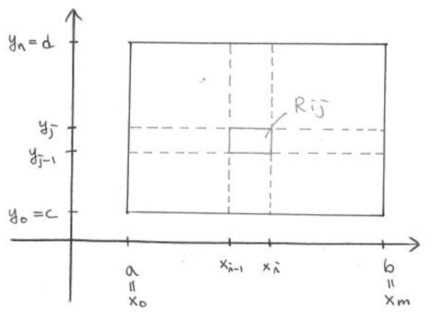
Double and Triple Integrals.

§1 Double Integrals

$f(x,y)$ continuous on $R=\{(x,y) \mid a \le x \le b,\ c \le y \le d\}$

Want to define

$$\iint_R f(x,y)\,dx\,dy.$$



Let M_{ij} be the maximum value of f on R_{ij}

m_{ij} 〃 〃 minimum value 〃 〃

and P be the partition of R

$$U_f(P) \triangleq \sum_{i=1}^{m} \sum_{j=1}^{n} M_{ij}\,(\text{area of } R_{ij})$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} M_{ij}\,(x_i - x_{i-1})(y_j - y_{j-1})$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} M_{ij}\,\Delta x_i \Delta y_j$$

$$L_f(P) \triangleq \sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij}\,(\text{area of } R_{ij})$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij}\,\Delta x_i \Delta y_j$$

$U_f(P)$: the P upper sum of f

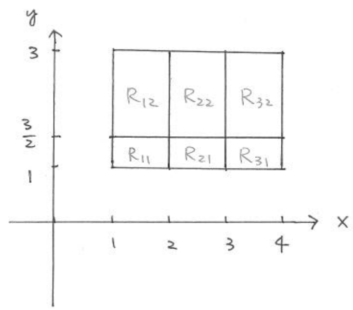
$L_f(P)$: the P lower sum of f.

Example

Set $f(x,y) = x+y-2$ on $R = \{(x,y) \mid 1 \le x \le 4,\ 1 \le y \le 3\}$

As a partition of $[1,4]$, take $P_1 = \{1,2,3,4\}$

〃 〃 $[1,3]$, take $P_2 = \{1, \frac{3}{2}, 3\}$.



$M_{11} = 2 + \frac{3}{2} - 2 = \frac{3}{2}$, $m_{11} = 0$

$M_{12} = 2+3-2 = 3$, $m_{12} = \frac{1}{2}$

$M_{21} = 3 + \frac{3}{2} - 2 = \frac{5}{2}$, $m_{21} = 1$

$M_{22} = 3+3-2 = 4$, $m_{22} = \frac{3}{2}$

$M_{31} = 4 + \frac{3}{2} - 2 = \frac{7}{2}$, $m_{31} = 2$

$M_{32} = 4+3-2 = 5$, $m_{32} = \frac{7}{2}$

$$U_f(P) = \frac{3}{2}\cdot\frac{1}{2} + 3\cdot\frac{3}{2} + \frac{5}{2}\cdot\frac{1}{2} + 4\cdot\frac{3}{2} + \frac{7}{2}\cdot\frac{1}{2} + 5\cdot\frac{3}{2} = \frac{87}{4}$$

$$L_f(P) = 0\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{3}{2} + 1\cdot\frac{1}{2} + \frac{3}{2}\cdot\frac{3}{2} + 2\cdot\frac{1}{2} + \frac{5}{2}\cdot\frac{3}{2} = \frac{33}{4}.$$

Definition 1.1

f continuous on a closed rectangle R.

The unique number I that satisfies

$$L_f(P) \le I \le U_f(P) \quad \text{for all } P$$

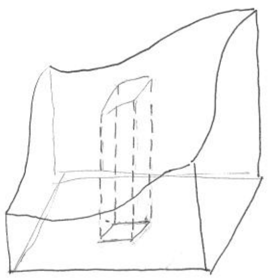
is called the double integral of f over R.

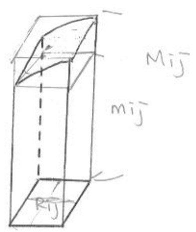
say $\iint_R f(x,y)\,dxdy = I$

• The double integral as a volume

$z = f(x,y)$, R : rectangle

$\iint_R f(x,y)\,dxdy$ gives the volume of the solid that is bounded below by R





$$m_{ij}(\text{area of } R_{ij}) \le \text{volume of } T_{ij}$$

$$\le M_{ij}(\text{area of } R_{ij})$$

$$\Rightarrow L_f(P) \le \text{volume of } T \le U_f(P)$$

$$\Rightarrow \text{volume of } T = \iint_R f(x,y)\,dxdy.$$

Example

Evaluate $\iint_R (x+y-2)\,dxdy$, $R = \{(x,y) \mid 1 \le x \le 4,\ 1 \le y \le 3\}$

Let P be any partition of R

On each rectangle R_{ij} : $x_{i-1} \le x \le x_i$, $y_{j-1} \le y \le y_j$

$$M_{ij} = x_i + y_j - 2 \ , \quad m_{ij} = x_{i-1} + y_{j-1} - 2$$

$$L_f(P) = \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{i-1} + y_{j-1} - 2)\,\Delta x_i \Delta y_j$$

$$U_f(P) = \sum_{i=1}^{m} \sum_{j=1}^{n} (x_i + y_j - 2)\,\Delta x_i \Delta y_j$$

Note

$$x_{i-1} + y_{j-1} - 2 \le \tfrac{1}{2}(x_i + x_{i-1}) + \tfrac{1}{2}(y_j + y_{j-1}) - 2 \le x_i + y_j - 2$$

$$\Rightarrow L_f(P) \le \sum_{i=1}^{m} \sum_{j=1}^{n} \left(\tfrac{1}{2}(x_i + x_{i-1}) + \tfrac{1}{2}(y_j + y_{j-1}) - 2\right)\Delta x_i \Delta y_j \le U_f(P)$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \left(\tfrac{1}{2}(x_i+x_{i-1}) + \tfrac{1}{2}(y_j+y_{j-1}) - 2 \right) \Delta x_i \Delta y_j$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \overset{(1)}{\tfrac{1}{2}(x_i+x_{i-1})} \Delta x_i \Delta y_j + \sum_{i=1}^{m} \sum_{j=1}^{n} \overset{(2)}{\tfrac{1}{2}(y_j+y_{j-1})} \Delta x_i \Delta y_j$$

$$- \sum_{i=1}^{m} \sum_{j=1}^{n} \overset{(3)}{2} \Delta x_i \Delta y_j$$

$$= 15 + 12 - 12 = 15$$

(1) $\sum_{i=1}^{m} \sum_{j=1}^{n} \tfrac{1}{2}(x_i+x_{i-1}) \Delta x_i \Delta y_j$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \tfrac{1}{2}(x_i+x_{i-1})(x_i-x_{i-1}) \Delta y_j$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \tfrac{1}{2}(x_i^2-x_{i-1}^2) \Delta y_j$$

$$= \sum_{i=1}^{m} \tfrac{1}{2}(x_i^2-x_{i-1}^2) \sum_{j=1}^{n} \Delta y_j = \tfrac{1}{2}(16-1)(3-1) = 15$$

(2) $\sum_{i=1}^{m} \sum_{j=1}^{n} \tfrac{1}{2}(y_j+y_{j-1}) \Delta x_i \Delta y_j$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \tfrac{1}{2}(y_j+y_{j-1})(y_j-y_{j-1}) \Delta x_i$$

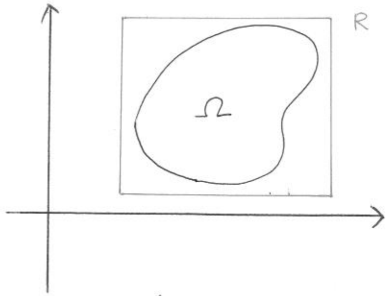
$$= \sum_{i=1}^{m} \Delta x_i \sum_{j=1}^{n} \tfrac{1}{2}(y_j^2-y_{j-1}^2) = (4-1)\cdot\tfrac{1}{2}(9-1) = 12$$

(3) $\sum_{i=1}^{m} \sum_{j=1}^{n} 2\Delta x_i \Delta y_j = \sum_{i=1}^{m} 2\Delta x_i \cdot \sum_{j=1}^{n} \Delta y_j$

$$= 2(4-1)(3-1) = 12$$

- The double integral over a region

Ω : closed and bounded set



f : function continuous on Ω

Extend f to R

$$f(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in \Omega \\ 0, & \text{if } (x,y) \notin \Omega \end{cases}$$

We have $\iint_R f(x,y)\,dx\,dy$.

Define

$$\iint_\Omega f(x,y)\,dx\,dy = \iint_R f(x,y)\,dx\,dy.$$

Properties of double integral

1. Linearity :

$$\iint_{\Omega} (\alpha f(x,y) + \beta g(x,y))\,dxdy$$

$$= \alpha \iint_{\Omega} f(x,y)\,dxdy + \beta \iint_{\Omega} f(x,y)\,dxdy.$$

2. Order :

If $f \geq 0$ on Ω, then $\iint_{\Omega} f(x,y)\,dxdy \geq 0$

If $f \geq g$ on Ω, then $\iint_{\Omega} f(x,y)\,dxdy$

$$\geq \iint_{\Omega} g(x,y)\,dxdy.$$

3. Additivity :

If Ω is broken up into n nonoverlapping basic regions $\Omega_1, \ldots, \Omega_n$, then

$$\iint_{\Omega} f(x,y)\,dxdy = \iint_{\Omega_1} f(x,y)\,dxdy + \cdots + \iint_{\Omega_n} f(x,y)\,dxdy.$$

4. Mean-value condition :

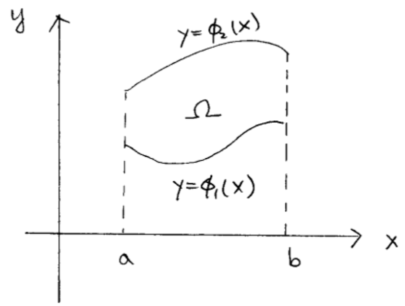
$\exists\,(x_0, y_0) \in \Omega$ such that

$$\iint_{\Omega} f(x,y)\,dxdy = f(x_0,y_0) \cdot (\text{area of } \Omega)$$

§2 The Evaluation of Double Integrals by Repeat Integrals.

We introduce a technique for evaluating double integrals of continuous functions over regions of type I & type II.

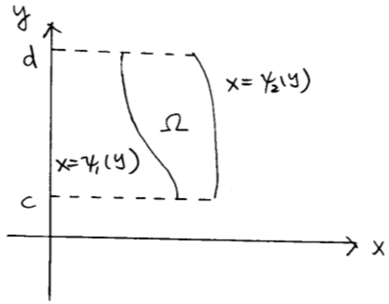
Type I



$\Omega = \{(x,y) \mid a \le x \le b,\ \phi_1(x) \le y \le \phi_2(x)\}$

$$\iint_\Omega f(x,y)\,dx\,dy = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx$$

Type II

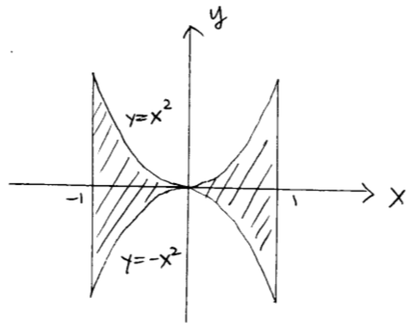


$\Omega = \{(x,y) \mid c \le y \le d,\ \psi_1(y) \le x \le \psi_2(y)\}$

$$\iint_\Omega f(x,y)\,dx\,dy = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\,dy.$$

Examples

1. Evaluate $\iint_\Omega (x^4-2y)\,dx\,dy$.



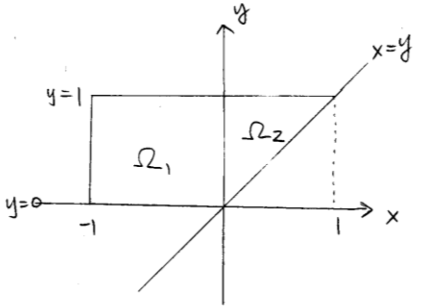
$\Omega = \{(x,y) \mid -1 \le x \le 1,\ -x^2 \le y \le x^2\}$

$$\iint_\Omega (x^4-2y)\,dx\,dy$$

$$= \int_{-1}^{1} \int_{-x^2}^{x^2} (x^4-2y)\,dy\,dx = \int_{-1}^{1} x^4y - y^2\Big|_{-x^2}^{x^2}\,dx$$

$$= \int_{-1}^{1} x^6 - x^4 - (-x^6 - x^4)\,dx = \int_{-1}^{1} 2x^6\,dx = \frac{2}{7}x^7\Big|_{-1}^{1} = \frac{2}{7} - \left(-\frac{2}{7}\right) = \frac{4}{7}$$

2. $\iint_{\Omega} (xy - y^3)\, dx dy$



$\Omega_1 = \{(x,y) \mid -1 \le x \le 0,\ 0 \le y \le 1\}$

$\Omega_2 = \{(x,y) \mid 0 \le y \le 1,\ 0 \le x \le y\}$

$$\iint_{\Omega} (xy - y^3)\, dx dy = \int_{-1}^{0} \int_{0}^{1} (xy - y^3)\, dy\, dx + \int_{0}^{1} \int_{0}^{y} (xy - y^3)\, dx\, dy$$

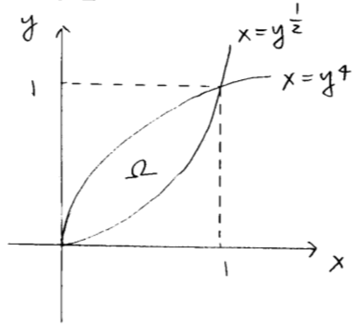
$$= \int_{-1}^{0} \tfrac{1}{2}xy^2 - \tfrac{1}{4}y^3 \Big|_{0}^{1} dx + \int_{0}^{1} \tfrac{1}{2}x^2 y - xy^3 \Big|_{0}^{y} dy$$

$$= \int_{-1}^{0} \tfrac{1}{2}x - \tfrac{1}{4}\, dx + \int_{0}^{1} \tfrac{1}{2}y^3 - y^4\, dy$$

$$= \tfrac{1}{4}x^2 - \tfrac{1}{4}x \Big|_{-1}^{0} + \tfrac{1}{8}y^4 - \tfrac{1}{5}y^5 \Big|_{0}^{1}$$

$$= -\left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} - \tfrac{1}{5}\right) = -\tfrac{1}{2} - \tfrac{3}{40} = -\tfrac{23}{40}$$

3. $\iint_{\Omega} (x^{\frac{1}{2}} - y^2)\, dx dy$



$\Omega = \{(x,y) \mid 0 \le x \le 1,\ x^2 \le y \le x^{\frac{1}{4}}\}$

$= \{(x,y) \mid 0 \le y \le 1,\ y^4 \le x \le y^{\frac{1}{2}}\}$

① $$\iint_{\Omega} (x^{\frac{1}{2}} - y^2)\, dx dy = \int_0^1 \int_{x^2}^{x^{\frac{1}{4}}} (x^{\frac{1}{2}} - y^2)\, dy\, dx$$

$$= \int_0^1 x^{\frac{1}{2}} y - \tfrac{1}{3} y^3 \Big|_{x^2}^{x^{\frac{1}{4}}} dx = \int_0^1 \tfrac{2}{3} x^{\frac{3}{4}} - x^{\frac{5}{2}} - \tfrac{1}{3} x^6\, dx$$

$$= \tfrac{8}{21} x^{\frac{7}{4}} - \tfrac{2}{7} x^{\frac{7}{2}} + \tfrac{1}{21} x^7 \Big|_0^1 = \tfrac{8}{21} - \tfrac{2}{7} + \tfrac{1}{21} = \tfrac{1}{7}.$$

② $$\iint_{\Omega} (x^{\frac{1}{2}} - y^2)\, dx dy = \int_0^1 \int_{y^4}^{y^{\frac{1}{2}}} (x^{\frac{1}{2}} - y^2)\, dx\, dy$$

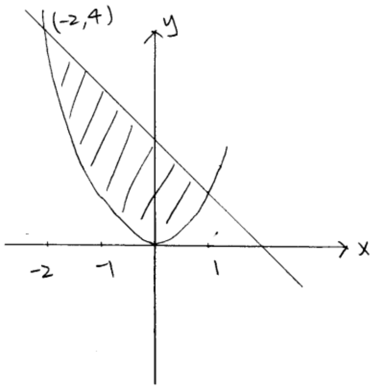
$$= \int_0^1 \tfrac{2}{3} x^{\frac{3}{2}} - y^2 x \Big|_{y^4}^{y^{\frac{1}{2}}} dy = \int_0^1 \tfrac{2}{3} y^{\frac{3}{4}} - y^{\frac{5}{2}} + \tfrac{1}{3} y^6\, dy$$

$$= \tfrac{8}{21} y^{\frac{7}{4}} - \tfrac{2}{7} y^{\frac{7}{2}} + \tfrac{1}{21} y^7 \Big|_0^1 = \tfrac{8}{21} - \tfrac{2}{7} + \tfrac{1}{21} = \tfrac{1}{7}$$

4. Find the area of the region Ω enclosed

$y = x^2$ and $x+y=2$

Note: area of $\Omega = \iint_\Omega dx\,dy$



$\Omega = \{(x,y) \mid -2 \le x \le 1,\ x^2 \le y \le 2-x\}$

$$\iint_\Omega dx\,dy = \int_{-2}^{1}\int_{x^2}^{2-x} dy\,dx = \int_{-2}^{1} y \Big|_{x^2}^{2-x} dx$$

$$= \int_{-2}^{1} 2-x-x^2\,dx = 2x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 \Big|_{-2}^{1}$$

$$= \left(2-\tfrac{1}{2}-\tfrac{1}{3}\right) - \left(-4-2+\tfrac{1}{3}\cdot 8\right) = \tfrac{9}{2}$$

Suppose Ω is symmetric about y-axis

If f is odd in x, i.e. $f(-x,y) = -f(x,y)$

then $\iint_\Omega f(x,y)\,dx\,dy = 0$

If f is even in x, i.e. $f(-x,y) = f(x,y)$

then $\iint_\Omega f(x,y)\,dx\,dy = 2\iint_{\text{right half of } \Omega} f(x,y)\,dx\,dy$.

Suppose Ω is symmetric about x-axis

If f is odd in y, i.e. $f(x,-y) = -f(x,y)$

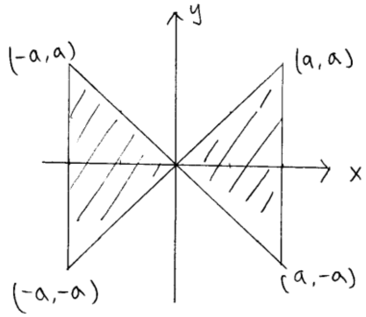
then $\iint_\Omega f(x,y)\,dx\,dy = 0$

If f is even in y, i.e. $f(x,-y) = f(x,y)$

then $\iint_\Omega f(x,y)\,dx\,dy = 2\iint_{\text{upper half of } \Omega} f(x,y)\,dx\,dy$.

Example

$f(x,y) = 2x - \sin x^2 y$



$$\iint_\Omega f(x,y)\,dx\,dy$$

$$= \iint_\Omega (2x - \sin x^2 y)\,dx\,dy$$

$$= \iint_\Omega 2x\,dx\,dy - \iint_\Omega \sin x^2 y\,dx\,dy$$

Note Ω is symmetric about y-axis and $2(-x) = -2x$

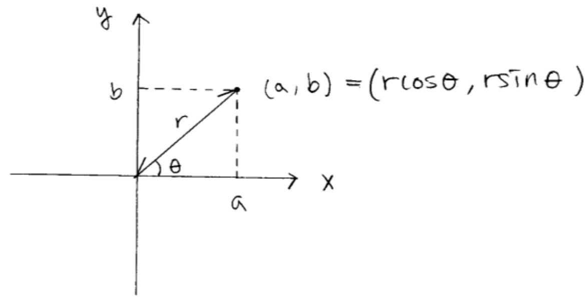
$$\Rightarrow \iint_\Omega 2x\,dx\,dy = 0$$

Ω is symmetric about x-axis and

$\sin x^2(-y) = -\sin x^2 y \Rightarrow \iint_\Omega \sin x^2 y\,dx\,dy = 0$

$$\therefore \iint_\Omega f(x,y)\,dx\,dy = 0$$

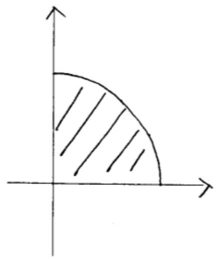
§3 Polar Coordinates



$$\boxed{\iint_\Omega f(x,y)\,dx\,dy = \iint_\Gamma f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta}$$

Examples

1. Use polar coordinates to evaluate $\iint_\Omega xy\,dx\,dy$ where Ω is the portion of the unit disc that lies in the first quadratic



$\Gamma: 0 \le \theta \le \frac{\pi}{2}, \quad 0 \le r \le 1$

$$\iint_\Omega xy\,dx\,dy = \int_0^{\frac{\pi}{2}}\int_0^1 r\cos\theta \cdot r\sin\theta \cdot r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 r^3\cos\theta\sin\theta\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{4}r^4\cos\theta\sin\theta\Big|_0^1\,d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{4}\cos\theta\sin\theta\,d\theta$$

$$= \frac{1}{8}\sin^2\theta\Big|_0^{\frac{\pi}{2}} = \frac{1}{8}$$

2. Use polar coordinates to calculate the volume of a sphere of radius R.

$$V = 2\iint_{\Omega} \sqrt{R^2-(x^2+y^2)}\, dxdy$$

Ω: the disk of radius R centered at the origin.

Γ: $0 \le \theta \le 2\pi$, $0 \le r \le R$

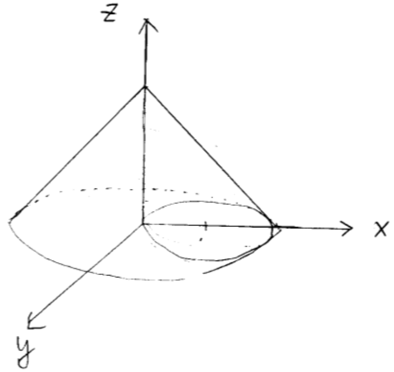
$$V = 2\int_0^{2\pi}\int_0^R \sqrt{R^2-(r^2\cos^2\theta+r^2\sin^2\theta)} \cdot r\,dr\,d\theta$$

$$= 2\int_0^{2\pi}\int_0^R r\cdot\sqrt{R^2-r^2}\, dr\,d\theta$$

$$= 2\int_0^{2\pi} \frac{2}{3}(R^2-r^2)^{\frac{3}{2}}\cdot\left(-\frac{1}{2}\right)\Big|_0^R\, d\theta$$

$$= 2\int_0^{2\pi} \frac{2}{3}\cdot\frac{1}{2}(R^2)^{\frac{3}{2}}\, d\theta = 2\cdot 2\pi\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot R^3 = \frac{4}{3}\pi R^3$$

3. Find the volume of the solid bounded above by the cone $z = 2-\sqrt{x^2+y^2}$ and below by the disk Ω: $(x-1)^2+y^2 \le 1$.



$$V = \iint_{\Omega} 2-\sqrt{x^2+y^2}\, dxdy$$

$$= 2\iint_{\Omega} dxdy - \iint_{\Omega}\sqrt{x^2+y^2}\, dxdy$$

$$= 2\cdot\pi - \iint_{\Omega}\sqrt{x^2+y^2}\, dxdy$$

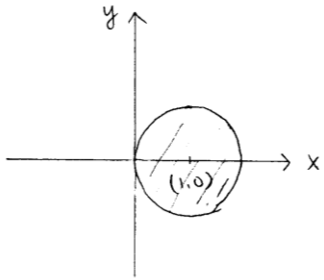
$$= 2\pi - \frac{32}{9}$$

$(x-1)^2+y^2=1 \Rightarrow x^2-2x+y^2=0 \Rightarrow x^2+y^2=2x$

let $x=r\cos\theta$, $y=r\sin\theta$

$\Rightarrow r^2\cos^2\theta+r^2\sin^2\theta = 2r\cos\theta \Rightarrow r^2=2r\cos\theta$

$\Rightarrow r=2\cos\theta$



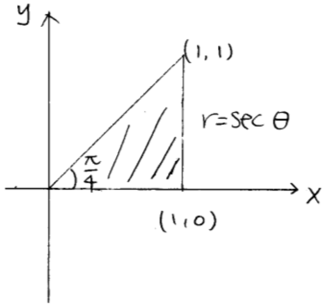
Γ: $-\frac{1}{2}\pi \le \theta \le \frac{1}{2}\pi$, $0 \le r \le 2\cos\theta$

$$\iint_{\Omega}\sqrt{x^2+y^2}\, dxdy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} r\cdot r\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{3}r^3\Big|_0^{2\cos\theta} d\theta = \frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^3\theta\, d\theta$$

$$= \frac{32}{9}$$

4. $\iint_\Omega \frac{1}{(1+x^2+y^2)^{\frac{3}{2}}}\,dxdy$



$\Gamma:\ 0\le\theta\le\frac{\pi}{4}\ ,\ 0\le r\le\sec\theta$

$$\iint_\Omega \frac{1}{(1+x^2+y^2)^{\frac{3}{2}}}\,dxdy = \int_0^{\frac{\pi}{4}}\int_0^{\sec\theta} \frac{1}{(1+r^2\cos^2\theta+r^2\sin^2\theta)^{\frac{3}{2}}}\,rdrd\theta$$

$$= \int_0^{\frac{\pi}{4}}\int_0^{\sec\theta} \frac{r}{(1+r^2)^{\frac{3}{2}}}\,drd\theta = \int_0^{\frac{\pi}{4}} \frac{-1}{\sqrt{1+r^2}}\Big|_0^{\sec\theta} d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left(1-\frac{1}{\sqrt{1+\sec^2\theta}}\right)d\theta$$

Note $\sec\theta = \frac{1}{\cos\theta}$. For $\theta\in[0,\frac{\pi}{4}]$.

$$\frac{1}{\sqrt{1+\sec^2\theta}} = \frac{\cos\theta}{\sqrt{\cos^2\theta+1}} = \frac{\cos\theta}{\sqrt{2-\sin^2\theta}}$$

$$= \int_0^{\frac{\pi}{4}} \left(1-\frac{\cos\theta}{\sqrt{2-\sin^2\theta}}\right)d\theta$$

$$= \theta - \arcsin\left(\frac{\sin\theta}{\sqrt{2}}\right)\Big|_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{4}-\frac{\pi}{6} = \frac{\pi}{12}.$$

5. $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$

Note $\int_{-\infty}^{\infty} e^{-x^2}dx = \int_{-\infty}^{\infty} e^{-y^2}dy \overset{\text{let}}{=} I$

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}dx \int_{-\infty}^{\infty} e^{-y^2}dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dxdy$$

$\left(\begin{array}{l}\Omega:\ \text{the } xy\text{-plane}\\ \Gamma:\ 0\le\theta\le 2\pi\ ,\ 0\le r<\infty\end{array}\right)$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2\cos^2\theta-r^2\sin^2\theta}\cdot rdrd\theta$$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\cdot rdrd\theta = \int_0^{\infty}\int_0^{2\pi} e^{-r^2}\cdot rd\theta dr$$

$$= \int_0^{\infty} \theta\cdot e^{-r^2}\cdot r\Big|_0^{2\pi} dr = 2\pi\cdot\int_0^{\infty} re^{-r^2}dr$$

$$= 2\pi\cdot\lim_{b\to\infty}\int_0^b r\cdot e^{-r^2}dr = 2\pi\cdot\lim_{b\to\infty} -\frac{1}{2}e^{-r^2}\Big|_0^b$$

$$= 2\pi\cdot\lim_{b\to\infty}\left(-\frac{1}{2}e^{-b^2}+\frac{1}{2}\right) = 2\pi\cdot\frac{1}{2} = \pi$$

$\Rightarrow I^2=\pi \ \Rightarrow\ I=\sqrt{\pi}.$