\sum_{k=1}^{\infty} \frac{1}{k \ln(k+1)}$ diverges.

Remark

- For each $j \geq 0$, $\sum_{k=0}^{\infty} a_k$ converges $\Leftrightarrow$ $\sum_{k=j}^{\infty} a_k$ converges
- If $\sum a_k$ & $\sum b_k$ both converges $\Rightarrow$ $\sum (a_k + b_k)$ converges.
- For each $\alpha \neq 0$, $\sum \alpha a_k$ converges $\Leftrightarrow$ $\sum a_k$ converges

Theorem 2.2 (The Basic Comparison Theorem)

Suppose that $\sum a_k$ & $\sum b_k$ are series with nonnegative terms and $a_k \leq b_k$ for all k sufficiently large.

(i) If $\sum b_k$ converges, then $\sum a_k$ converges

(ii) If $\sum a_k$ diverges, then $\sum b_k$ diverges.

<proof> exercise.

Examples

1. $\sum \frac{1}{2k^3+1}$ converges

$\frac{1}{2k^3+1} < \frac{1}{2k^3} < \frac{1}{k^3}$ for all $k > 1$ and $\sum \frac{1}{k^3}$ converges

By comparison thm, $\sum \frac{1}{2k^3+1}$ converges.

2. $\sum \frac{k^3}{k^5+5k^4+7}$ converges

$\frac{k^3}{k^5+5k^4+7} < \frac{k^3}{k^5} = \frac{1}{k^2}$ for all $k>1$ and $\sum \frac{1}{k^2}$ converges

By thm 2.2, $\sum \frac{k^3}{k^5+5k^4+7}$ converges.

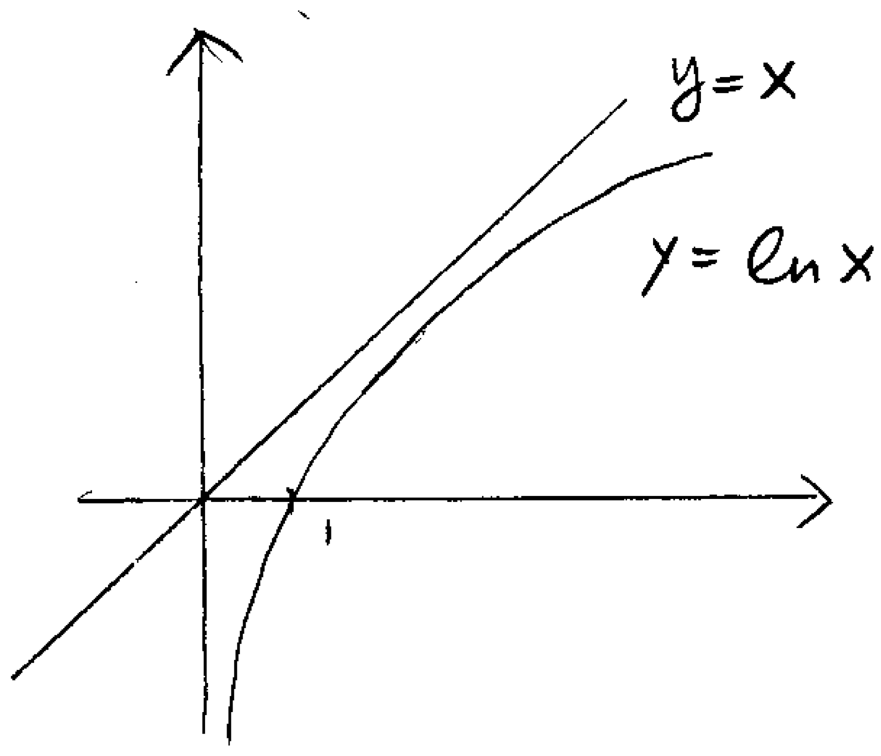
3. $\sum \frac{1}{3k+1}$ diverges

$\frac{1}{4k} = \frac{1}{3k+k} < \frac{1}{3k+1}$ for all $k \geq 1$

$\sum \frac{1}{4k} = \frac{1}{4} \sum \frac{1}{k}$ diverges

By Thm 2.2, $\sum \frac{1}{3k+1}$ diverges

4. $\sum \frac{1}{\ln(k+6)}$ diverges



Hence $\ln x < x$ for $x > 1$

consider $\ln(k+6) < \ln(2k) < 2k$ for $k > 6$

$\Rightarrow \frac{1}{\ln(k+6)} > \frac{1}{2k}$ for $k > 6$ and $\sum \frac{1}{2k}$ diverges.

By thm 2.2, $\sum \frac{1}{\ln(k+6)}$ diverges.

Theorem 2.3 (The Limit Comparison Theorem)

Let $\sum a_k$, $\sum b_k$ be series with positive terms. If $\frac{a_k}{b_k} \to L$ as $k \to \infty$ and $L > 0$, then ($\sum a_k$ converges $\Leftrightarrow$ $\sum b_k$ converges.)

<proof>

Let $0 < \varepsilon < L$.

Since $\frac{a_k}{b_k} \to L \Rightarrow \left| \frac{a_k}{b_k} - L \right| < \varepsilon$ for k sufficiently large

$\Rightarrow -\varepsilon < \frac{a_k}{b_k} - L < \varepsilon \Rightarrow L - \varepsilon < \frac{a_k}{b_k} < L + \varepsilon \Rightarrow b_k(L-\varepsilon) < a_k < b_k(L+\varepsilon)$

($\Rightarrow$) If $\sum a_k$ converges $\Rightarrow \sum b_k(L-\varepsilon)$ converges $\Rightarrow \sum b_k$ converges.

($\Leftarrow$) If $\sum b_k$ converges $\Rightarrow \sum b_k(L+\varepsilon)$ converges $\Rightarrow \sum a_k$ converges. //

Examples

Determine the following series converge or diverge

1. $\sum_{k=1}^{\infty} \frac{1}{5^k - 3}$

$$\frac{\frac{1}{5^k-3}}{\frac{1}{5^k}} = \frac{5^k}{5^k-3} = \frac{1}{1-\frac{3}{5^k}} \longrightarrow 1 \quad \text{as} \quad k \to \infty$$

Since $\sum \frac{1}{5^k}$ converges.

By thm 2.3, $\sum_{k=1}^{\infty} \frac{1}{5^k-3}$ converges

2. $\sum \frac{3k^2+2k+1}{k^3+1}$

Observe that $\frac{3k^2+2k+1}{k^3+1}$ differs little from $\frac{3k^2}{k^3} = \frac{3}{k}$

$$\frac{\frac{3k^2+2k+1}{k^3+1}}{\frac{3}{k}} = \frac{3k^2+2k+1}{k^3+1} \cdot \frac{k}{3} = \frac{3k^3+2k^2+k}{3k^3+3} = \frac{3+\frac{2}{k}+\frac{1}{k^2}}{3+\frac{3}{k^3}} \longrightarrow 1$$

as $k \to \infty$.

Since $\sum \frac{1}{k}$ diverges $\Rightarrow \sum \frac{3}{k}$ diverges

By thm 2.3, $\sum \frac{3k^2+2k+1}{k^3+1}$ diverges.

3. $\sum \frac{2k+5}{\sqrt{k^6+3k^3}}$

$$\frac{\frac{2k+5}{\sqrt{k^6+3k^3}}}{\frac{2}{k^2}} = \frac{2k+5}{\sqrt{k^6+3k^3}} \cdot \frac{k^2}{2} = \frac{2k^3+5k^2}{2\sqrt{k^6+3k^3}} = \frac{2+5\cdot\frac{1}{k}}{2\sqrt{1+3\cdot\frac{1}{k^3}}} \longrightarrow 1 \quad \text{as} \quad k \to \infty$$

Since $\sum \frac{1}{k^2}$ converges $\Rightarrow \sum \frac{2}{k^2}$ converges

By thm 2.3, $\sum \frac{2k+5}{\sqrt{k^6+3k^3}}$ converges.

Theorem 2.4 (The Root Test)

Let $\sum a_k$ be a series with nonnegative terms and suppose that

$$(a_k)^{\frac{1}{k}} \longrightarrow \rho \quad \text{as} \quad k \to \infty.$$

(i) If $\rho < 1$, then $\sum a_k$ converges

(ii) If $\rho > 1$, then $\sum a_k$ diverges

(iii) If $\rho = 1$, then the test is inconclusive.

<proof>

(i) Suppose $\rho < 1$ and choose μ, so that $\rho < \mu < 1$

$\because (a_k)^{\frac{1}{k}} \to \rho$ as $k \to \infty$

$\therefore (a_k)^{\frac{1}{k}} < \mu$ for k large enough.

$\Rightarrow a_k < \mu^k$ for all k large enough

Since $\sum \mu^k$ converges, $(\mu < 1)$

By comparison thm, $\sum a_k$ converges

(ii) Suppose $\rho > 1$ and choose μ, so that $\rho > \mu > 1$

$\because (a_k)^{\frac{1}{k}} \to \rho$ as $k \to \infty$

$\therefore (a_k)^{\frac{1}{k}} > \mu$ for all k large enough

$\Rightarrow a_k > \mu^k$ for all k large enough

Since $\sum \mu^k$ diverges $(\mu > 1)$

By comparison thm, $\sum a_k$ diverges.

(iii) Consider $\sum \frac{1}{k}$ & $\sum \frac{1}{k^2}$

$\left(\frac{1}{k}\right)^{\frac{1}{k}} \to 1$ as $k \to \infty$

$\left(\frac{1}{k^2}\right)^{\frac{1}{k}} \to 1$ as $k \to \infty$

$\sum \frac{1}{k}$ diverges and $\sum \frac{1}{k^2}$ converges.

Examples

1. $\sum \frac{1}{(\ln k)^k}$

$$\left(\frac{1}{(\ln k)^k}\right)^{\frac{1}{k}} = \frac{1}{\ln k} \to 0 \quad \text{as } k \to \infty$$

$\Rightarrow \sum \frac{1}{(\ln k)^k}$ converges.

2. $\sum \frac{2^k}{k^3}$

$$\left(\frac{2^k}{k^3}\right)^{\frac{1}{k}} = 2 \cdot \left(\frac{1}{k^3}\right)^{\frac{1}{k}} = 2 \cdot \left(\frac{1}{k^{\frac{1}{k}}}\right)^3 \to 2 \cdot 1^3 = 2 > 1$$

$\Rightarrow \sum \frac{2^k}{k^3}$ diverges.

3. $\sum \left(1-\frac{1}{k}\right)^k$

$$\left(\left(1-\frac{1}{k}\right)^k\right)^{\frac{1}{k}} = 1-\frac{1}{k} \to 1 \quad \text{as } k \to \infty \quad , \text{ inconclusive.}$$

Note that $\left(1-\frac{1}{k}\right)^k \to \underset{\neq 0}{\frac{1}{e}}$ as $k \to \infty$

$\Rightarrow \sum \left(1-\frac{1}{k}\right)^k$ diverges.

<u>Theorem 2.4</u> (The Ratio Test)

Let $\sum a_k$ be a series with positive terms and suppose that

$$\frac{a_{k+1}}{a_k} \to \lambda \quad \text{as } k \to \infty$$

(i) If $\lambda < 1$, then $\sum a_k$ converges

(ii) If $\lambda > 1$, then $\sum a_k$ diverges

(iii) If $\lambda = 1$, then the test is inconclusive.

<proof>

(i) Suppose $\lambda < 1$ and choose μ so that $\lambda < \mu < 1$.

$\Rightarrow \frac{a_{k+1}}{a_k} < \mu$ for k large enough, say $k \geq k_0$

$\Rightarrow a_{k_0+1} < \mu a_{k_0}$, $a_{k_0+2} < \mu a_{k_0+1} < \mu^2 a_{k_0}$, ..., $a_{k_0+j} < \mu^j a_{k_0}$

For any $k > k_0$, let $j = k - k_0$

$$a_k < \mu^j a_{k_0} = \mu^{k-k_0} \cdot a_{k_0} = \frac{a_{k_0}}{\mu^{k_0}} \cdot \mu^k$$

$$\Sigma \frac{a_{k_0}}{\mu^{k_0}} \cdot \mu^k = \frac{a_{k_0}}{\mu^{k_0}} \Sigma \mu^k \text{ converges } (\mu < 1)$$

By the comparison thm , Σa_k converges.

(ii) Suppose $\lambda > 1$ and choose μ so that $1 < \mu < \lambda$

$\Rightarrow \mu < \frac{a_{k+1}}{a_k}$ for k large enough , say $k \geq k_0$

$\Rightarrow \mu \cdot a_k < a_{k+1}$

$a_{k_0+1} > \mu \cdot a_{k_0}$, $a_{k_0+2} > \mu \cdot a_{k_0+1} > \mu \mu \cdot a_{k_0} = \mu^2 a_{k_0}$, ..., $a_{k_0+j} > \mu^j a_{k_0}$

For any $k > k_0$, let $j = k - k_0$

$$a_k > \mu^j a_{k_0} = \frac{a_{k_0}}{\mu^{k_0}} \cdot \mu^k$$

$$\Sigma \frac{a_{k_0}}{\mu^{k_0}} \cdot \mu^k = \frac{a_{k_0}}{\mu^{k_0}} \Sigma \mu^k \text{ diverges } (\mu > 1)$$

By the comparison thm, Σa_k diverges.

(iii) Use $\Sigma \frac{1}{k}$ & $\Sigma \frac{1}{k^2}$ //

Examples

1. $\Sigma \frac{1}{k!}$

$$\frac{1}{(k+1)!} \Big/ \frac{1}{k!} = \frac{k!}{(k+1)!} = \frac{1}{k+1} \longrightarrow 0 \text{ as } k \to \infty$$

$\Rightarrow \Sigma \frac{1}{k!}$ converges.

2. $\Sigma \frac{k}{10^k}$

$$\frac{k+1}{10^{k+1}} \Big/ \frac{k}{10^k} = \frac{k+1}{k} \cdot \frac{10^k}{10^{k+1}} = \frac{k+1}{k} \cdot \frac{1}{10} \longrightarrow \frac{1}{10} \text{ as } k \to \infty$$

$\Rightarrow \Sigma \frac{k}{10^k}$ converges.

3. $\sum \frac{1}{2k+1}$

$$\frac{\frac{1}{2(k+1)+1}}{\frac{1}{2k+1}} = \frac{2k+1}{2k+3} \longrightarrow 1 \quad \text{as } k \to \infty \text{ , inconclusive.}$$

$\frac{1}{2k+1} > \frac{1}{3k}$ for $k > 1$ and $\sum \frac{1}{3k}$ diverges

By comparison thm, $\sum \frac{1}{2k+1}$ diverges.

§3 Absolute Convergence and Conditional convergence ; Alternating Series

Let $\sum a_k$ be a series with positive and negative terms.

Theorem 3.1

If $\sum |a_k|$ converges, then $\sum a_k$ converges

<proof>

Note that $-|a_k| \le a_k \le |a_k|$, $\forall k$

$\Rightarrow 0 \le a_k + |a_k| \le 2|a_k|$, $\forall k$

If $\sum |a_k|$ converges, then $\sum 2|a_k|$ converges.

By comparison thm, $\sum (a_k + |a_k|)$ converges

$\sum (a_k + |a_k|) - \sum |a_k| = \sum a_k$ converges

converges

//.

* Series $\sum a_k$ for which $\sum |a_k|$ converges are called absolutely convergent

The theorem says that

$$\boxed{\text{absolutely convergent series are convergent}}$$

Examples

1. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$

consider $\sum_{k=1}^{\infty} \left|\frac{(-1)^{k+1}}{k^2}\right| = \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges

$\Rightarrow \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$ converges.

2. There are convergent series that are not absolutely convergent. Such series are called <u>conditionally convergent</u>.

$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ is convergent (See Next Thm).

$\sum_{k=1}^{\infty} \left|\frac{(-1)^{k-1}}{k}\right| = \sum_{k=1}^{\infty} \frac{1}{k}$ is divergent.

Hence $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$ is conditionally convergent.

A series of the form $\sum_{k=0}^{\infty} (-1)^k a_k$ is called an alternating series.

<u>Theorem 3.2</u>

Let $a_0, a_1, a_2, \dots$ be a decreasing sequence of positive numbers.

$\sum_{k=0}^{\infty} (-1)^k a_k$ converges $\Leftrightarrow$ $a_k \to 0$.

<proof>

($\Rightarrow$) $\sum_{k=0}^{\infty} (-1)^k a_k$ converges $\Rightarrow (-1)^k a_k \to 0$ as $k \to \infty$

$\Rightarrow a_k \to 0$ as $k \to \infty$.

($\Leftarrow$) Let $S_{2m} = \underbrace{(a_0 - a_1)}_{\vee\ 0} + \underbrace{(a_2 - a_3)}_{\vee\ 0} + \dots + \underbrace{(a_{2m-2} - a_{2m-1})}_{\vee\ 0} + \underbrace{a_m}_{\vee\ 0} > 0$

$\Rightarrow S_{2m+2} = S_{2m} - (a_{2m+1} - a_{2m+2})$ and $a_{2m+1} - a_{2m+2} > 0$

$\Rightarrow S_{2m+2} < S_{2m}$

Consider the sequence $\{S_{2m}\}_{m=0}^{\infty}$ is decreasing and $S_{2m} > 0, \forall m$

$\Rightarrow$ the sequence converges, say $S_{2m} \longrightarrow L$ as $m \to \infty$

Note that $S_{2m+1} = S_{2m} - a_{2m+1} \longrightarrow L - 0$ as $m \to \infty$

$\Rightarrow S_{2m+1} \longrightarrow L$ as $m \to \infty$

Hence $S_m \longrightarrow L$ as $m \to \infty$

$\Rightarrow \sum_{k=0}^{\infty} (-1)^k a_k = L$ converges //

Examples

1. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ and $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$ are convergent alternating series.

2. Give a numerical estimate for the sum of the series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}$$

correct to three decimal places.

<pf>

By Thm 3.2, the series is convergent, say $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} = L$

Let $S_n = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}$

For S_n to approximate L to three decimal places, we must have $|S_n - L| < 0.0005$

Observe the series

$$1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \dots = 1 - \frac{1}{6} + \frac{1}{120} - \frac{1}{5040} + \dots$$

$$\frac{1}{5040} < \frac{1}{5000} = 0.0002$$

$\Rightarrow S_2 = 1 - \frac{1}{3!} + \frac{1}{5!} = \frac{101}{120}$ approximate L to within 0.0005. //