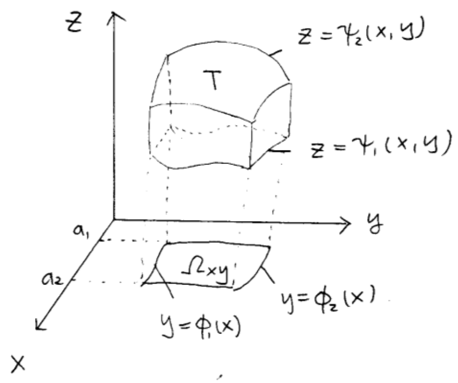
§4 Triple Integrals



$$T=\{(x,y,z) \mid (x,y)\in\Omega_{xy},\ \psi_1(x,y)\le z\le \psi_2(x,y)\}$$

$$\iiint_T f(x,y,z)\,dx\,dy\,dz=\iint_{\Omega_{xy}}\left(\int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)\,dz\right)dx\,dy$$

$$=\int_{a_1}^{a_2}\int_{\phi_1(x)}^{\phi_2(x)}\int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

Examples

1. $\displaystyle\int_0^2\int_0^x\int_0^{4-x^2} xyz\,dz\,dy\,dx$

$$=\int_0^2\int_0^x \frac{1}{2}xyz^2\Big|_0^{4-x^2}dy\,dx$$

$$=\int_0^2\int_0^x \frac{1}{2}xy(4-x^2)^2\,dy\,dx$$

$$=\int_0^2 \frac{1}{4}xy^2(4-x^2)^2\Big|_0^x dx$$

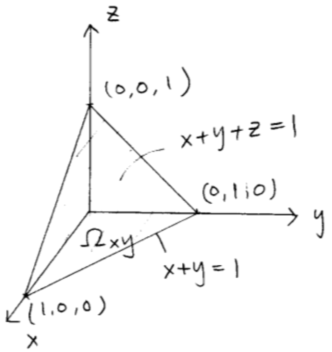
$$=\int_0^2 \frac{1}{4}x\cdot x^2(4-x^2)^2\,dx$$

$$=\int_0^2 \frac{1}{4}x^3(4-x^2)^2\,dx=\frac{1}{4}\int_0^2 16x^3-8x^5+x^7\,dx$$

$$=\frac{1}{4}\left(4x^4-\frac{8}{6}x^6+\frac{1}{8}x^8\Big|_0^2\right)=\frac{1}{4}\left(4\cdot16-\frac{8}{6}\cdot64+\frac{1}{8}\cdot2^8\right)$$

$$=\frac{8}{3}$$

2. Find the volume of T



The volume of T

$$= \iiint_T dx\,dy\,dz$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} z \Big|_0^{1-x-y} dy\,dx$$

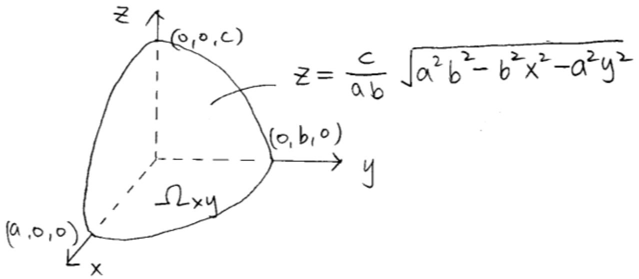
$$= \int_0^1 \int_0^{1-x} (1-x-y)\,dy\,dx$$

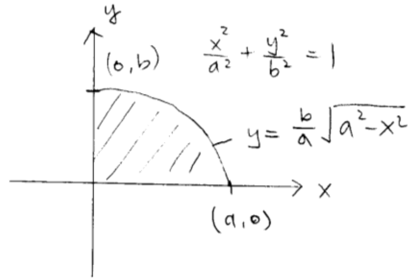
$$= \int_0^1 y - xy - \frac{1}{2}y^2 \Big|_0^{1-x} dx$$

$$= \int_0^1 (1-x) - x(1-x) - \frac{1}{2}(1-x)^2\,dx$$

$$= \int_0^1 \frac{1}{2}(1-x)^2\,dx = \frac{-1}{6}(1-x)^3\Big|_0^1 = \frac{1}{6}$$

3. Integrate $f(x,y,z) = yz$ over the first-octant solid bounded by the coordinates planes and the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$





$0 \le x \le a,\ 0 \le y \le \frac{b}{a}\sqrt{a^2 - x^2}$

$$\iiint_T f(x,y,z)\,dx\,dy\,dz = \iint_{\Omega_{xy}} \int_0^{\frac{c}{ab}\sqrt{a^2b^2 - b^2x^2 - a^2y^2}} yz\,dz\,dx\,dy$$

$$= \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \int_0^{\frac{c}{ab}\sqrt{a^2b^2-b^2x^2-a^2y^2}} yz\,dz\,dy\,dx$$

$$= \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \frac{1}{2}yz^2 \Big|_0^{\frac{c}{ab}\sqrt{a^2b^2-b^2x^2-a^2y^2}} dy\,dx$$

$$= \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \frac{1}{2}y \cdot \frac{c^2}{a^2b^2}(a^2b^2 - b^2x^2 - a^2y^2)\,dy\,dx$$

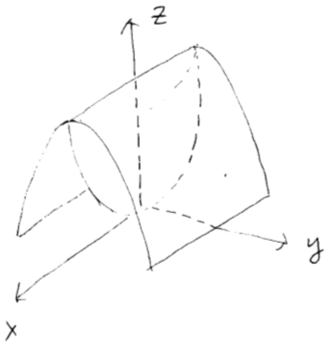
$$= \frac{1}{2}\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} c^2y - \frac{c^2}{a^2}yx^2 - \frac{c^2}{b^2}y^3\,dy\,dx$$

$$= \frac{1}{2}\int_0^a \frac{1}{2}c^2y^2 - \frac{1}{2}\frac{c^2}{a^2}y^2x^2 - \frac{1}{4}\frac{c^2}{b^2}y^4 \Big|_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a \frac{1}{2}c^2 \cdot \frac{b^2}{a^2}(a^2-x^2) - \frac{1}{2}\frac{c^2}{a^2} \cdot \frac{b^2}{a^2}(a^2-x^2) - \frac{1}{4} \cdot \frac{c^2}{b^2} \cdot \frac{b^4}{a^4}(a^2-x^2)^2\,dx$$

$$= \frac{1}{15}ab^2c^2$$

4. Use triple integral to find the volume of the solid T bounded above by the parabolic cylinder $z=4-y^2$ and bounded below by the elliptic paraboloid $z=x^2+3y^2$



$$\begin{cases} z=4-y^2 \\ z=x^2+3y^2 \end{cases} \Rightarrow 4-y^2=x^2+3y^2$$

$$\Rightarrow x^2+4y^2=4$$

$$T=\{(x,y,z) \mid (x,y)\in\Omega_{xy},\ x^2+3y^2\le z\le 4-y^2\}$$

where $\Omega_{xy}=\{(x,y) \mid x^2+4y^2\le 4\}$

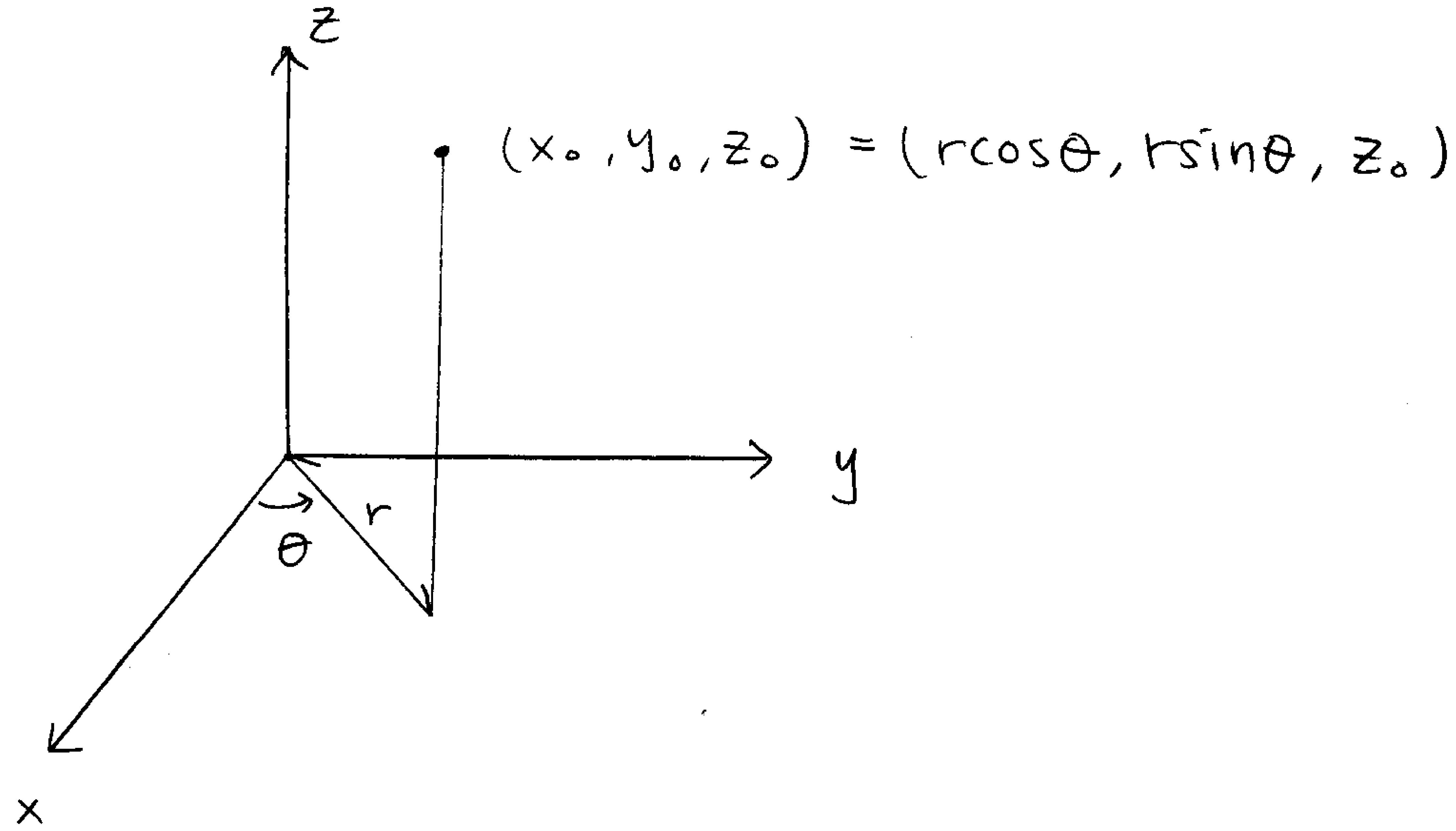
$$=\{(x,y) \mid -2\le x\le 2,\ -\tfrac{1}{2}\sqrt{4-x^2}\le y\le \tfrac{1}{2}\sqrt{4-x^2}\}$$

$$V=\iiint_T dx\,dy\,dz=\int_{-2}^{2}\int_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}}\int_{x^2+3y^2}^{4-y^2} dz\,dy\,dx$$

$$=\int_{-2}^{2}\int_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}} z\Big|_{x^2+3y^2}^{4-y^2} dy\,dx=\int_{-2}^{2}\int_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}} 4-y^2-x^2-3y^2\,dy\,dx$$

$$=\int_{-2}^{2} 4y-\tfrac{4}{3}y^3-yx^2\Big|_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}} dx \ = 4\pi$$

§5 Cylindrical Coordinates

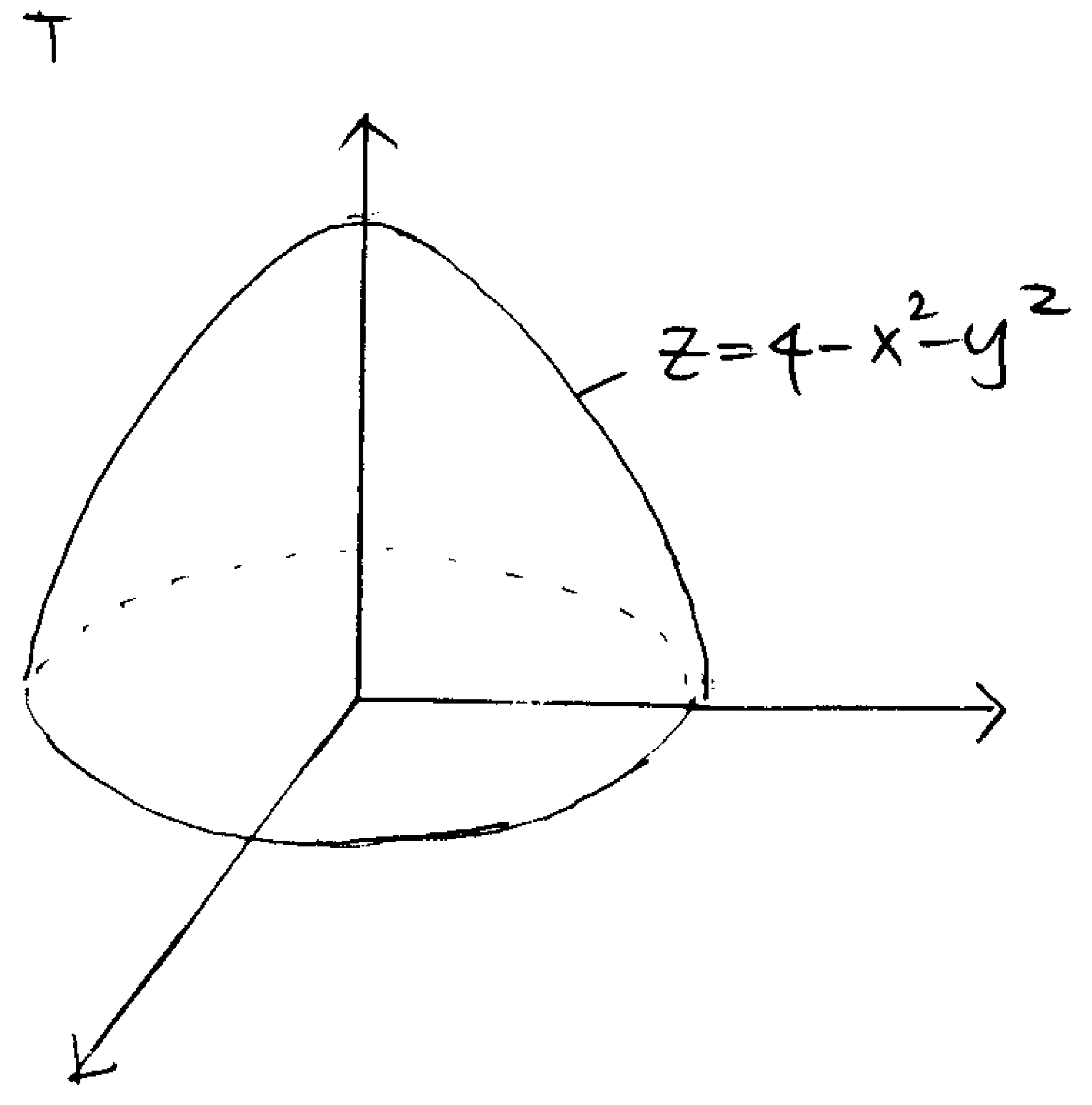


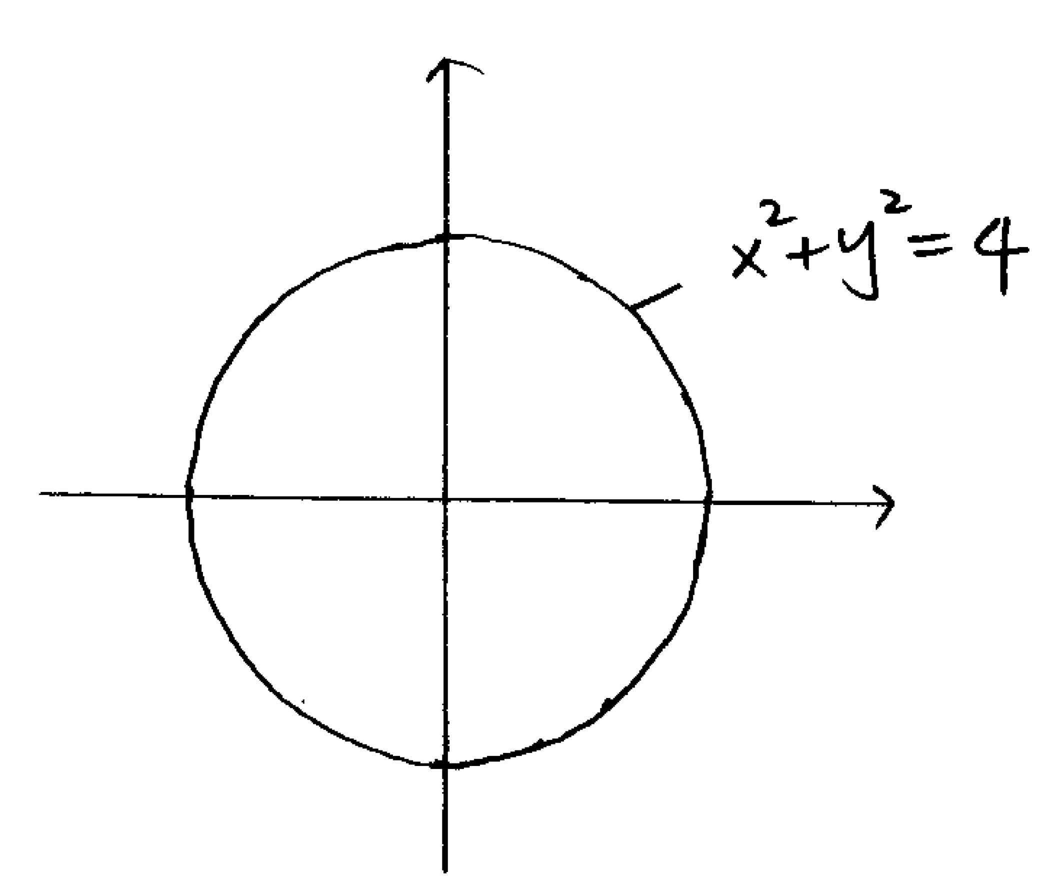
Suppose that T is some basic solid in xyz-space, and S is the basic solid with cylindrical coordinates of T in $r\theta z$-space. Then

$$\boxed{\iiint_T f(x,y,z)\,dx\,dy\,dz = \iiint_S f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz}$$

Examples

1. $\iiint_T (x^2+y^2)\,dx\,dy\,dz$, $T: -2 \le x \le 2$, $-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$, $0 \le z \le 4-x^2-y^2$





$S: \ 0 \le r \le 2 \ , 0 \le \theta \le 2\pi \ , 0 \le z \le 4 - r^2$

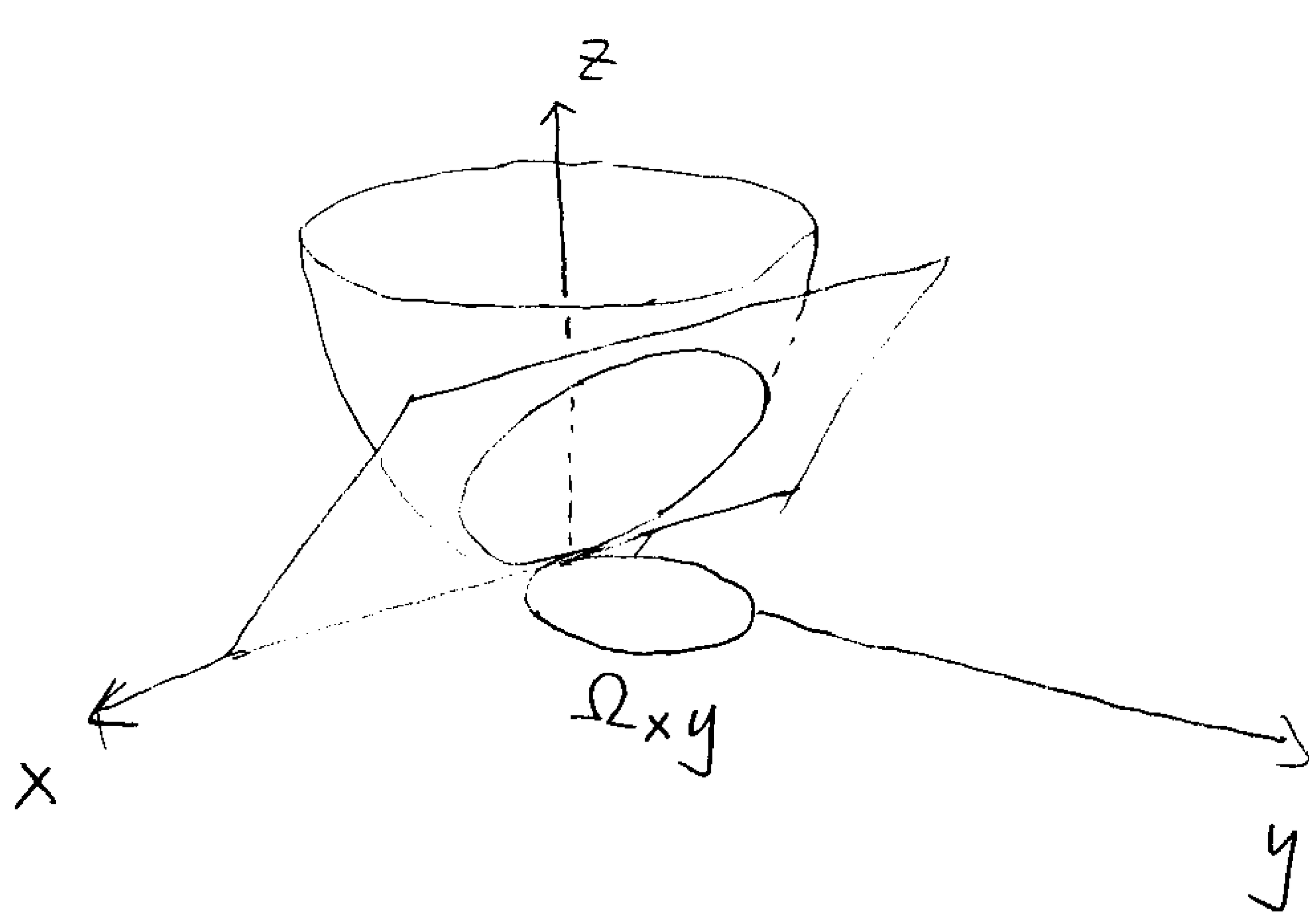
$$\iiint_T (x^2+y^2)\, dx dy dz = \iiint_S r^2 \cdot r dr d\theta dz$$

$$= \int_0^{2\pi}\int_0^{2}\int_0^{4-r^2} r^3 \cdot dz dr d\theta = \int_0^{2\pi}\int_0^{r} r^3 \cdot z \Big|_0^{4-r^2} dr d\theta$$

$$= \int_0^{2\pi}\int_0^{0} r^3(4-r^2)\, dr d\theta = \int_0^{2\pi} r^5 - \frac{1}{6} r^6 \Big|_0^2 = \int_0^{2\pi} 32 - \frac{1}{6}\cdot 64\, d\theta$$

$$= 2\pi \cdot \frac{16}{3} = \frac{32}{3}\pi$$

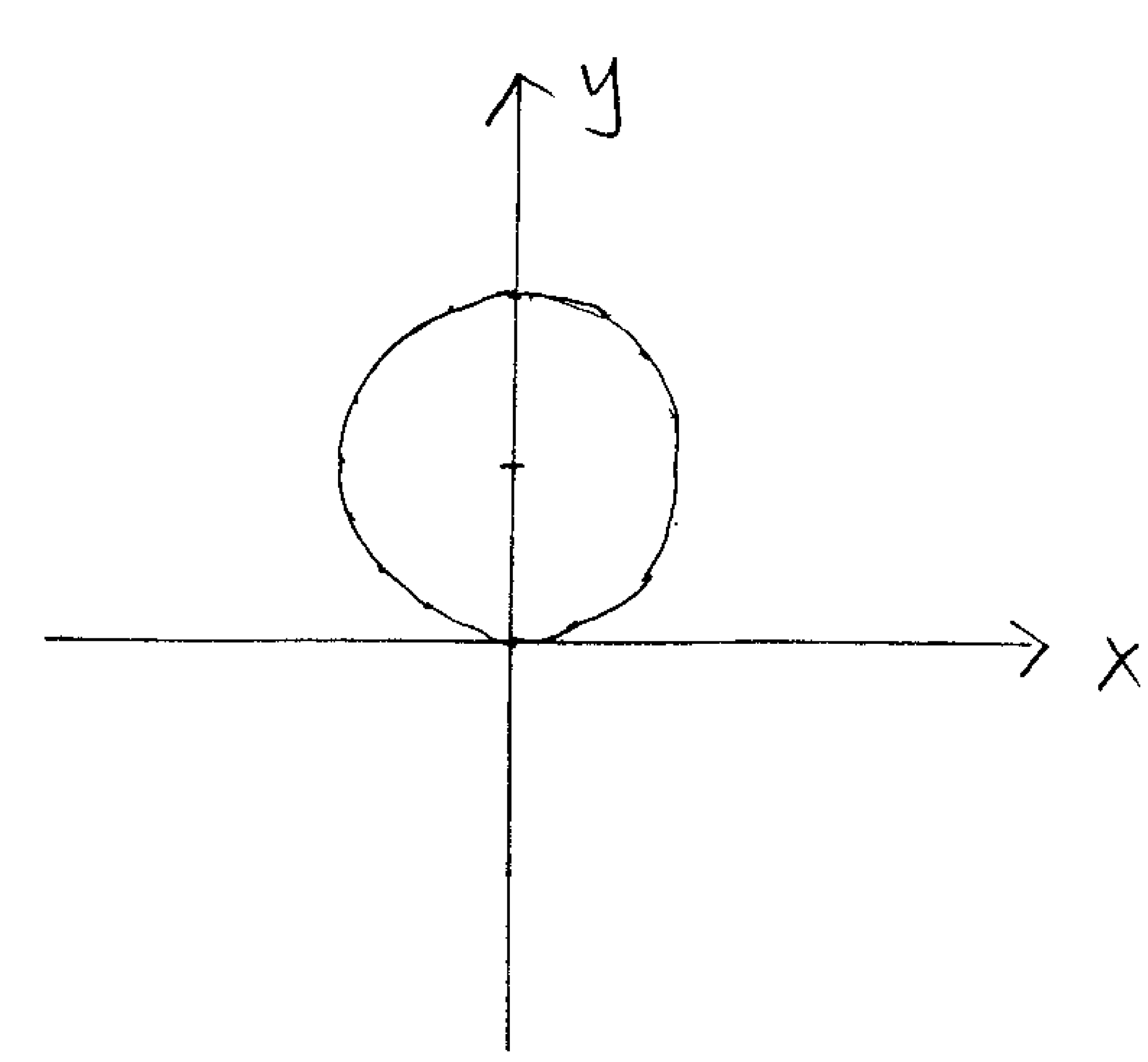
2. Use cylindrical coordinates to find the volume of the solid T bounded above by the plane $z = y$ and below by the paraboloid $z = x^2 + y^2$



$x^2+y^2 \le z \le y$

$y = x^2 + y^2$

$\Rightarrow \ x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$



$$r^2\cos^2\theta + r^2\sin^2\theta = r\sin\theta$$

$$\Rightarrow r^2 = r\sin\theta \quad \Rightarrow \quad r = \sin\theta$$

$$\Rightarrow 0 \le \theta \le \pi \;, \quad 0 \le r \le \sin\theta$$

$$V = \iiint_T dx\,dy\,dz = \iiint_S r\,dr\,d\theta\,dz = \int_0^{\pi}\int_0^{\sin\theta}\int_{r^2}^{r\sin\theta} r\,dz\,dr\,d\theta$$

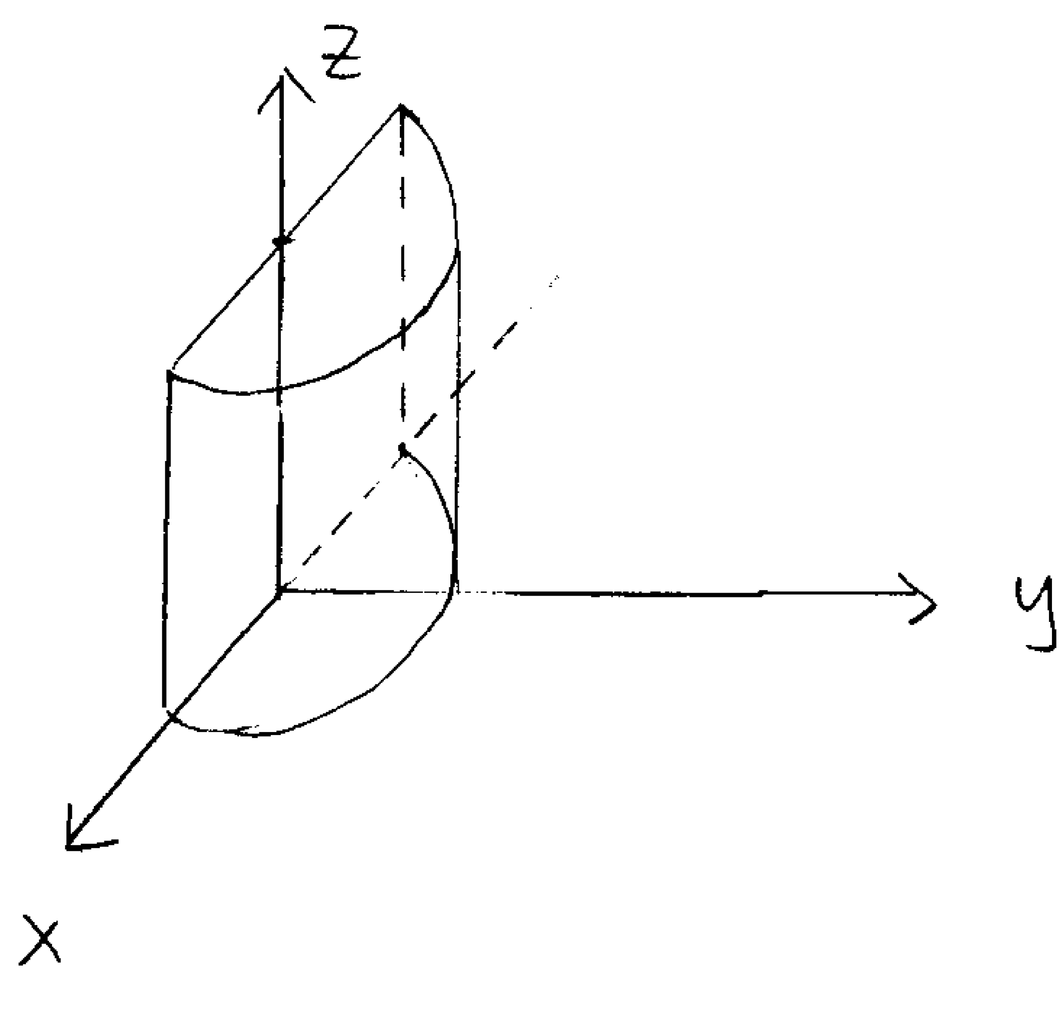
$$= \int_0^{\pi}\int_0^{\sin\theta} rz\Big|_{r^2}^{r\sin\theta} dr\,d\theta = \int_0^{\pi}\int_0^{\sin\theta} r(r\sin\theta - r^2)\,dr\,d\theta$$

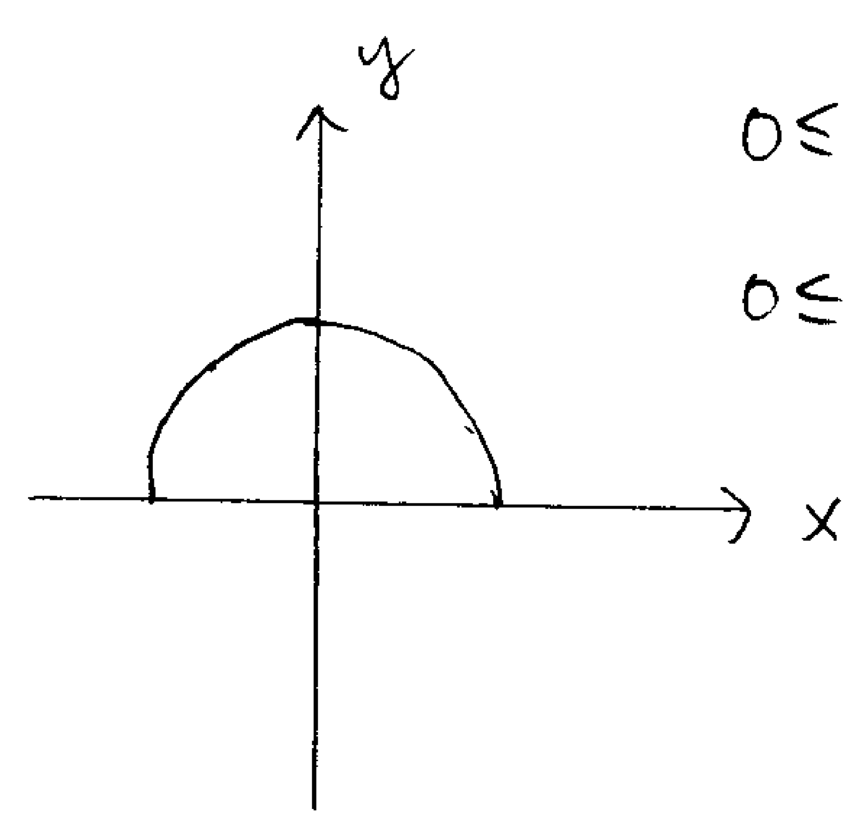
$$= \int_0^{\pi} \frac{1}{3}r^3\sin\theta - \frac{1}{4}r^4 \Big|_0^{\sin\theta} d\theta = \int_0^{\pi} \frac{1}{3}\sin^4\theta - \frac{1}{4}\sin^4\theta\, d\theta$$

$$= \frac{1}{12}\int_0^{\pi} \sin^4\theta\, d\theta$$

$$= \frac{1}{32}\pi$$

3、 $\iiint_T \sin(x^2+y^2)\,dx\,dy\,dz \;;\; T:\; 0 \le x \le 1 \,,\; 0 \le y \le \sqrt{1-x^2} \,,\; 0 \le z \le 2$





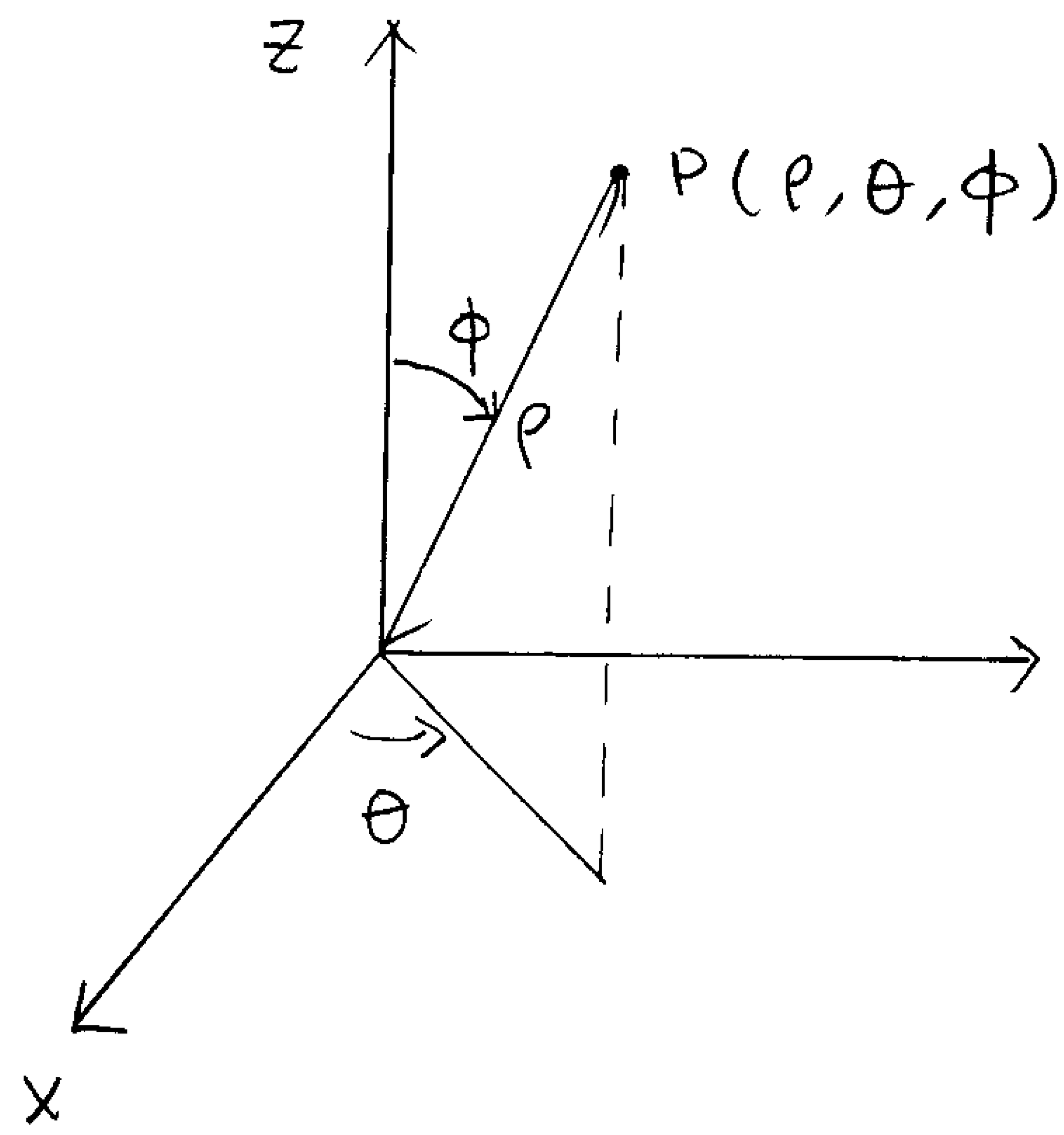
$0 \le \theta \le \pi$

$0 \le r \le 1$

$$\iiint_T \sin(x^2+y^2)\,dx\,dy\,dz = \int_0^{\pi}\int_0^{1}\int_0^{2} \sin r^2 \cdot r\,dz\,dr\,d\theta = \int_0^{\pi}\int_0^{1} 2r\sin r^2\,dr\,d\theta$$

$$= \int_0^{\pi} -\cos r^2\Big|_0^1 d\theta = \pi\,(-\cos 1 + \cos 0) = \pi\,(1 - \cos 1)$$

§6 Spherical Coordinates



$\rho = \sqrt{x^2+y^2+z^2}$

$\tan\theta = \frac{y}{x}$

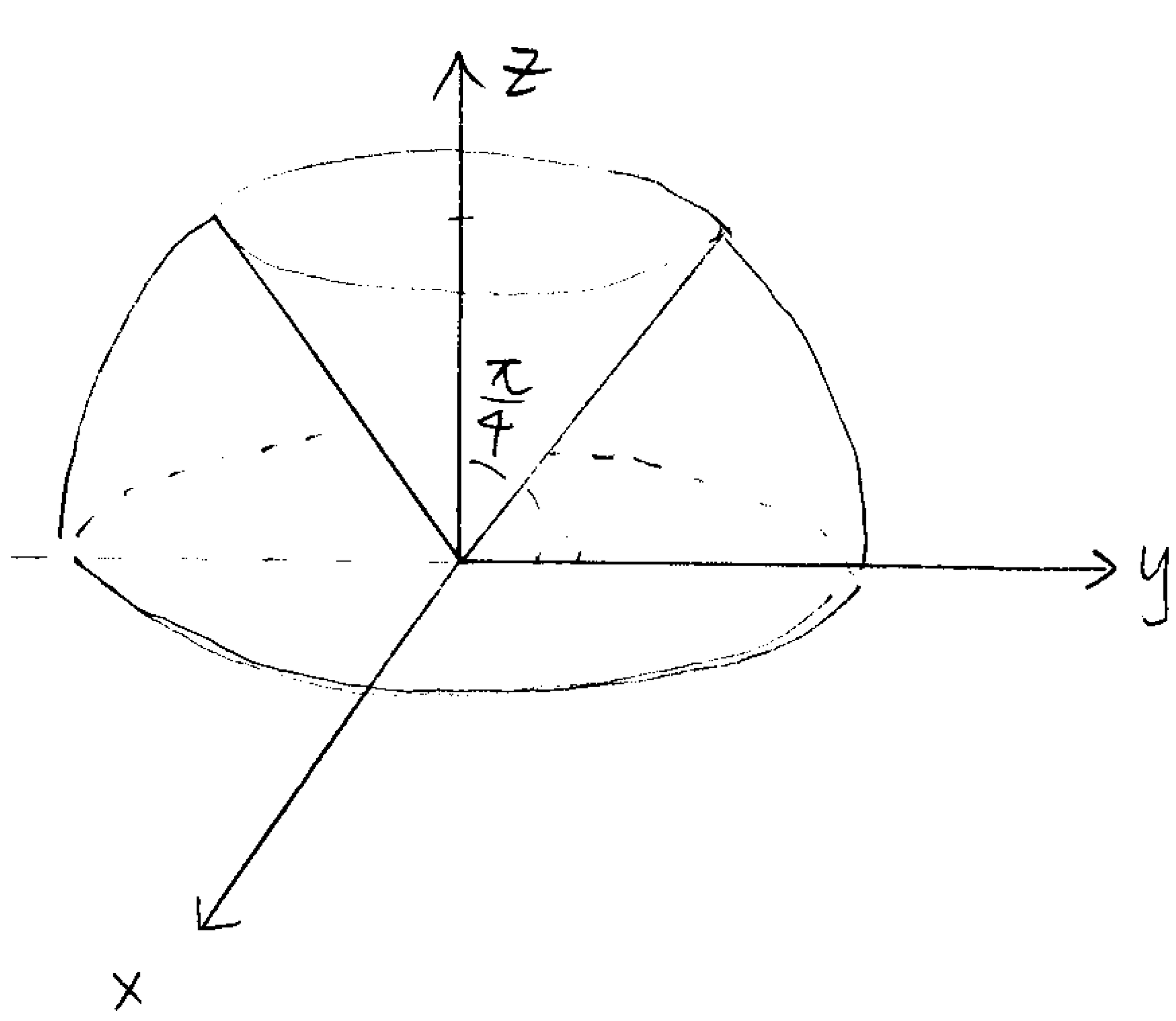
$\cos\phi = \frac{z}{\sqrt{x^2+y^2+z^2}}$

Suppose that T is a basic solid in xyz-space with spherical coordinates in some basic solid S of $\rho\theta\phi$-space.

$$\boxed{\iiint_T f(x,y,z)\,dx\,dy\,dz = \iiint_S f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)\ \rho^2\sin\phi\ d\rho\, d\theta\, d\phi}$$

Examples

1. Find the volume of the solid bounded above by the cone $z^2 = x^2+y^2$ below by the xy-plane, and on the sides by the hemisphere $z=\sqrt{4-x^2-y^2}$



$S:\ 0 \le \rho \le 2$

$0 \le \theta \le 2\pi$

$\frac{\pi}{4} \le \phi \le \frac{\pi}{2}$

$$V = \iiint_T dx\,dy\,dz = \iiint_S \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^2 \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \rho^2 \sin\phi \, d\phi\, d\theta\, d\rho = \int_0^2 \int_0^{2\pi} -\rho^2 \cos\phi \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta\, d\rho$$

$$= \int_0^2 \int_0^{2\pi} \rho^2 \cdot \frac{1}{\sqrt{2}} \, d\theta\, d\rho = \int_0^2 2\pi \cdot \frac{1}{\sqrt{2}} \rho^2 \, d\rho = \sqrt{2}\pi \left(\frac{1}{3}\rho^2 \Big|_0^2 \right)$$

$$= \sqrt{2}\pi \cdot \frac{1}{3} \cdot 8 = \frac{8\sqrt{2}}{3}\pi.$$

2. Find the volume of the solid T enclosed by the surface

$$(x^2+y^2+z^2)^2 = 2z\,(x^2+y^2)$$

In spherical coordinates,

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

$$\Rightarrow \left(\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta + \rho^2 \cos^2\phi\right)^2 = 2\rho\cos\phi \cdot \left(\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta\right)$$

$$\Rightarrow \rho^4 = 2\rho\cos\phi \cdot \rho^2 \sin^2\phi \Rightarrow \rho = 2\cos\phi \sin^2\phi$$

observe that the equation have no restriction on θ.

Hence $0 \le \theta \le 2\pi$

Note $\rho \geq 0 \Rightarrow 2\cos\phi \sin^2\phi \geq 0 \Rightarrow 0 \le \phi \le \frac{\pi}{2}$

S: $0 \le \theta \le 2\pi$, $0 \le \phi \le \frac{1}{2}\pi$, $0 \le \rho \le 2\sin^2\phi \cos\phi$

$$V = \iiint_T dx\,dy\,dz = \iiint_S \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2\sin^2\phi\cos\phi} \rho^2\sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \frac{1}{3}\rho^3\sin\phi \Big|_0^{2\sin^2\phi\cos\phi} d\phi\, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} 8\sin^6\phi\cos^3\phi\,\sin\phi \, d\phi\, d\theta$$

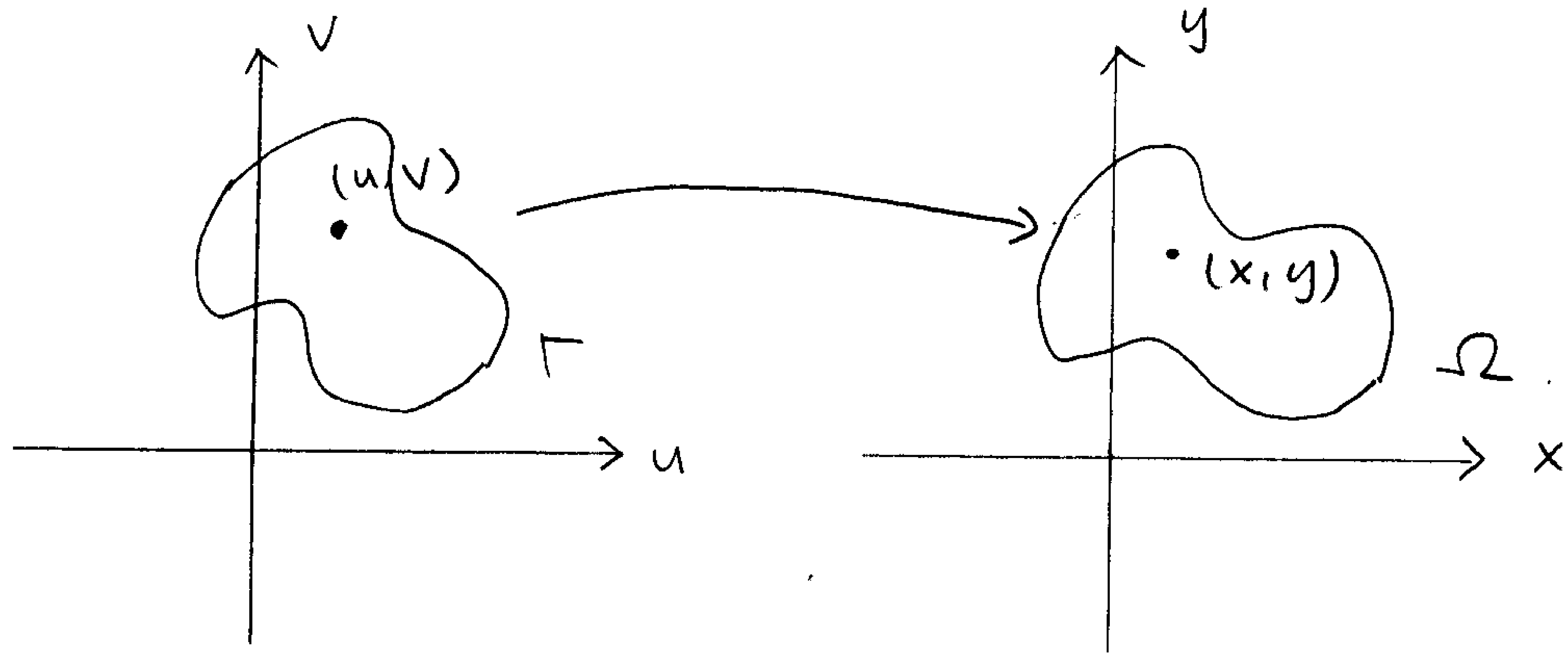
$$= \frac{8}{3}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin^7\phi\cos^3\phi\, d\phi\, d\theta \qquad (\text{Note } \cos^2\phi = 1-\sin^2\phi)$$

$$= \frac{8}{3}\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\frac{\pi}{2}} \sin^7\phi\cos\phi - \sin^9\phi\cos\phi \, d\phi\right)$$

$$= \frac{16}{3}\pi \cdot \left(\frac{1}{8}\sin^8\phi - \frac{1}{10}\sin^{10}\phi \Big|_0^{\frac{\pi}{2}}\right) = \frac{16}{3}\pi \cdot \left(\frac{1}{8} - \frac{1}{10}\right) = \frac{2}{15}\pi$$

§7 Jacobians ; Changing variables in Multiple Integration

Suppose that $x = x(u,v)$, $y = y(u,v)$



If the map is 1-1 on the interior of Γ , and

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \cdot \frac{\partial x}{\partial v} \neq 0 \text{ on the}$$

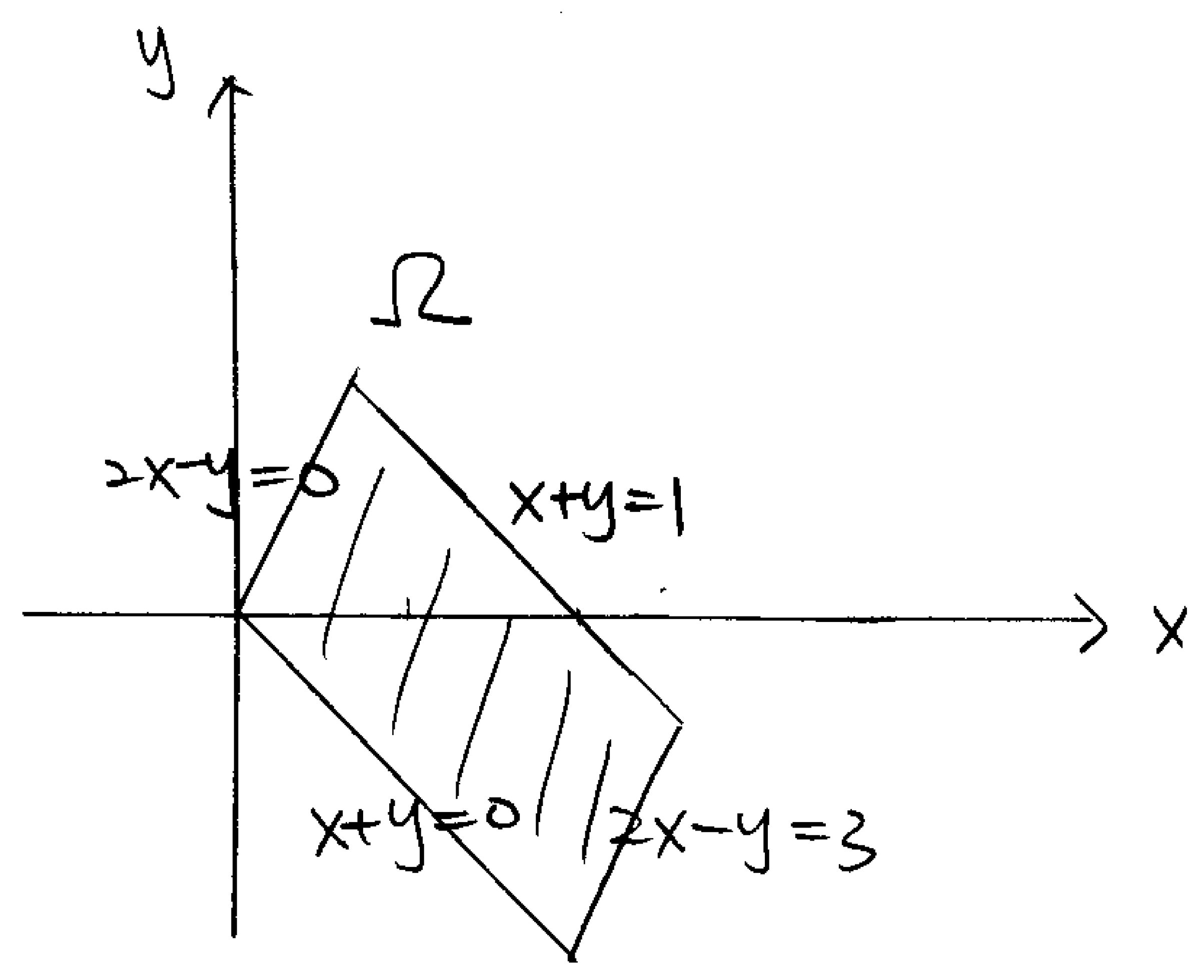
interior of Γ , then

$$\boxed{\text{area of } \Omega = \iint_{\Gamma} |J(u,v)|\, du\, dv}$$

and $$\boxed{\iint_{\Omega} f(x,y)\, dx\, dy = \iint_{\Gamma} f(x(u,v), y(u,v))\, |J(u,v)|\, du\, dv}$$

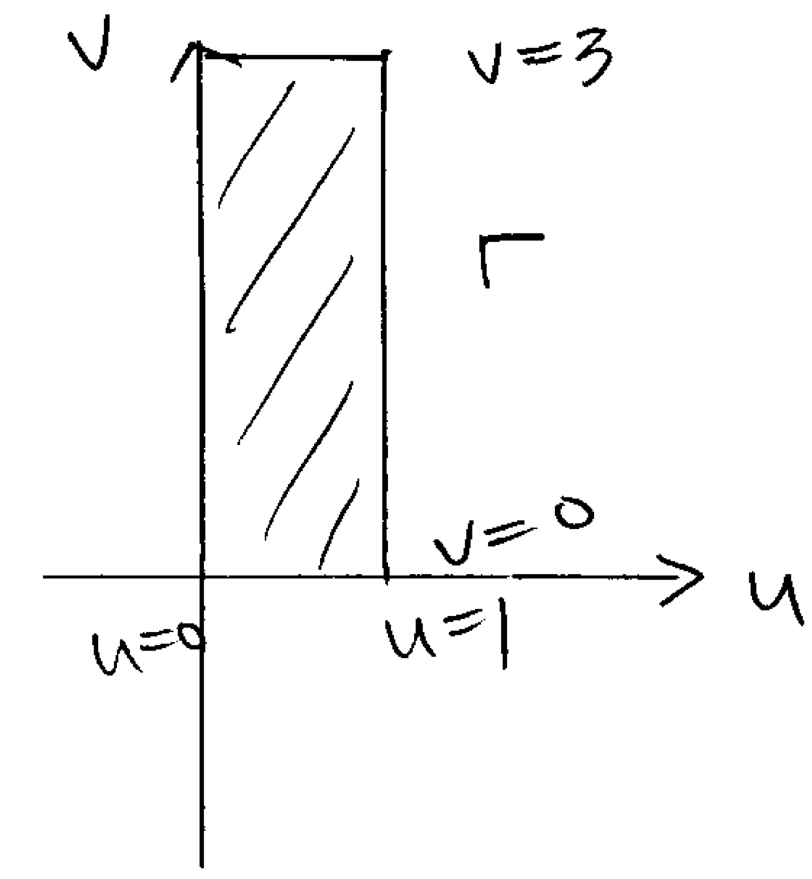
Examples

1. Evaluate $\iint_{\Omega} (x+y)^2\, dx\, dy$ where Ω is the parallelogram bounded by the lines $x+y=0$, $x+y=1$, $2x-y=0$, $2x-y=3$.



let $u=x+y$

$v=2x-y$



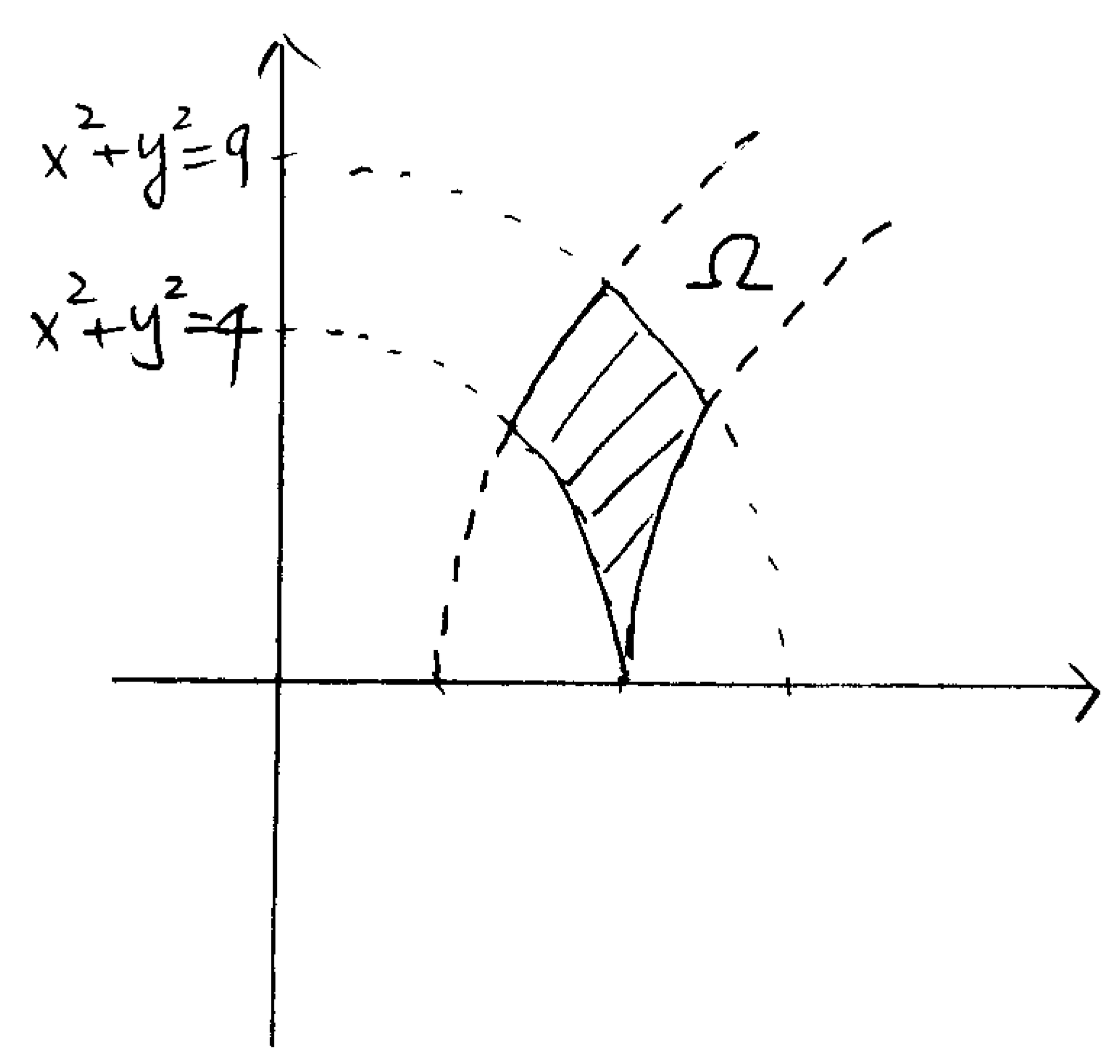
we have $x=\frac{1}{3}(u+v)$, $y=\frac{1}{3}(2u-v)$

$$J(u,v)=\begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{9}-\frac{2}{9}=-\frac{1}{3}$$

$$\iint_\Omega (x+y)^2\,dxdy = \int_0^1\int_0^3 \left(\tfrac{1}{3}(u+v)+\tfrac{1}{3}(2u-v)\right)^2\cdot\left|-\tfrac{1}{3}\right|\,dvdu$$

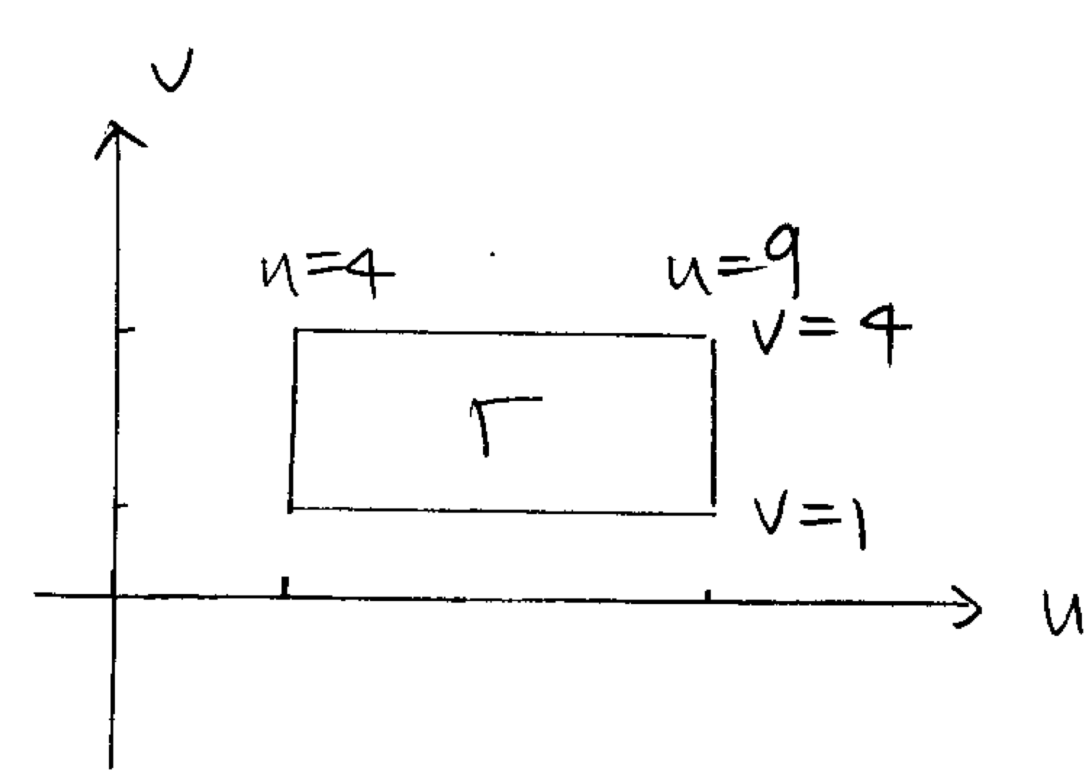
$$=\int_0^1\int_0^3 u^2\,dvdu = \frac{1}{3}\cdot 3\int_0^1 u^2\,du = 1\cdot\left(\frac{1}{3}u^3\Big|_0^1\right)=\frac{1}{3}$$

2. Evaluate $\iint_\Omega xy\,dxdy$ where Ω is the first-quadrant region bounded by the curves $x^2+y^2=4$, $x^2+y^2=9$, $x^2-y^2=1$, $x^2-y^2=4$



let $x^2+y^2=u$

$x^2-y^2=v$



We have $x = \sqrt{\frac{u+v}{2}}$, $y = \sqrt{\frac{u-v}{2}}$

$$J(u,v) = \begin{vmatrix} \frac{1}{2}\cdot\frac{1}{2}\left(\frac{u+v}{2}\right)^{-\frac{1}{2}} & \frac{1}{2}\cdot\frac{1}{2}\left(\frac{u-v}{2}\right)^{-\frac{1}{2}} \\ \frac{1}{2}\cdot\frac{1}{2}\left(\frac{u+v}{2}\right)^{-\frac{1}{2}} & \frac{1}{2}\left(-\frac{1}{2}\right)\left(\frac{u-v}{2}\right)^{-\frac{1}{2}} \end{vmatrix} = \frac{-1}{4}\cdot\frac{1}{\sqrt{u^2-v^2}}$$

$$\iint_\Omega xy\,dx\,dy = \iint_\Gamma \sqrt{\frac{u+v}{2}}\cdot\sqrt{\frac{u-v}{2}}\left(\frac{1}{4}\frac{1}{\sqrt{u^2-v^2}}\right) du\,dv$$

$$= \int_1^4\int_4^9 \frac{1}{8}\,du\,dv = \frac{15}{8}.$$

◎ Polar coordinates

$x = r\cos\theta$, $y = r\sin\theta$

$$J(r,\theta) = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$$\iint_\Omega f(x,y)\,dx\,dy = \iint_\Gamma f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$$

◎ Spherical coordinates

$x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$

$$J(\rho,\phi,\theta) = \begin{vmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ \rho\cos\phi\cos\theta & \rho\cos\phi\sin\theta & -\rho\sin\phi \\ -\rho\sin\phi\sin\theta & \rho\sin\phi\cos\theta & 0 \end{vmatrix} = \rho^2\sin\phi$$

$$\iiint_T f(x,y,z)\,dx\,dy\,dz = \iiint_S f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$