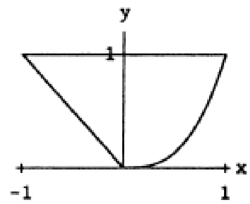
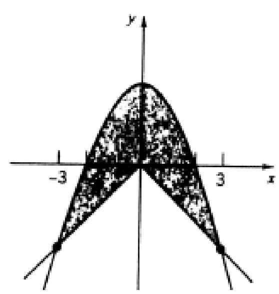
SECTION 6.1

30.



$$A = \int_0^1 \left[y^{1/3} - (-y) \right] \, dy$$
$$= \int_0^1 (y^{1/3} + y) \, dy$$
$$= \left[\frac{3}{4} y^{4/3} + \frac{y^2}{2} \right]_0^1 = \frac{5}{4}$$

31.



$$A = \int_{-3}^0 \left[6 - x^2 - x \right] \, dx + \int_0^3 \left[6 - x^2 - (-x) \right] \, dx$$
$$= \left[6x - \frac{1}{3} x^3 - \frac{1}{2} x^2 \right]_{-3}^0 + \left[6x - \frac{1}{3} x^3 + \frac{1}{2} x^2 \right]_0^3 = 27$$

34. We want $\displaystyle \int_0^c \cos x \, dx = \frac{1}{2} \int_0^{\pi/2} \cos x \, dx \quad \Longrightarrow \quad [\sin x]_0^c = \frac{1}{2} [\sin x]_0^{\pi/2} = \frac{1}{2}$

$$\Longrightarrow \sin c = \frac{1}{2} \quad \Longrightarrow \quad c = \frac{\pi}{6}$$

SECTION 6.2

37. $\displaystyle V = \int_{-a}^a \pi \left(b \sqrt{1 - \frac{x^2}{a^2}} \right)^2 dx = \frac{2\pi b^2}{a^2} \int_0^a \pi \left(a^2 - x^2 \right) \, dx = \frac{2\pi b^2}{a^2} \pi \left[a^2 x - \frac{1}{3} x^3 \right]_0^a$

$$= \frac{2\pi b^2}{a^2} \left(\frac{2}{3} a^3 \right) = \frac{4}{3} \pi a b^2$$

53. $V = \int_0^{\pi} \pi \left[2\sin x - \sin^2 x\right]\, dx = \pi\left[-2\cos x\right]_0^{\pi} - \frac{\pi}{2}\int_0^{\pi}(1-\cos 2x)\,dx$

$= 4\pi - \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = 4\pi - \frac{1}{2}\pi^2$

59. (a) $V = \int_0^4 \pi\left(x^{3/2}\right)^2\, dx = \pi\int_0^4 x^3\, dx = \pi\left[\frac{1}{4}x^4\right]_0^4 = 64\pi$

(b) $V = \int_0^8 \pi\left(4 - y^{2/3}\right)^2\, dy = \pi\int_0^8\left(16 - 8y^{2/3} + y^{4/3}\right)\, dy$

$= \pi\left[16y - \frac{24}{5}y^{5/3} + \frac{3}{7}y^{7/3}\right]_0^8 = \frac{1024}{35}\pi$

(c) $V = \int_0^4 \pi\left[(8)^2 - \left(8 - x^{3/2}\right)^2\right]\, dx = \pi\int_0^4\left(16x^{3/2} - x^3\right)\, dx$

$= \pi\left[\frac{32}{5}x^{5/2} - \frac{1}{4}x^4\right]_0^4 = \frac{704}{5}\pi$

(d) $V = \int_0^8 \pi\left[(4)^2 - \left(y^{2/3}\right)^2\right]\, dy = \pi\int_0^8\left(16 - y^{4/3}\right)\, dy = \pi\left[16y - \frac{3}{7}y^{7/3}\right]_0^8 = \frac{512}{7}\pi$

SECTION 6.3

37. (a) $F'(x) = \sin x + x\cos x - \sin x = x\cos x = f(x)$.

(b) $V = \int_0^{\pi/2} 2\pi x \cdot \cos x\, dx = 2\pi\left[x\sin x + cos x\right]_0^{\pi/2} = \pi^2 - 2\pi$

39. (a) $V = \int_0^1 2\sqrt{3}\pi x^2\, dx + \int_1^2 2\pi x\sqrt{4 - x^2}\, dx$

(b) $V = \int_0^{\sqrt{3}} \pi\left[4 - \frac{4}{3}y^2\right]\, dy$

(c) $V = \int_0^{\sqrt{3}} \pi\left[4 - \frac{4}{3}y^2\right]\, dy = \pi\left[4y - \frac{4}{9}y^3\right]_0^{\sqrt{3}} = \frac{8\pi\sqrt{3}}{3}$

SECTION 6.4

25. (a) $(0,0)$ by symmetry

(b) Ω_1 smaller quarter disc, Ω_2 the larger quarter disc

$$A_1 = \frac{1}{16}\pi, \quad A_2 = \pi; \quad \overline{x}_1 = \overline{y}_1 = \frac{2}{3\pi}, \quad \overline{x}_2 = \overline{y}_2 = \frac{8}{3\pi} \quad \text{(Example 1)}$$

$$\overline{x}A = \left(\frac{8}{3\pi}\right)(\pi) - \frac{2}{3\pi}\left(\frac{1}{16}\pi\right) = \frac{63}{24}, \quad A = \frac{15}{16}\pi$$

$$\overline{x} = \left(\frac{63}{24}\right) \Big/ \left(\frac{15\pi}{16}\right) = \frac{14}{5\pi}, \quad \overline{y} = \overline{x} = \frac{14}{5\pi} \quad \text{(symmetry)}$$

(c) $\overline{x} = 0, \quad \overline{y} = \dfrac{14}{5\pi}$

26. $A = \frac{1}{2}\pi ab; \quad \overline{x} = 0$ by symmetry.

$$\overline{y}A = \int_{-a}^{a} \frac{1}{2}\left(\frac{b}{a}\sqrt{a^2 - x^2}\right)^2 dx = \frac{2}{3}ab^2 \Longrightarrow \overline{y} = \frac{4b}{3\pi}.$$

SECTION 10.7

38. $s = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (1-\cos t)^2}\, dt = \int_0^{2\pi} \sqrt{2 - 2\cos t}\, d\theta = \left[-4\cos\frac{t}{2}\right]_0^{2\pi} = 8$

initial speed $= v(0) = 0$, terminal speed $= v(2\pi) = 0$

39. $c = 1$; the curve $y = e^x$ is the curve $y = \ln x$ reflected in the line $y = x$

40. By the hint:

$$L = \int_1^4 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\, dx = \int_1^4 \sqrt{1 + \frac{9}{4}x}\, dx = \frac{8}{27}\left[\left(1 + \frac{9}{4}x\right)^{3/2}\right]_1^4$$

$$= \frac{1}{27}\left[80\sqrt{10} - 13\sqrt{13}\right] \cong 7.63$$

SECTION 10.8

23. (a) The centroids of the 3, 4, 5 sides are the midpoints $\left(\frac{3}{2}, 0\right)$, $(3, 2)$, $\left(\frac{3}{2}, 2\right)$.

(b) $\overline{x}(3+4+5) = \frac{3}{2}(3) + 3(4) + \frac{3}{2}(5)$, $\quad 12\overline{x} = 24$, $\quad \overline{x} = 2$

$\overline{y}(3+4+5) = 0(3) + 2(4) + 2(5)$, $\quad 12\overline{y} = 18$, $\quad \overline{y} = \dfrac{3}{2}$

(c) $A = \frac{1}{3}(3)(4) = 6$

$$\overline{x}A = \int_0^3 x\left(\frac{4}{3}x\right)\, dx = \int_0^3 \frac{4}{3}x^2 dx = \frac{4}{9}\left[x^3\right]_0^3 = 12, \quad \overline{x} = 2$$

$$\overline{y}A = \int_0^3 \frac{1}{2}\left(\frac{4}{3}x\right)^2 dx = \int_0^3 \frac{8}{9}x^2 dx = \frac{8}{27}\left[x^3\right]_0^3 = 8, \quad \overline{y} = \frac{4}{3}$$

(d) $\overline{x}\,(4+5) = 3(4) + \dfrac{3}{2}(5)$, $\quad 9\overline{x} = \dfrac{39}{2}$, $\quad \overline{x} = \dfrac{13}{6}$

$\overline{y}\,(4+5) = 2(4) + 2(5)$, $\quad 9\overline{y} = 18$, $\quad \overline{y} = 2$

(e) $A_x = 2\pi(2)(5) = 20\pi$

28. C revolved about the y-axis generates a surface of area

$$A_y = 2\pi\overline{x}L = 2\pi\overline{x}r(\theta_2 - \theta_1).$$

C revolved about the x-axis generates a surface of area

$$A_x = 2\pi x\overline{y}L = 2\pi\overline{y}r(\theta_2 - \theta_1).$$

By our solution to Exercise 27.

$$A_y = 2\pi r^2(\sin\theta_2 - \sin\theta_1)$$

and

$$A_x = 2\pi r^2(\cos\theta_1 - \cos\theta_2).$$

Therefore

$$\overline{x} = r\left(\frac{\sin\theta_2 - \sin\theta_1}{\theta_2 - \theta_1}\right), \quad \overline{y} = r\left(\frac{\cos\theta_1 - \cos\theta_2}{\theta_2 - \theta_1}\right).$$

29. (a) Parameterize the upper half of the ellipse by

$$x(t) = a\cos t, \quad y(t) = b\sin t; \qquad t \in [0, \pi].$$

Here

$$\sqrt{[x'(t)]^2 + [y'(t)]^2} = \sqrt{a^2\sin^2 t + b^2\cos^2 t} = \sqrt{a^2 - (a^2 - b^2)\cos^2 t},$$

which, with $c = \sqrt{a^2 - b^2}$, can be written $\sqrt{a^2 - c^2\cos^2 t}$. Therefore,

$$A = \int_0^{\pi} 2\pi b\sin t\sqrt{a^2 - c^2\cos^2 t}\,dt = 4\pi b\int_0^{\pi/2} \sin t\sqrt{a^2 - c^2\cos^2 t}\,dt.$$

Setting $u = c\cos t$, we have $du = -c\sin t$ and

$$A = -\frac{4\pi b}{c}\int_c^0 \sqrt{a^2 - u^2}\,du = \frac{4\pi b}{c}\left[\frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right)\right]_0^c$$

$$= 2\pi b^2 + \frac{2\pi a^2 b}{c}\sin^{-1}\left(\frac{c}{a}\right) = 2\pi b^2 + \frac{2\pi ab}{e}\sin^{-1} e$$

where e is the eccentricity of ellipse: $e = c/a$.

(b) Parameterize the right half of the ellipse by

$$x(t) = a\cos t, \quad y(t) = b\sin t; \qquad t \in \left[-\frac{1}{2}\pi, \frac{1}{2}\pi\right].$$

Again $\sqrt{[x'(t)]^2 + [y'(t)]^2} = \sqrt{a^2 - c^2\cos^2 t}$ where $c = \sqrt{a^2 - b^2}$.

Therefore

$$A = \int_{-\pi/2}^{\pi/2} 2\pi a\cos t\sqrt{a^2 - c^2\cos^2 t}\,dt.$$

Set $u = c\sin t$. Then $du = c\cos t\,dt$ and

$$A = \frac{2\pi a}{c}\int_{-c}^{c} \sqrt{b^2 + u^2}\,du = \frac{2\pi a}{c}\left[\frac{u}{2}\sqrt{b^2 + u^2} + \frac{b^2}{2}\ln\left|u + \sqrt{b^2 + u^2}\right|\right]_{-c}^{c}$$

Routine calculation gives

$$A = 2\pi a^2 + \frac{\pi b^2}{e}\ln\left|\frac{1+e}{1-e}\right|.$$