Functions of several variables

§1 Elementary Examples

$f : D \longrightarrow \mathbb{R}$ where D is the domain of f.

1. $D = \{(x,y) \mid x, y \in \mathbb{R}\}$, the entire xy-plane

   $f(x,y) = x \cdot y$

2. $D = \{(x,y) \mid y \neq 0\}$

   $f(x,y) = \arctan\left(\frac{x}{y}\right)$

3. $D = \{(x,y) \mid x^2+y^2<1\}$, the open unit disk

   $f(x,y) = \dfrac{1}{\sqrt{1-(x^2+y^2)}}$

4. $D = \{(x,y,z) \mid x, y, z \in \mathbb{R}\}$

   $f(x,y,z) = x \cdot y \cdot z$

5. $D = \{(x,y,z) \mid x+y \neq z\}$

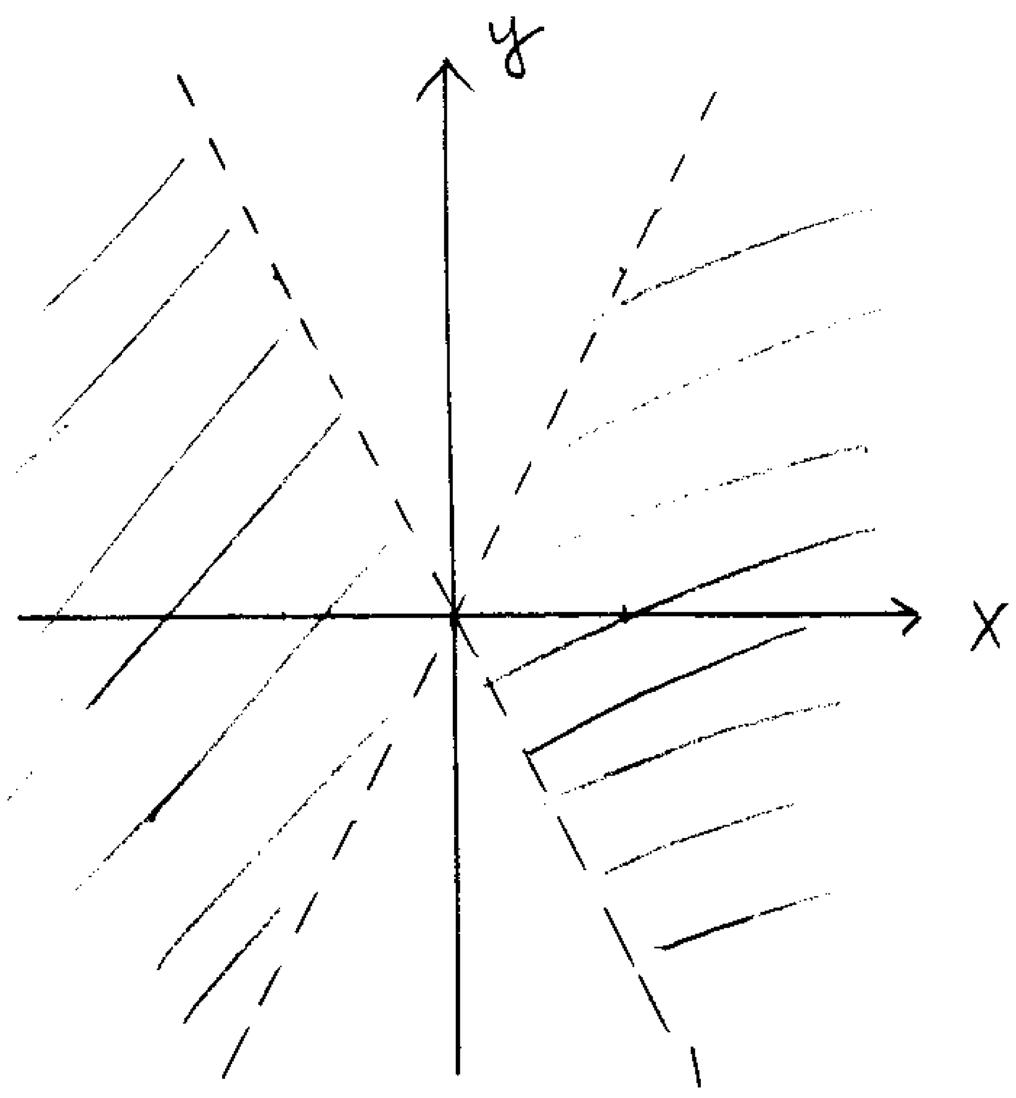
   $f(x,y,z) = \cos\left(\dfrac{1}{x+y-z}\right)$

6. $D = \{(x,y,z) \mid x^2+y^2+z^2<1\}$

   $f(x,y,z) = \dfrac{1}{\sqrt{1-(x^2+y^2+z^2)}}$

7. Find the domain and range of the function $f(x,y)=\frac{1}{\sqrt{4x^2-y^2}}$

For (x,y) in the domain of $f \iff 4x^2-y^2>0$



$\iff y^2<4x^2=(2x)^2$

$\iff -2|x|<y<2|x|$

Hence $\sqrt{4x^2-y^2}$ takes all positive values

$\Rightarrow$ range of $f=(0,\infty)$

§2 The Quadratic Surfaces ; Projections

$*\quad Ax^2+By^2+Cz^2+Dxy+Exz+Fyz+Hx+Iy+Jz+K=0$

Note A, B, C are not all zero !

xy, xz, yz can be eliminated by a suitable change of coordinates

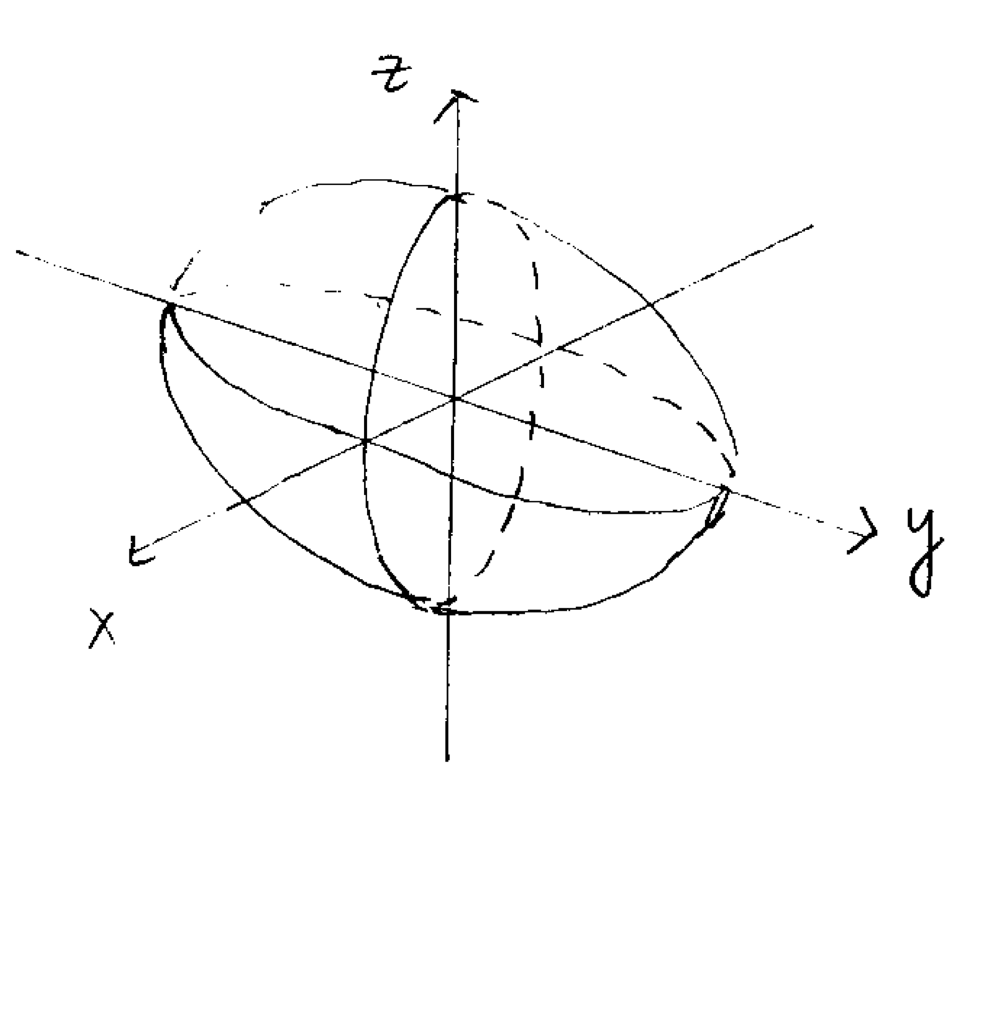
The quadratic surfaces are given by

$$Ax^2+By^2+Cz^2+Dx+Ey+Fz+H=0.$$

The quadratic surfaces fall into nine distinct types.

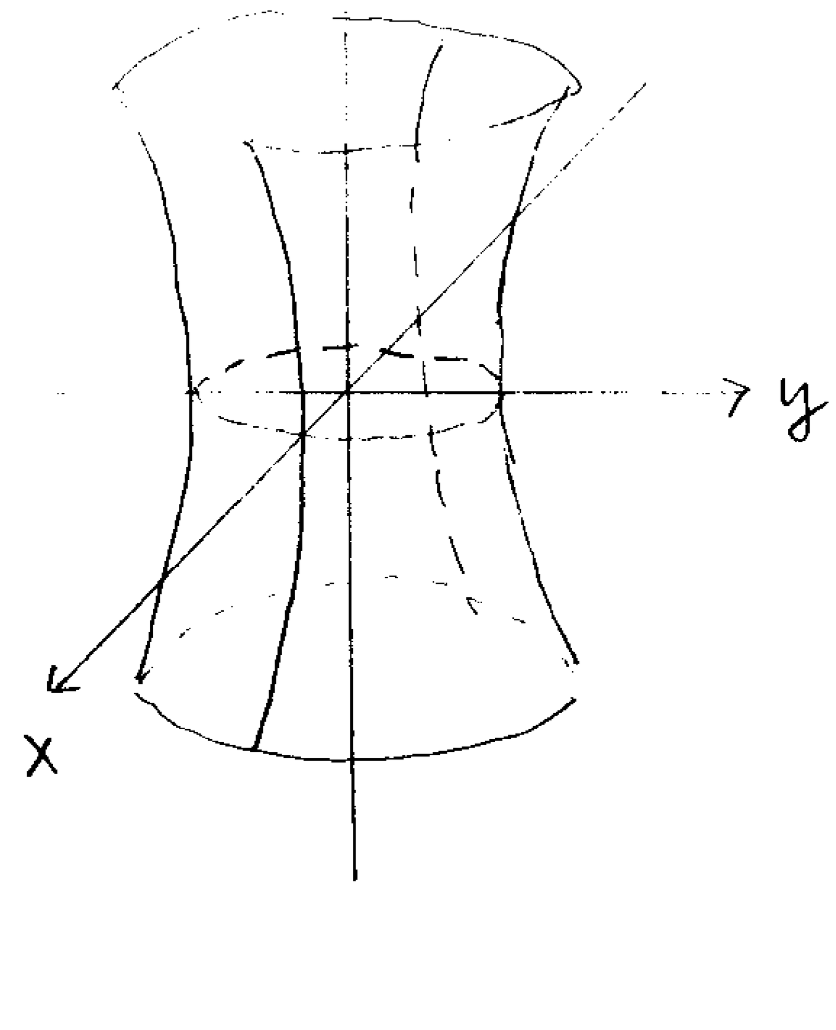
1. The ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$



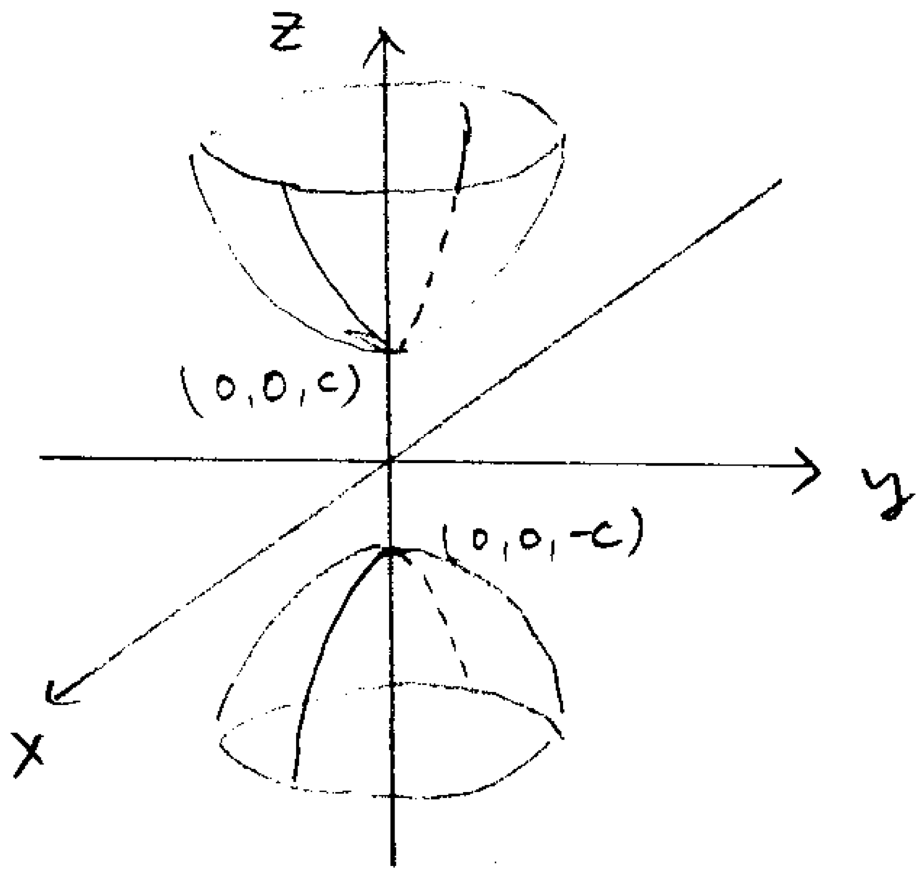
2. The hyperboloid of one sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$



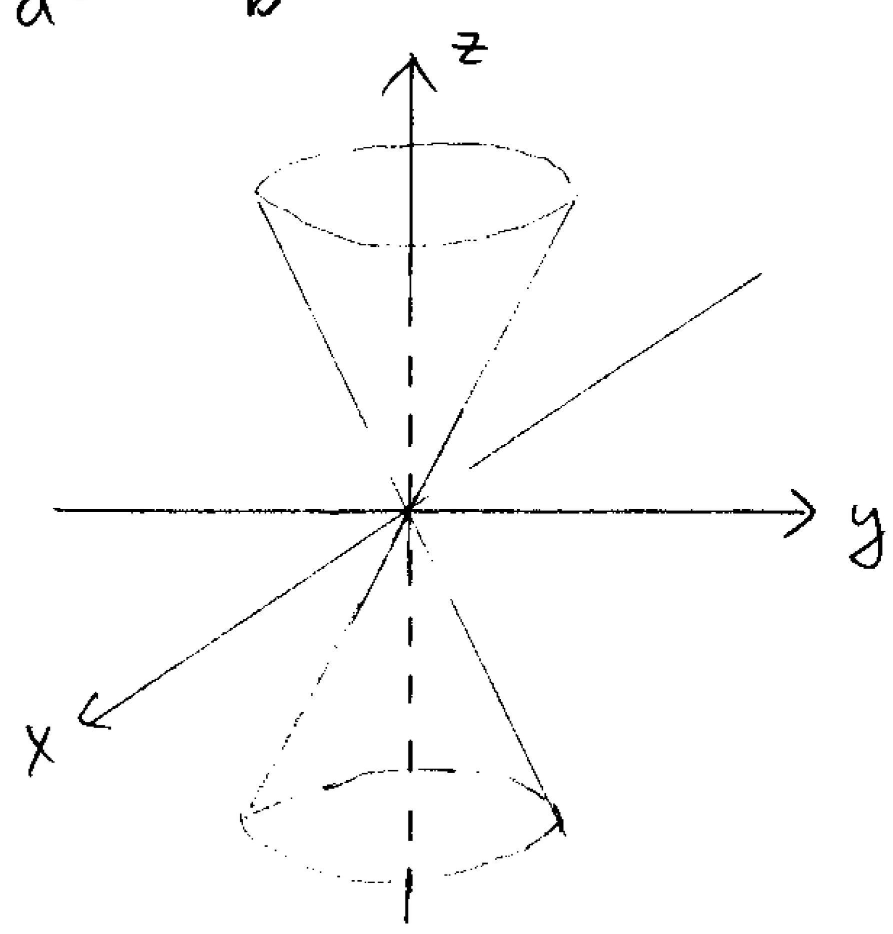
3. The hyperboloid of two sheets

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$$



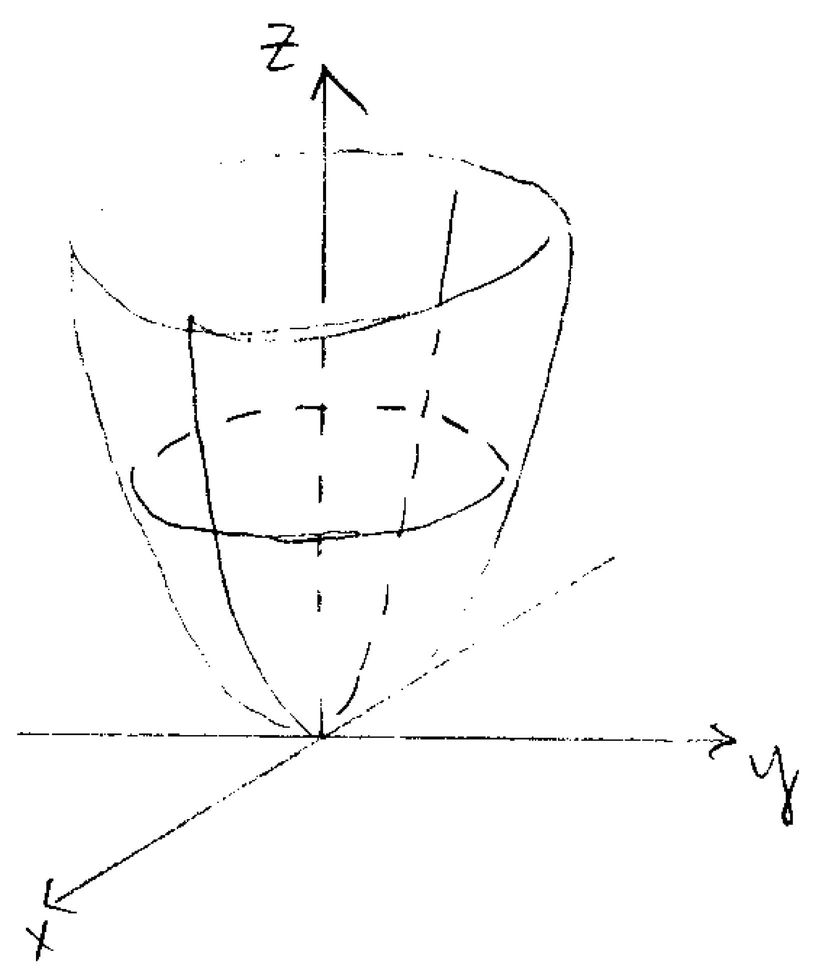
4. The elliptic cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=z^2$$



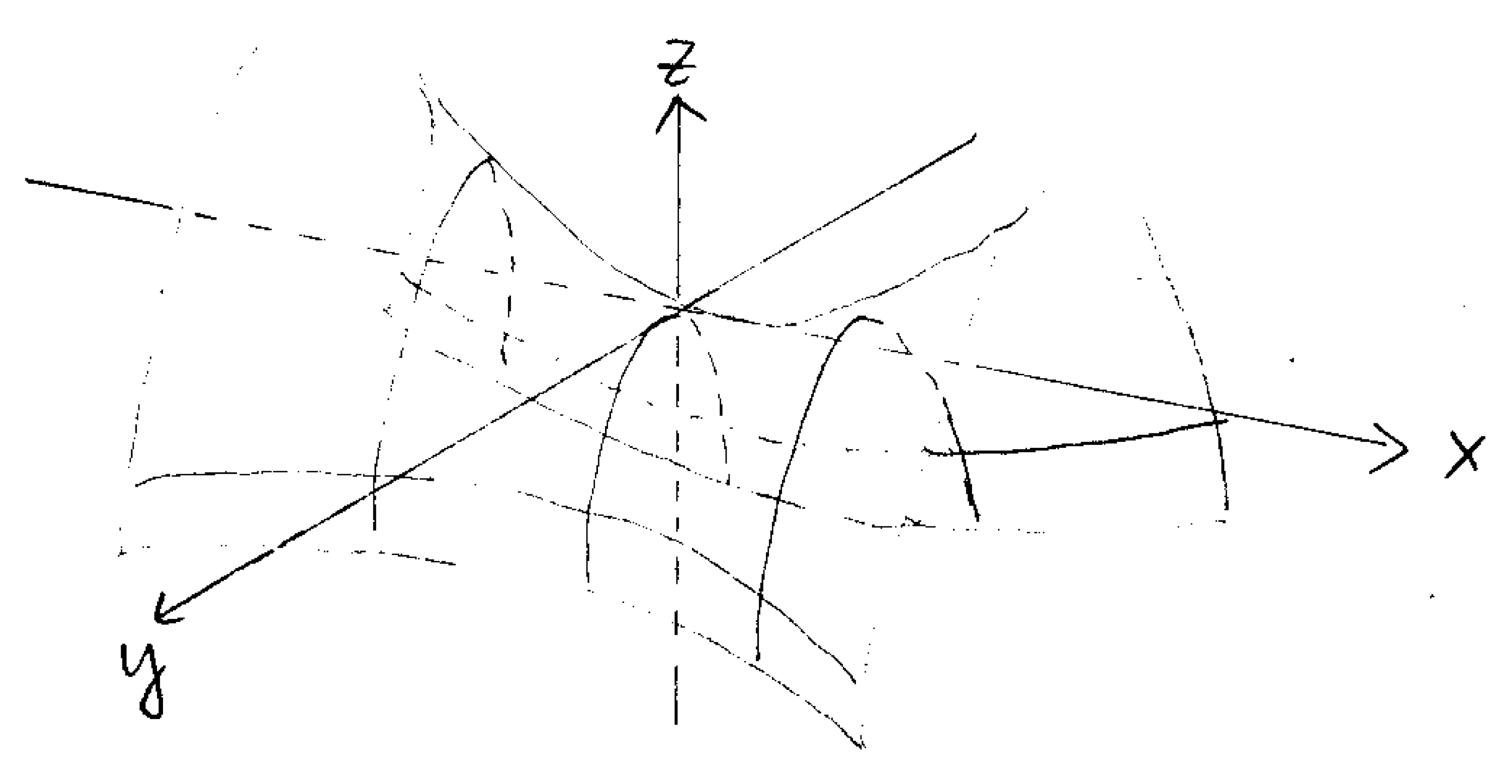
5. The elliptic paraboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=z$$



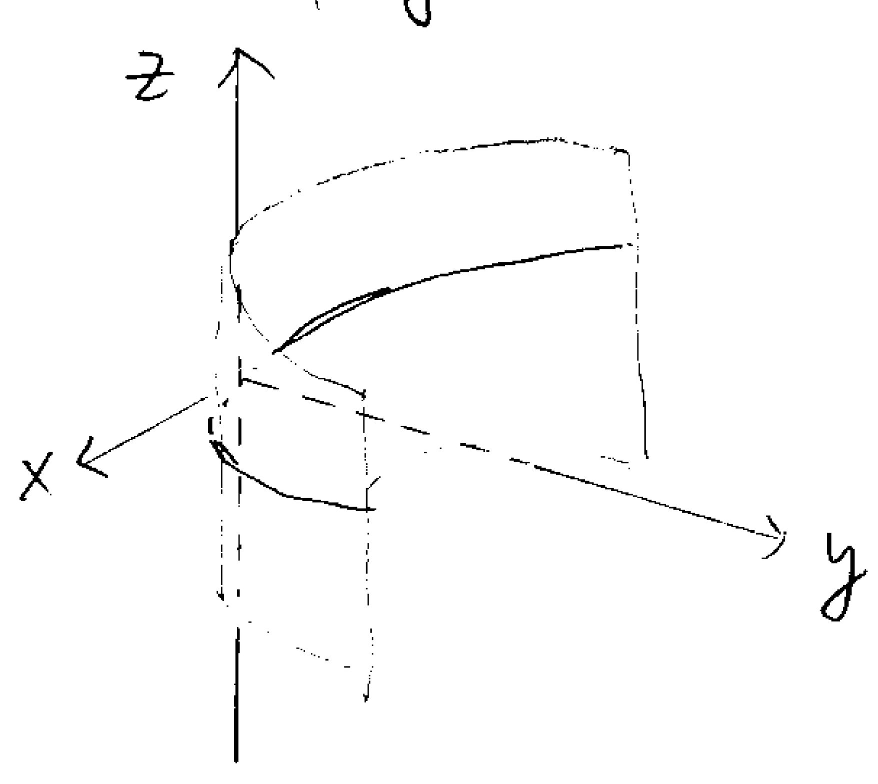
6. The hyperbolic paraboloid

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=z$$



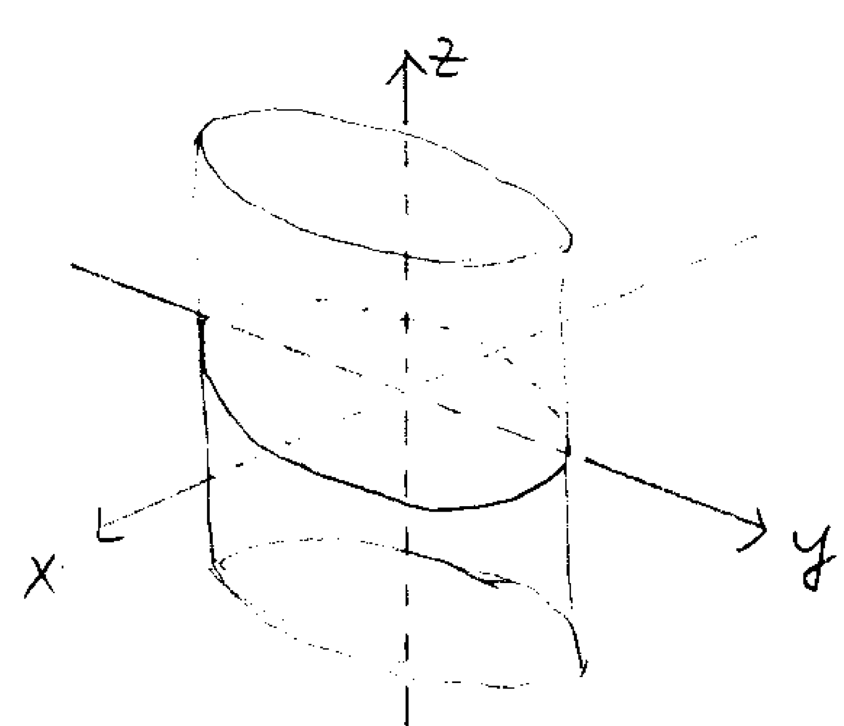
7. The parabolic cylinder

$$x^2=4cy$$



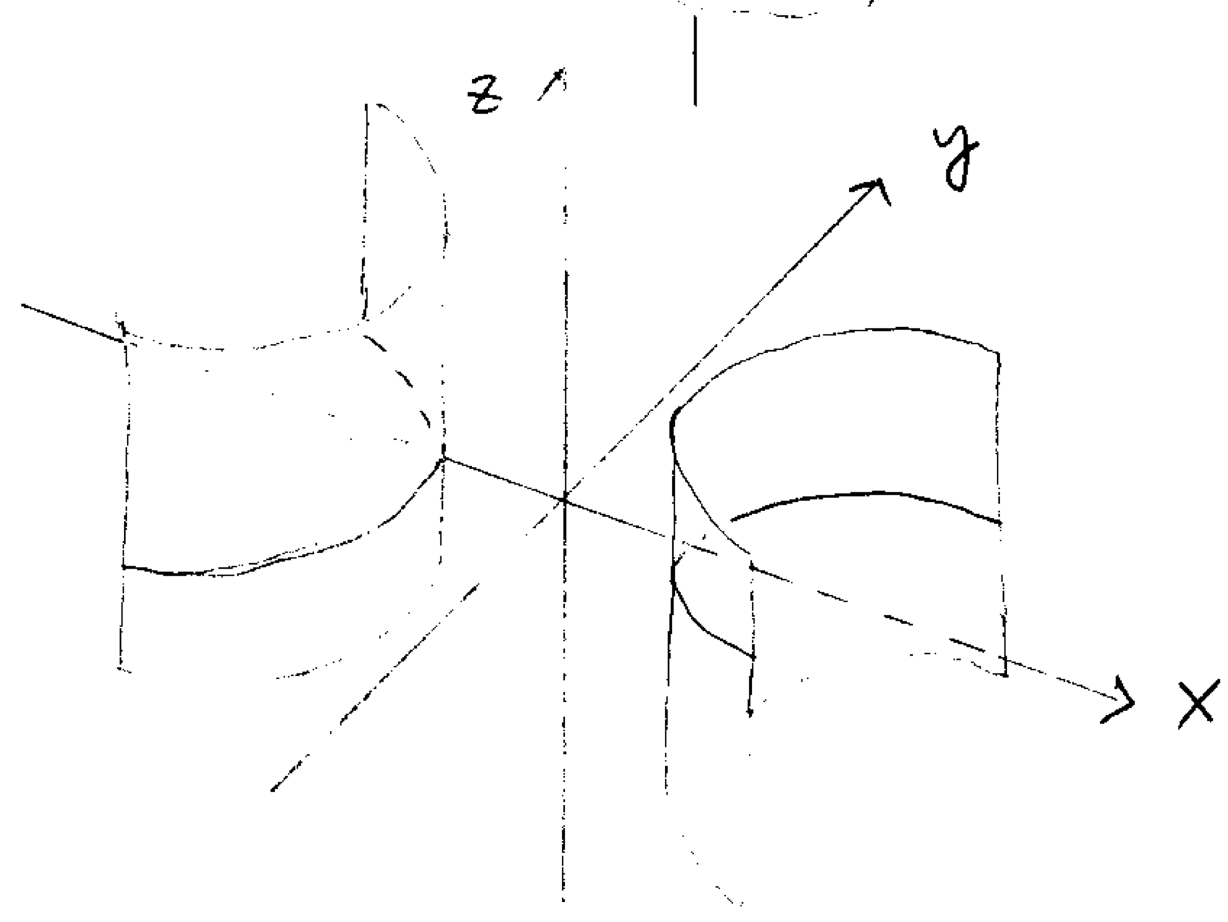
8. The elliptic cylinder

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



9. The hyperbolic cylinder

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

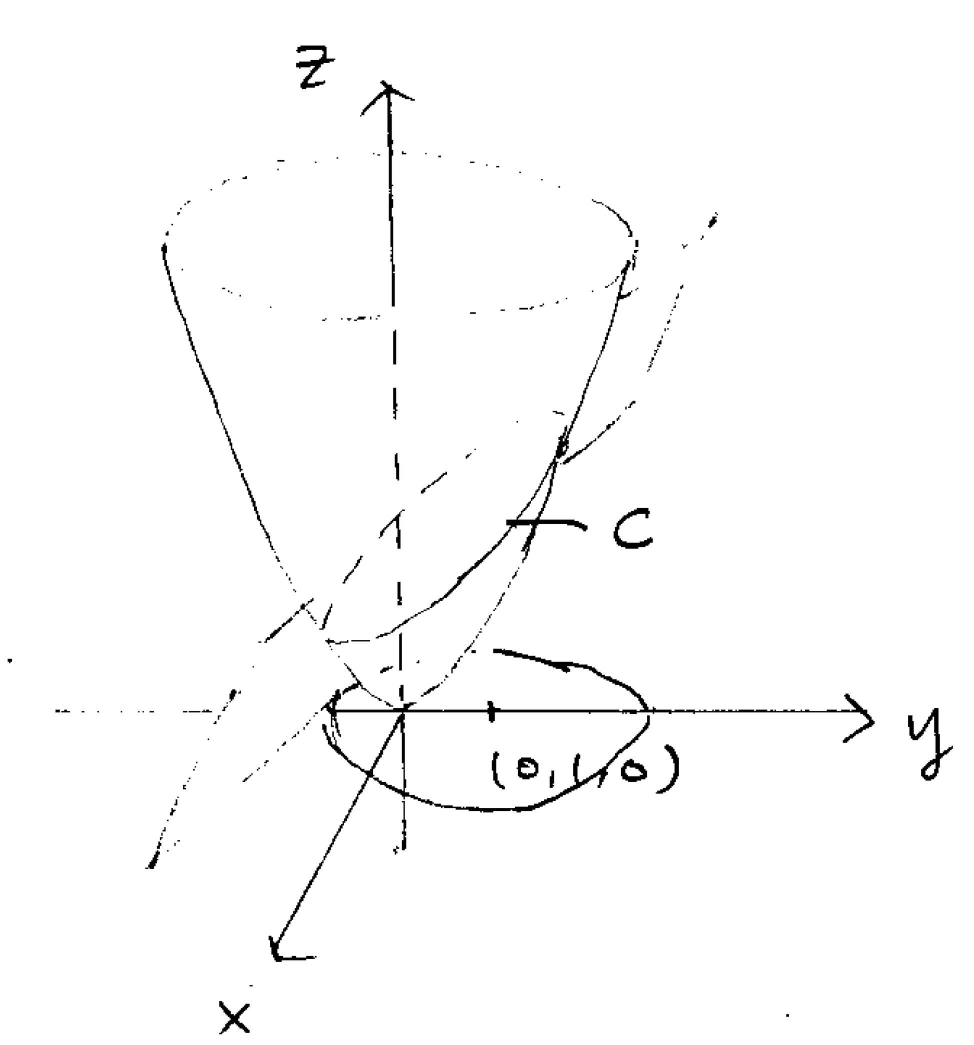


Let $S_1: z=f(x,y)$ and $S_2: z=g(x,y)$ are surfaces that intersect in a space curve C.

The set of all points $(x,y,0)$ with $f(x,y)=g(x,y)$ is called the projection of C onto the xy-plane.

Example

$z=x^2+y^2$ & $z=2y+3$



$x^2+y^2=2y+3 \Rightarrow x^2+y^2-2y=3$

$\Rightarrow x^2+(y-1)^2=4=2^2$

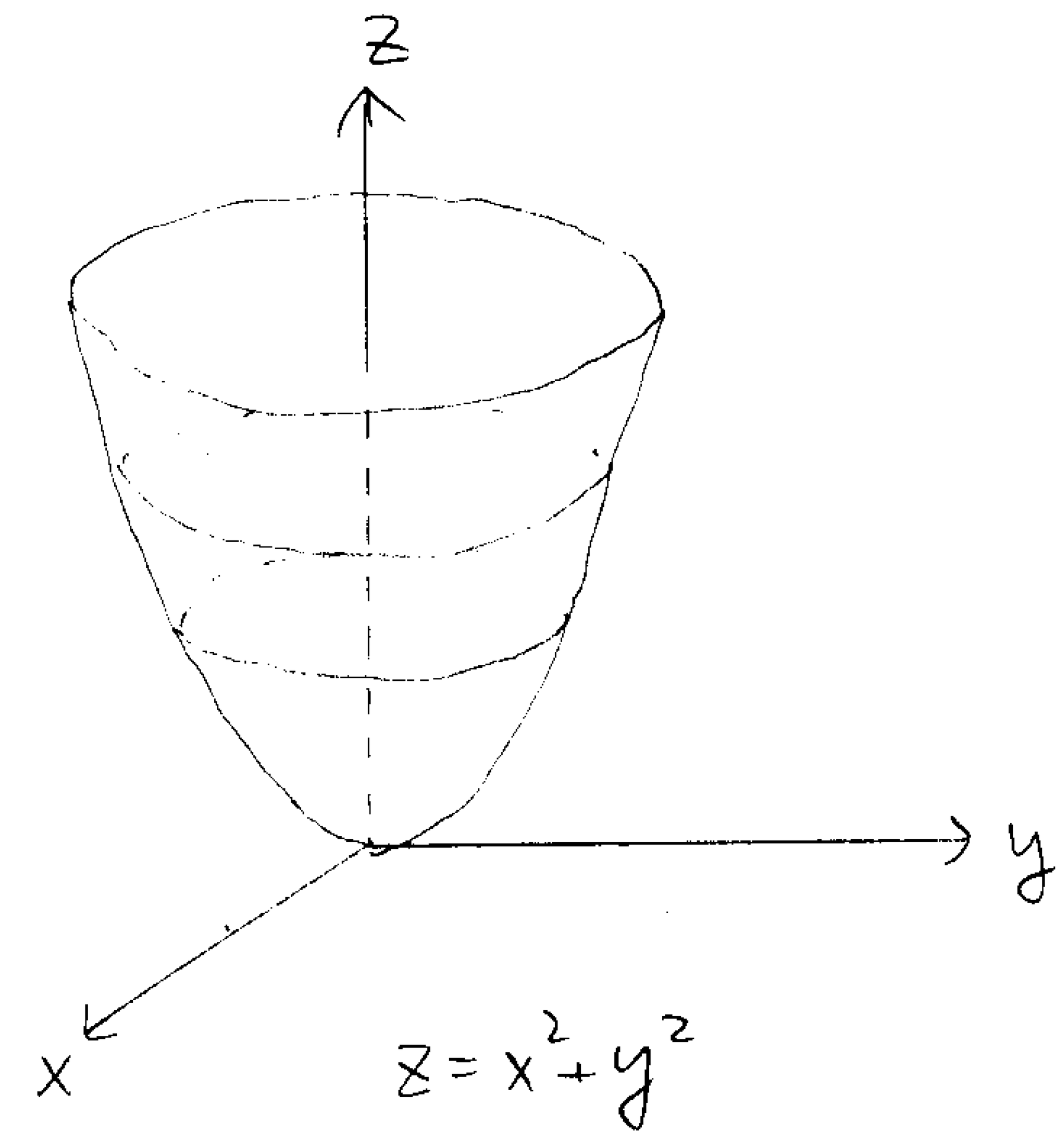
The projection of C onto the xy-plane is the circle of radius 2 centered at $(0,1,0)$

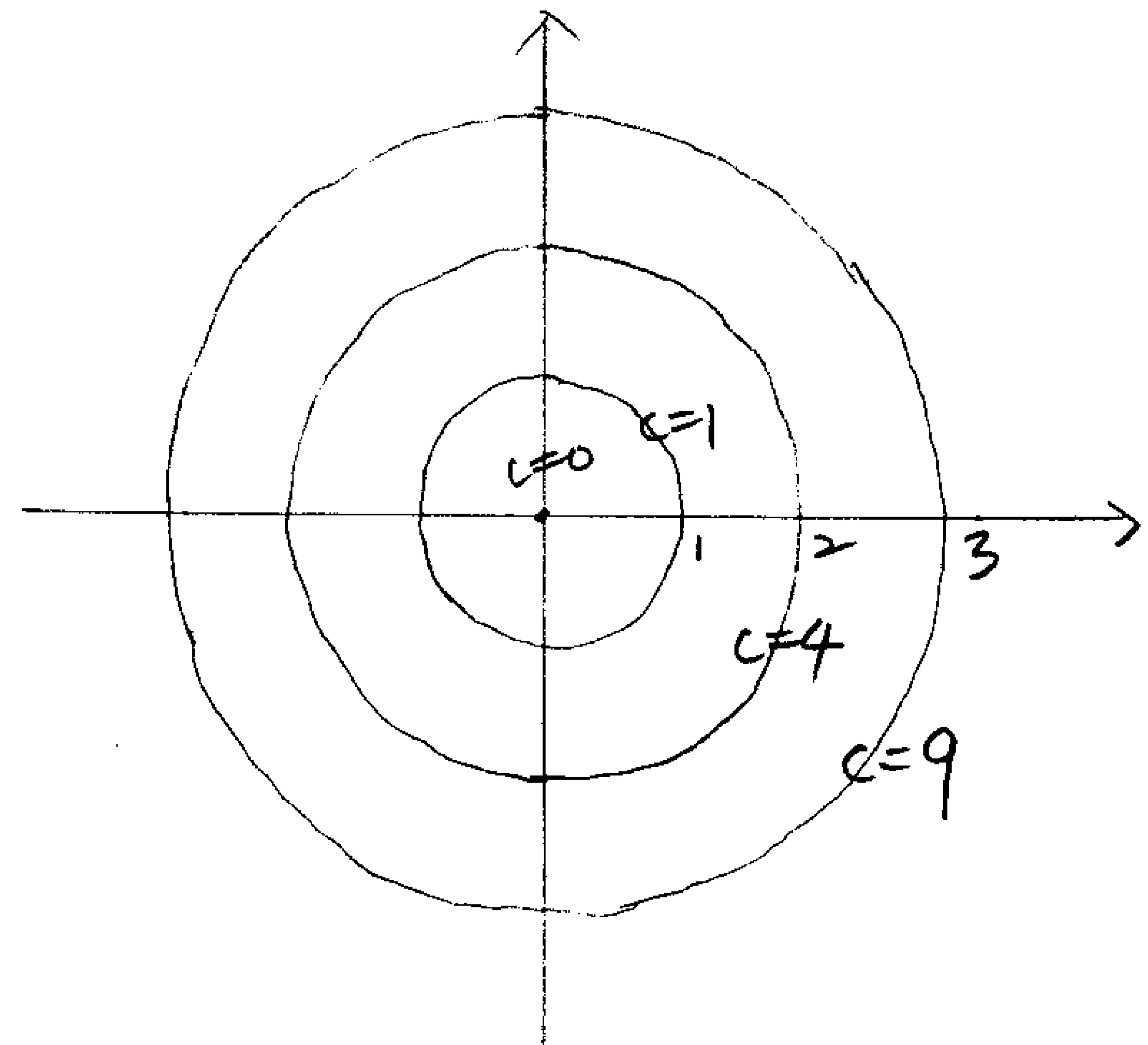
§3 Level curves & Level surfaces

The level curve $f(x,y)=c$ lies entirely in the domain of f

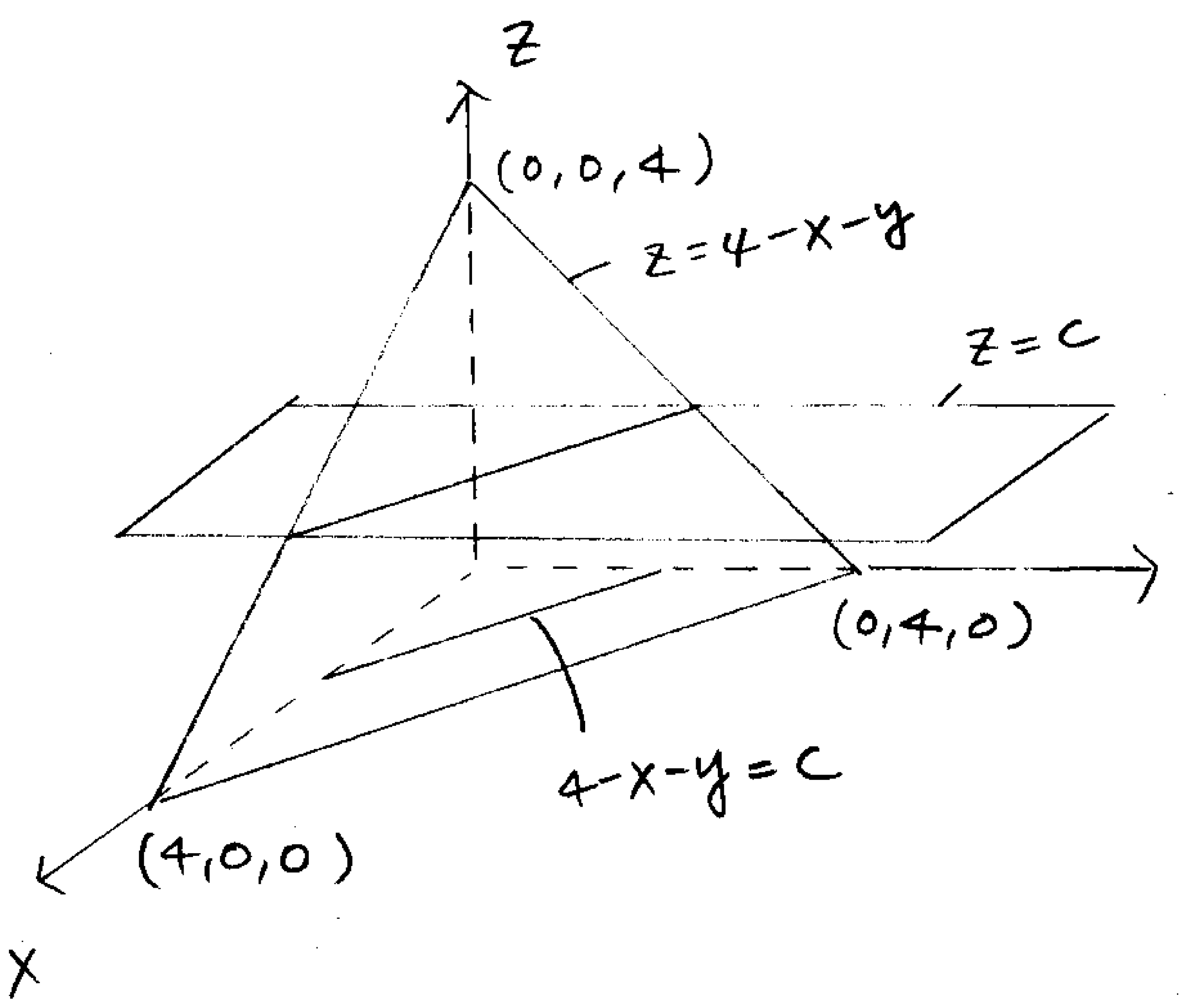
Example

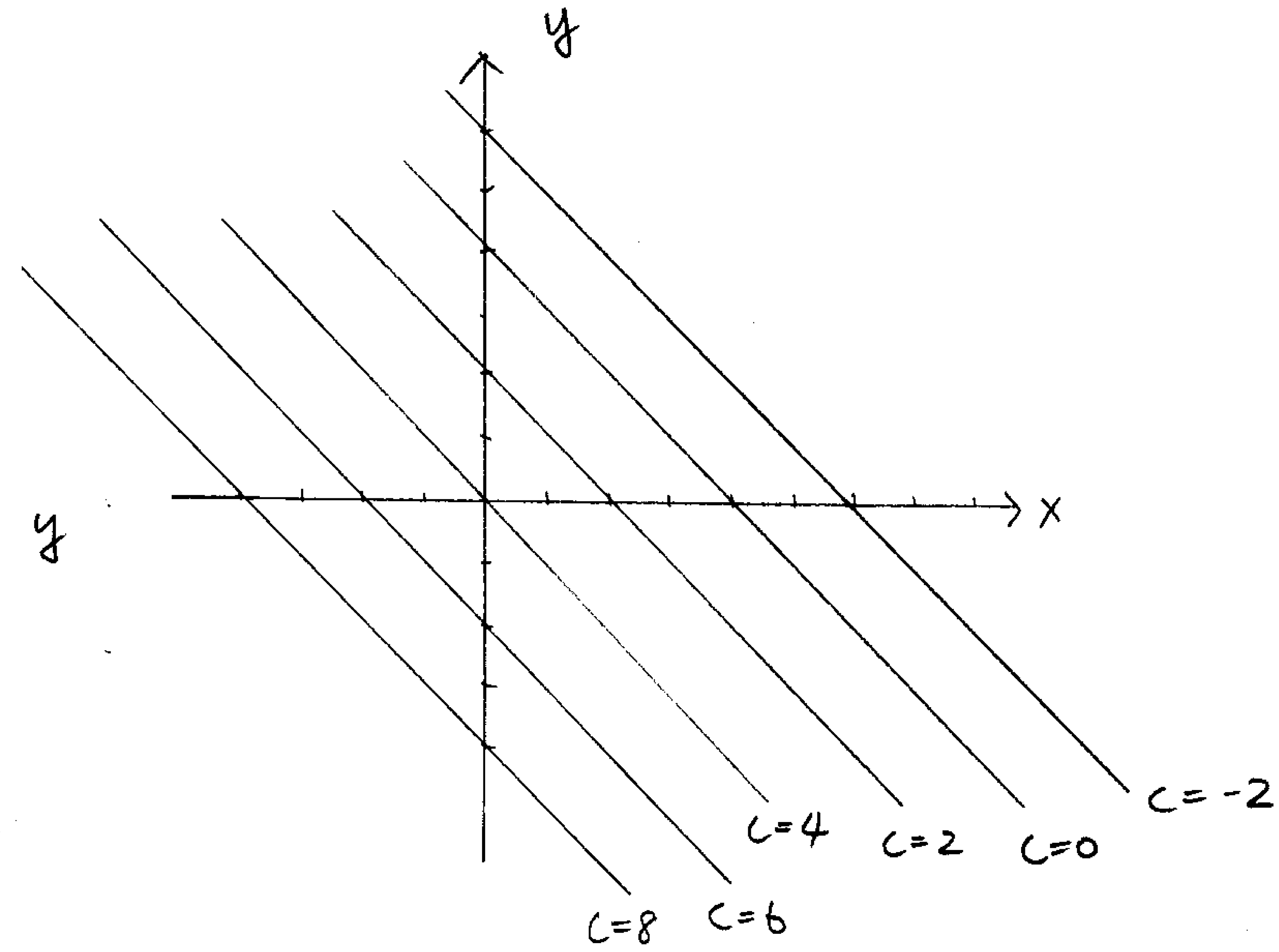
1. $f(x,y)=x^2+y^2$, The level curve are circles centered at O, $x^2+y^2=c$, $c \geq 0$





2. $g(x,y) = 4-x-y$ is a plane. The level curves are parallel lines of the form $4-x-y=c$.





3. $h(x,y) = \begin{cases} \sqrt{x^2+y^2} & , x \geq 0 \\ |y| & , x < 0 \end{cases}$

$h(x,y) = c$

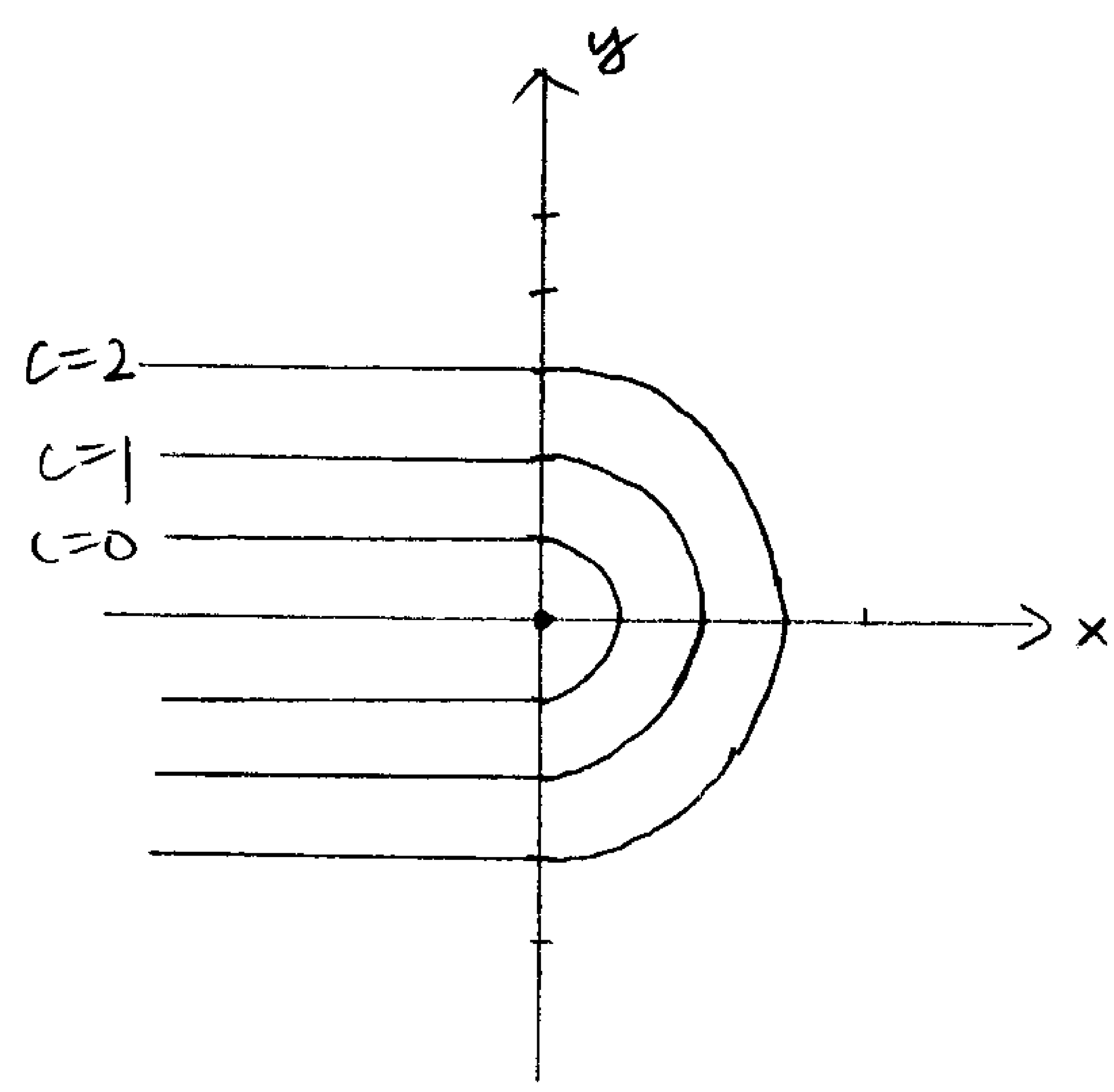
$c=0 \Rightarrow x, y = 0$

$c=1$, $x \geq 0$, $\sqrt{x^2+y^2} = 1$

$x<0$, $|y| = 1$

$c=2$, $x \geq 0$, $\sqrt{x^2+y^2} = 2$

$x<0$, $|y|=2$



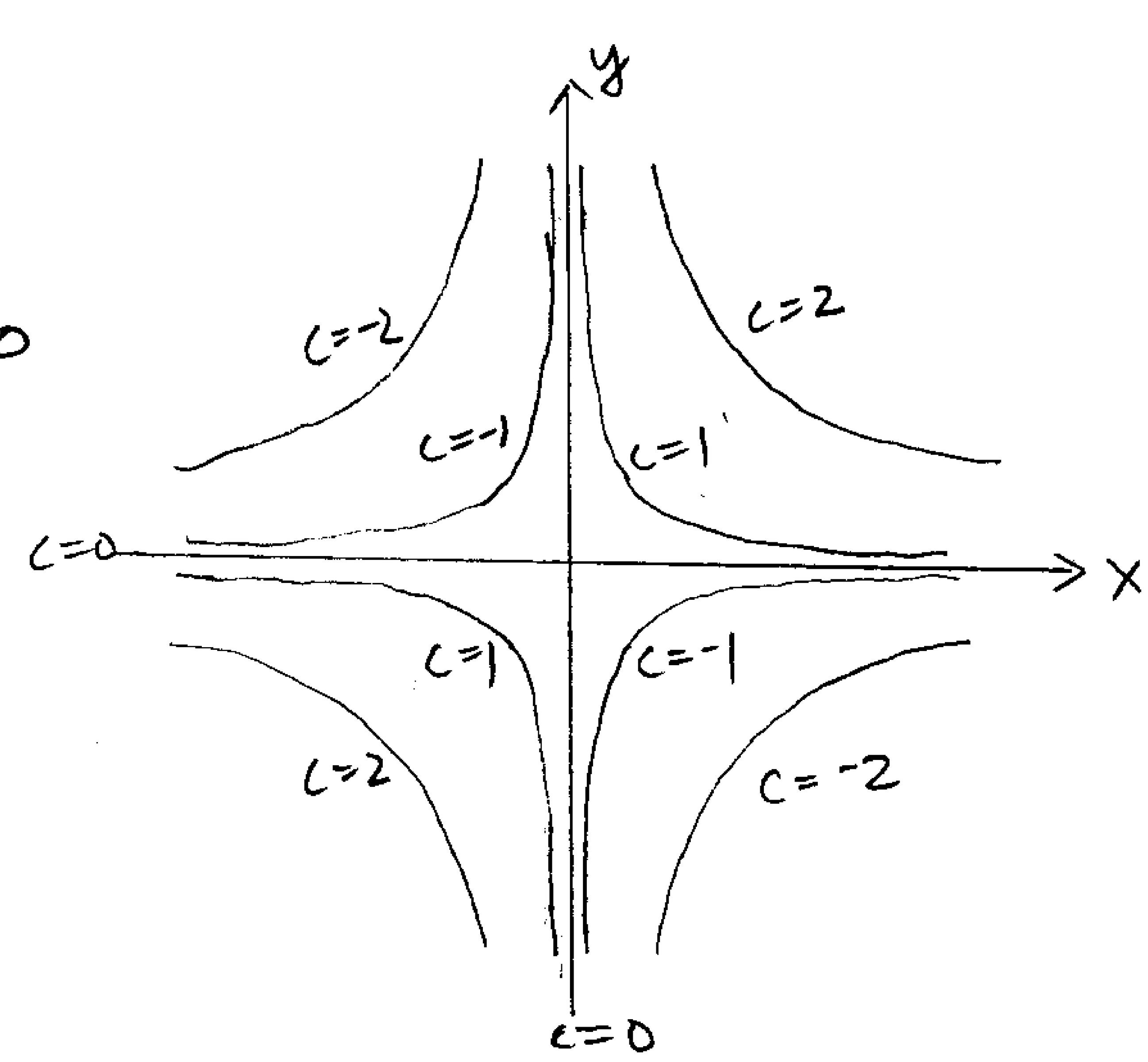
4. $f(x,y) = xy$

$c=0 \Rightarrow xy=0 \Rightarrow x=0$ or $y=0$

$\Rightarrow$ x-axis & y-axis

$c=1 \Rightarrow xy=1$

$c=-1 \Rightarrow xy=-1$



5. $f(x,y,z) = Ax + By + Cz$, the level surfaces are parallel planes $Ax+By+Cz = c$

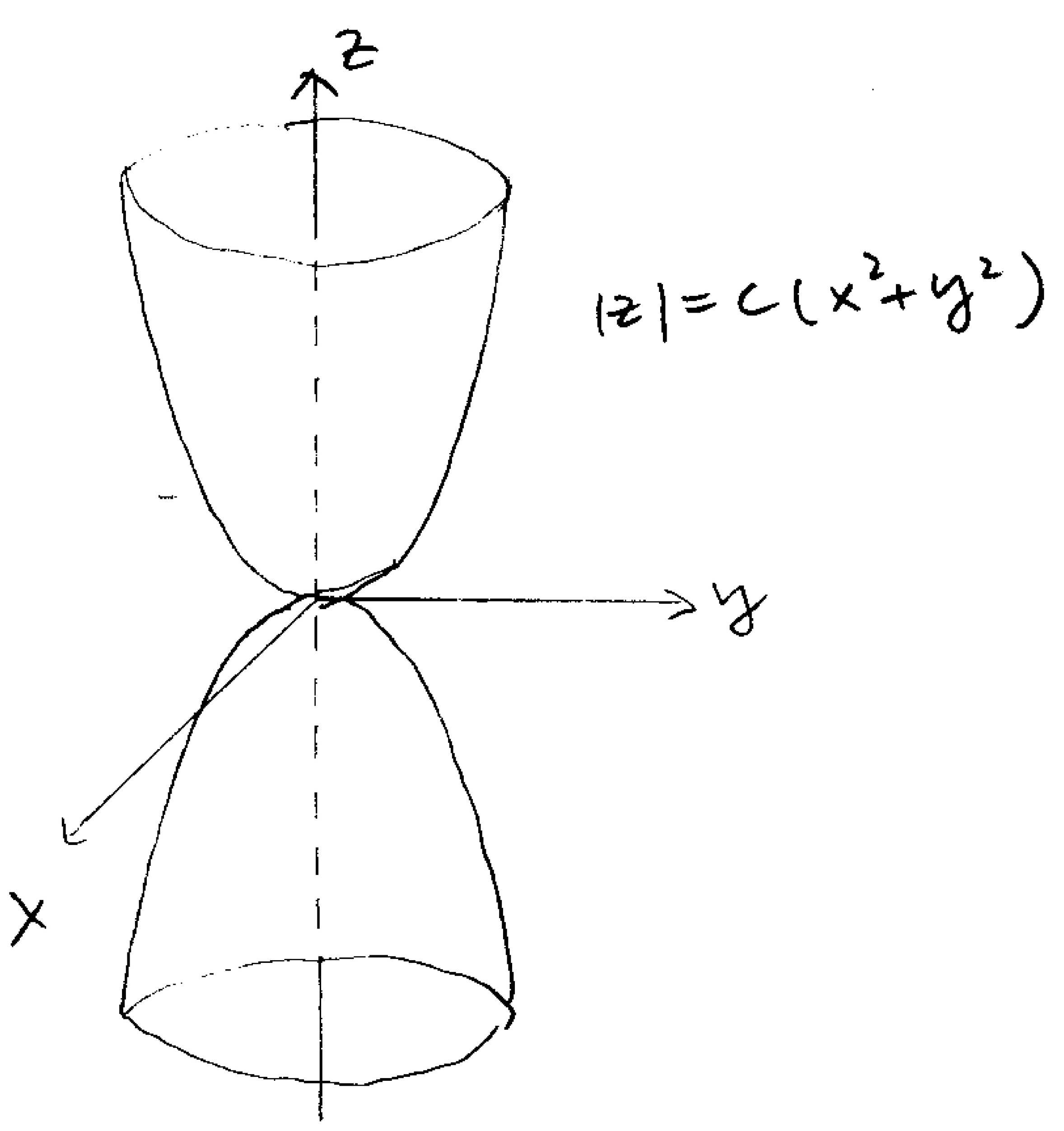
6. $g(x,y,z) = \sqrt{x^2+y^2+c^2}$, the level surfaces are concentric spheres $x^2+y^2+z^2 = c$

7. $f(x,y,z) = \begin{cases} \dfrac{|z|}{x^2+y^2} & , x^2+y^2 \neq 0 \\ 0 & , (x,y,z) = (0,0,0) \end{cases}$, At other points of z-axis

we leave f undefined.

$c = 0 \Rightarrow \begin{matrix} \dfrac{|z|}{x^2+y^2} = 0 & , x^2+y^2 \neq 0 \\ 0 & , (x,y,z) = 0 \end{matrix} \Rightarrow z = 0$ & (x,y) arbitrary. $\Rightarrow$ xy-plane.

For $c > 0$, $f(x,y,z) = c \Rightarrow \dfrac{|z|}{x^2+y^2} = c \Rightarrow c(x^2+y^2) = |z|$



§4 Partial Derivatives

Definition 4.1 (Partial derivatives in two variables)

f : function of two variables x, y.

f_x, f_y : the partial derivatives of f w.r.t x and w.r.t y.

$$f_x(x,y) \triangleq \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} , \quad f_y(x,y) \triangleq \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

provided these limit exist.

Examples

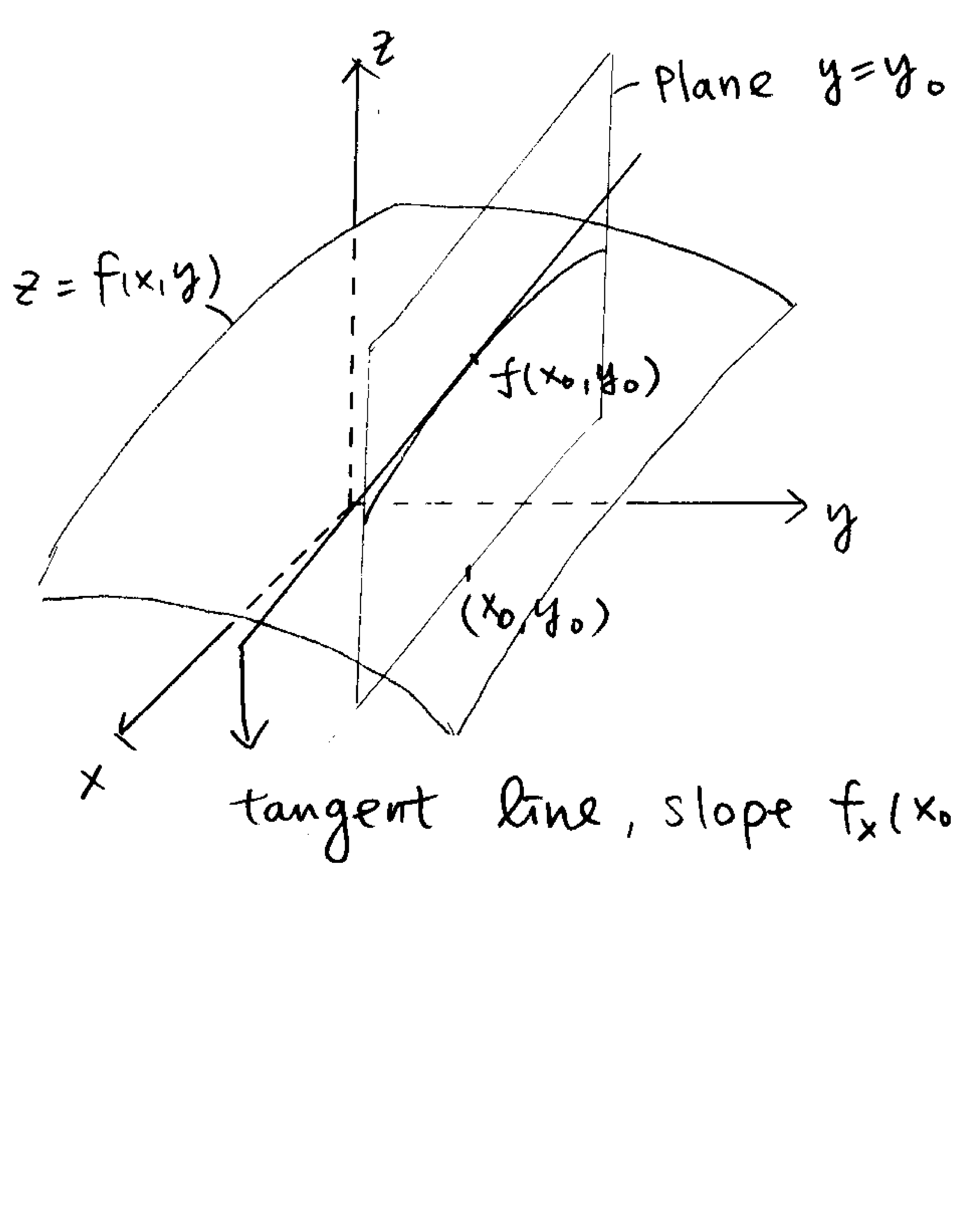
1. $f(x,y) = x\arctan(xy)$

$$f_x(x,y) = x\cdot\frac{y}{1+(xy)^2} + \arctan(xy) = \frac{xy}{1+(xy)^2} + \arctan(xy)$$

$$f_y(x,y) = x\cdot\frac{y}{1+(xy)^2} = \frac{x^2}{1+(xy)^2}$$

2. $f(x,y) = e^{xy} + \ln(x^2+y)$

$$f_x(x,y) = ye^{xy} + \frac{2x}{x^2+y} \quad \& \quad f_y(x,y) = xe^{xy} + \frac{1}{x^2+y}$$



Definition 4.2 (Partial derivatives in 3 variables)

f: function of 3 variables x, y, z

f_x, f_y, f_z: the partial derivatives of f with respect to x, y, z.

$$f_x(x,y,z) \triangleq \lim_{h\to 0} \frac{f(x+h,y,z)-f(x,y,z)}{h}$$

$$f_y(x,y,z) \triangleq \lim_{h\to 0} \frac{f(x,y+h,z)-f(x,y,z)}{h}$$

$$f_z(x,y,z) \triangleq \lim_{h\to 0} \frac{f(x,y,z+h)-f(x,y,z)}{h}$$

Provided these limit exist.

Examples

1. $f(x,y,z) = xy^2z^3$

$f_x(x,y,z) = y^2z^3$, $f_y(x,y,z) = 2xyz^3$, $f_z(x,y,z) = 3xy^2z^2$

2. $g(x,y,z) = x^2 e^{y/z}$

$g_x(x,y,z) = 2x\, e^{y/z}$, $g_y(x,y,z) = x^2 \cdot e^{y/z} \cdot \frac{1}{z} = \frac{x^2}{z} e^{y/z}$

$g_z(x,y,z) = x^2 \cdot e^{y/z} \cdot \left(\frac{-y}{z^2}\right) = \frac{-x^2y}{z^2} \cdot e^{y/z}$

Note:

We also denote f_x by $\frac{\partial f}{\partial x}$

f_y by $\frac{\partial f}{\partial y}$

f_z by $\frac{\partial f}{\partial z}$.

§5 Limits and continuity ; Equality of mixed partials.

In this section, we use $\underline{x}$ to denote a 2 or 3 variable point.

i.e. $\underline{x} = (x, y)$ or (x, y, z)

Definition 5.1

Let f be a function defined at least on some deleted neighborhood of $\underline{x}_0$.

$\lim_{\underline{x} \to \underline{x}_0} f(\underline{x}) = L$ provided that $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $0 < \|\underline{x} - \underline{x}_0\| < \delta$, then $|f(\underline{x}) - L| < \varepsilon$

Examples

1. $f(x, y) = \dfrac{xy + y^3}{x^2 + y^2}$, Find $\lim_{(x,y)\to(0,0)} f(x, y)$.

Note that f is not defined at $(0,0)$ but is defined for all $(x, y) \neq (0,0)$

Along x-axis, $y = 0$: $\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{x\to 0} f(x, 0) = \lim_{x \to 0} \dfrac{0}{x^2} = 0$

Along $y = 2x$, $\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{x\to 0} f(x, 2x) = \lim_{x\to 0} \dfrac{2x^2 + (2x)^3}{x^2 + (2x)^2}$

$$= \lim_{x\to 0} \frac{2x^2 + 8x^3}{x^2 + 4x^2} = \lim_{x \to 0} \frac{2 + 8x}{5} = \frac{2}{5}$$

Hence $\lim_{(x,y)\to(0,0)} f(x, y)$ doesn't exist.

2. $g(x, y) = \dfrac{x^2 y}{x^4 + y^2}$. Find $\lim_{(x,y)\to(0,0)} g(x, y)$

Along $y = mx$, $\lim_{(x,y)\to(0,0)} g(x, y) = \lim_{x\to 0} g(x, mx) = \lim_{x \to 0} \dfrac{x^2 (mx)}{x^4 + (mx)^2}$

$$= \lim_{x\to 0} \frac{mx^3}{x^4 + m^2x^2} = \lim_{x\to 0} \frac{mx}{x^2 + m^2} = 0$$

Along $y = x^2$, $\lim_{(x,y)\to(0,0)} g(x, y) = \lim_{x\to 0} g(x, x^2) = \lim_{x\to 0} \dfrac{x^2 \cdot x^2}{x^4 + x^4} = \lim_{x\to 0} \dfrac{x^4}{2x^4} = \dfrac{1}{2}$

Hence $\lim_{(x,y)\to(0,0)} g(x, y)$ doesn't exist.

Definition 5.2

f is continuous at $\underline{x}_0$ if $\lim_{\underline{x}\to\underline{x}_0} f(\underline{x}) = f(\underline{x}_0)$.

$$\left(\lim_{h\to 0} f(\underline{x}_0 + h) = f(\underline{x}_0) \right)$$

Theorem 5.3

If g is continuous at $\underline{x}_0$ and f is continuous at $g(\underline{x}_0)$, then $f \circ g$ is continuous at $\underline{x}_0$.

<proof>

$\forall \varepsilon > 0$

$\exists \delta_1 > 0$ s.t. if $|u - g(\underline{x}_0)| < \delta_1$, then $|f(u) - f(g(\underline{x}_0))| < \varepsilon$

$\exists \delta > 0$ s.t. if $\|\underline{x} - \underline{x}_0\| < \delta$, then $|g(\underline{x}) - g(\underline{x}_0)| < \delta_1$

$\Rightarrow |f(g(\underline{x})) - f(g(\underline{x}_0))| < \varepsilon$ //

Corollary 5.4

If $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$, then $\lim_{x\to x_0} f(x,y_0) = f(x_0,y_0)$ & $\lim_{y\to y_0} f(x_0,y) = f(x_0,y_0)$

Remark

For functions of several variables the existence of partial derivatives does not guarantee continuity.

Example

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

Since $f(x,0) = 0 = f(0,0)$ & $f(0,y) = 0 = f(0,0)$

$\Rightarrow f$ is continuous in x & continuous in y

Along $x=y$ $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{2x\cdot x}{x^2+x^2} = 1$

$\Rightarrow f$ is not continuous at $(0,0)$

Since both $f(x,0)$ & $f(0,y)$ are constantly zero $\Rightarrow$ both partials exist at $(0,0)$.

However, the function is discontinuous.

- The second-order partials

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \quad , \quad f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x}$$

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} \quad , \quad f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}.$$

Examples

1. $f(x,y) = \sin x^2 y$

$$\frac{\partial f}{\partial x} = 2xy\cos x^2 y \quad , \quad \frac{\partial f}{\partial y} = x^2 \cos x^2 y$$

$$\frac{\partial^2 f}{\partial x^2} = -4x^2y^2 \sin x^2 y + 2y\cos x^2 y \quad , \quad \frac{\partial^2 f}{\partial y\,\partial x} = -2x^3 y \sin x^2 y + 2x\cos x^2 y$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = -2x^3 y \sin x^2 y + 2x\cos x^2 y \quad , \quad \frac{\partial^2 f}{\partial y^2} = -x^4 \sin x^2 y.$$

2. $f(x,y) = \ln(x^2+y^3)$

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^3} \quad , \quad \frac{\partial f}{\partial y} = \frac{3y^2}{x^2+y^3}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{2(y^3-x^2)}{(x^2+y^3)^2} \quad , \quad \frac{\partial^2 f}{\partial y\,\partial x} = -\frac{6xy^2}{(x^2+y^3)^2}$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = -\frac{6xy^2}{(x^2+y^3)^2} \quad , \quad \frac{\partial^2 f}{\partial y^2} = \frac{3y(2x^2-y^3)}{(x^2+y^3)^2}.$$