SECTION 4.1

30. Set $P(x) = x^3 + ax + b$. It is obvious that for x sufficiently large, $P(x) > 0$ and for x sufficiently large negative, $P(x) < 0$. Thus, by the intermediate-value theorem, the equation $P(x) = 0$ has at least one real root.

If $a \geq 0$, then $P'(x) = 3x^2 + a$ is positive, except possibly at 0, where it remains nonnegative. It follows that P is everywhere increasing and therefore it cannot take on the value 0 more than once.

Suppose now that $a < 0$. Then $-\frac{1}{3}\sqrt{3}\,|a|$ and $\frac{1}{3}\sqrt{3}\,|a|$ are consecutive roots of the equation $P'(x) = 0$ and thus, by Exercise 27, P cannot take on the value zero more than once between these two numbers.

33. For $p(x) = x^n + ax + b$, $\quad p'(x) = nx^{n-1} + a$, which has at most one real zero for n even $\left(x = -\frac{a}{n}^{\frac{1}{n-1}}\right)$. If there were more than two distinct real roots of $p(x)$, then by Rolle's theorem there would be more than one zero of $p'(x)$. Thus there are at most two distinct real roots of $p(x)$.

38. (a) Let $f(x) = \cos x$. Choose any numbers x and y, (assume $x < y$). By the mean-value theorem, there is a number c between x and y such that

$$\frac{f(y) - f(x)}{y - x} = f'(c) \quad \Rightarrow \quad \frac{|\cos y - \cos x|}{|y - x|} = |-\sin c| \leq 1 \quad \Rightarrow \quad |\cos x - \cos y| \leq |x - y|$$

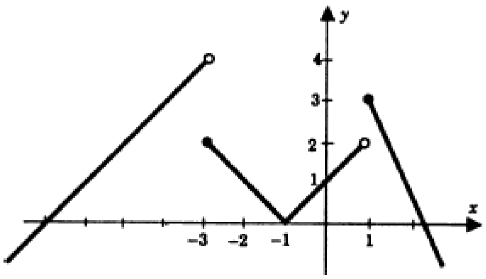
(b) Repeat the in part (a) with $f(x) = \sin x$.

SECTION 4.2

33. $$f'(x) = \begin{cases} 1, & x < -3 \\ -1, & -3 < x < -1 \\ 1, & -1 < x < 1 \\ -2, & 1 < x \end{cases}$$

f increases on $(-\infty, -3)$ and $[-1, 1]$;

decreases on $[-3, -1]$ and $[1, \infty)$



55. Let f and g be functions such that $f'(x) = -g(x)$ and $g'(x) = f(x)$. Then:

(a) Differentiating $f^2(x) + g^2(x)$ with respect to x, we have

$$2f(x)f'(x) + 2g(x)g'(x) = -2f(x)g(x) + 2g(x)f(x) = 0.$$

Thus, $f^2(x) + g^2(x) = C$ (constant).

(b) $f(0) = 0$ and $g(0) = 1$ implies $C = 1$.

(c) The functions $f(x) = \sin x$, $g(x) = \cos x$ have these properties.

58. Let $f(x) = \cos x - \left(1 - \frac{1}{2}x^2\right)$ for $x \in [0, \infty)$. Then $f(0) = 0$ and $f'(x) = -\sin x + x = x - \sin x > 0$ for $x \in (0, \infty)$ by Exercise 51 (b). Thus, $f(x) > 0$ for $x \in (0, \infty)$ which implies $\cos x > 1 - \frac{1}{2}x^2$ on $(0, \infty)$.

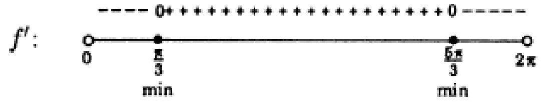
62. (a) Let $f(x) = \cos x - (1 - \dfrac{1}{2}x^2 + \dfrac{1}{24}x^4)$. Then $f(0) = 0$ and $f'(x) = -\sin x + x - \dfrac{x^3}{6} < 0$ by Exercise 60. Therefore, $f(x) < f(0) = 0$ on all $x \in (0, \infty)$, which implies $\cos x < 1 - \dfrac{1}{2}x^2 + \dfrac{1}{24}x^4$ on $(0, \infty)$.

(b) $6^\circ = \dfrac{\pi}{30}$. Using this for x in $1 - \dfrac{1}{2}x^2 < \cos x < 1 - \dfrac{1}{2}x^2 + \dfrac{1}{24}x^4$,

$\Longrightarrow \quad 0.994517 < \cos 6^\circ < 0.994522.$

SECTION 4.3

27. $f'(x) = \cos^2 x - \sin^2 x - 3\cos x + 2 = (2\cos x - 1)(\cos x - 1)$ critical pts $\frac{1}{3}\pi,\ \frac{5}{3}\pi$



$f\left(\frac{1}{3}\pi\right) = \frac{2}{3}\pi - \frac{5}{4}\sqrt{3}$ local min

$f\left(\frac{5}{3}\pi\right) = \frac{10}{3}\pi + \frac{5}{4}\sqrt{3}$ local max

35.
$$P(x) = x^4 - 8x^3 + 22x^2 - 24x + 4$$
$$P'(x) = 4x^3 - 24x^2 + 44x - 24$$
$$P''(x) = 12x^2 - 48x + 44$$

Since $P'(1) = 0$, $P'(x)$ is divisible by $x - 1$. Division by $x - 1$ gives

$$P'(x) = (x-1)\left(4x^2 - 20x + 24\right) = 4(x-1)(x-2)(x-3).$$

The critical pts are 1, 2, 3. Since

$$P''(1) > 0, \quad P''(2) < 0, \quad P''(3) > 0,$$

$P(1) = -5$ is a local min, $P(2) = -4$ is a local max, and $P(3) = -5$ is a local min.
Since $P'(x) < 0$ for $x < 0$, P decreases on $(-\infty, 0]$. Since $P(0) > 0$, P does not take on the value 0 on $(-\infty, 0]$.

Since $P(0) > 0$ and $P(1) < 0$, P takes on the value 0 at least once on $(0, 1)$. Since $P'(x) < 0$ on $(0, 1)$, P decreases on $[0, 1]$. It follows that P takes on the value zero only once on $[0, 1]$.

Since $P'(x) > 0$ on $(1, 2)$ and $P'(x) < 0$ on $(2, 3)$, P increases on $[1, 2]$ and decreases on $[2, 3]$. Since $P(1)$, $P(2)$, $P(3)$ are all negative, P cannot take on the value 0 between 1 and 3.

Since $P(3) < 0$ and $P(100) > 0$, P takes on the value 0 at least once on $(3, 100)$. Since $P'(x) > 0$ on $(3, 100)$, P increases on $[3, 100]$. It follows that P takes on the value zero only once on $[3, 100]$.

Since $P'(x) > 0$ on $(100, \infty)$, P increases on $[100, \infty)$. Since $P(100) > 0$, P does not take on the value 0 on $[100, \infty)$.

44. If $f(x) = \sin x + \dfrac{x^2}{2} - 2x$, then $f'(x) = \cos x + x - 2$ and $f''(x) = -\sin x + 1$. Since $f'(2) = -0.4161 < 0$ and $f'(3) = 0.01 > 0$, f' has at least one zero in $(2, 3)$. Since $f''(x) > 0$ for $x \in (2, 3)$, f' is increasing on this interval and so it has exactly one zero. Thus, f has exactly one critical point c in $(2, 3)$.

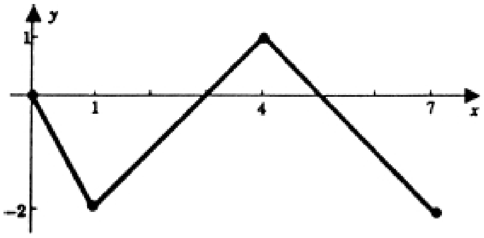
SECTION 4.4

22. $f'(x) = 2\cos 2x - 1, \quad x \in (0, \pi)$:

critical pts. $\frac{1}{6}\pi, \frac{5}{6}\pi$; $\quad f(0) = 0$ endpt min, $f\left(\frac{1}{6}\pi\right) = \frac{1}{2}\sqrt{3} - \frac{1}{6}\pi$ local and abs max,

$f\left(\frac{5}{6}\pi\right) = -\frac{1}{2}\sqrt{3} - \frac{5}{6}\pi$ local and abs min, $f(\pi) = -\pi$ endpt max

25.



$$f'(x) = \begin{cases} -2, & 0 < x < 1 \\ 1, & 1 < x < 4 \\ -1, & 4 < x < 7 \end{cases}$$

critical pts. 1, 4;

$f(0) = 0$ endpt max, $\quad f(1) = -2$ local and abs min,

$f(4) = 1$ local and absolute max, $\quad f(7) = -2$ endpt and abs min

39. If f is not differentiable on (a, b), then f has a critical point at each point c in (a, b) where $f'(c)$ does not exist. If f is differentiable on (a, b), then by the mean-value theorem there exists c in (a, b) where $f'(c) = [f(b) - f(a)]/(b - a) = 0$. This means c is a critical point of f.

44. Let R be a rectangle with its diagonals having length c, and let x be the length of one of its sides. Then the length of the other side is $y = \sqrt{c^2 - x^2}$ and the area of R is given by

$$A(x) = x\sqrt{c^2 - x^2}$$

Now

$$A'(x) = \sqrt{c^2 - x^2} - \frac{x^2}{\sqrt{c^2 - x^2}} = \frac{c^2 - 2x^2}{\sqrt{c^2 - x^2}},$$

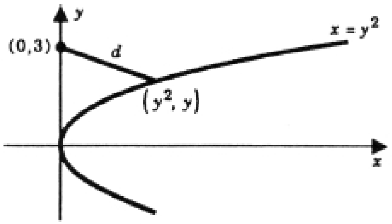
and

$$A'(x) = 0 \Longrightarrow x = \frac{\sqrt{2}}{2}\,c$$

It is easy to verify that A has a maximum at $x = \dfrac{\sqrt{2}}{2}\,c$. Since $y = \dfrac{\sqrt{2}}{2}\,c$ when $x = \dfrac{\sqrt{2}}{2}\,c$, it follows that the rectangle of maximum area is a square.

SECTION 4.5

19.



<u>Minimize d</u>

$d = \sqrt{(y^2 - 0)^2 + (y - 3)^2}$

The square-root function is increasing;

d is minimal when $D = d^2$ is minimal.

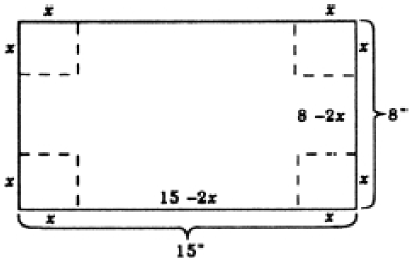
$D(y) = y^4 + (y - 3)^2, \quad y$ real.

$D'(y) = 4y^3 + 2(y - 3) = (y - 1)\left(4y^2 + 4y + 6\right), \quad D'(y) = 0$ at $y = 1$.

Since $D''(y) = 12y^2 + 2 > 0$, the local min at $y = 1$ is the abs min.

The point $(1, 1)$ is the point on the parabola closest to $(0, 3)$.

25.



Maximize V

$V = x(8-2x)(15-2x)$

$$\left.\begin{array}{r} x \geq 0 \\ 8-2x \geq 0 \\ 15-2x \geq 0 \end{array}\right\} \implies 0 \leq x \leq 4$$

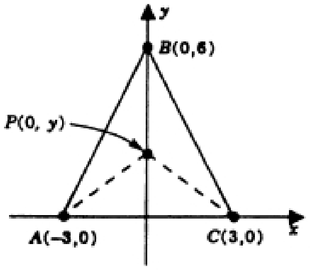
$V(x) = 120x - 46x^2 + 4x^3, \quad 0 \leq x \leq 4.$

$V'(x) = 120 - 92x + 12x^2 = 4(3x-5)(x-6), \quad V'(x) = 0 \ \text{ at } \ x = \frac{5}{3}.$

Since V increases on $(0, \frac{5}{3})$ and decreases on $[\frac{5}{3}, 4)$, the abs max of V occurs when $x = \frac{5}{3}$.

The box of maximal volume is made by cutting out squares 5/3 inches on a side.

27.



Minimize $\overline{AP} + \overline{BP} + \overline{CP} = S$

length $AP = \sqrt{9+y^2}$

length $BP = 6 - y$

length $CP = \sqrt{9+y^2}$

$S(y) = 6 - y + 2\sqrt{9+y^2}, \quad 0 \leq y \leq 6.$

$S'(y) = -1 + \dfrac{2y}{\sqrt{9+y^2}}, \quad S'(y) = 0 \quad \implies \quad y = \sqrt{3}.$

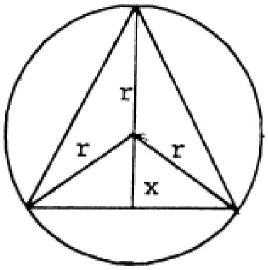
Since

$$S(0) = 12, \quad S\left(\sqrt{3}\right) = 6 + 3\sqrt{3} \cong 11.2, \quad \text{and} \quad S(6) = 6\sqrt{5} \cong 13.4,$$

the abs min of S occurs when $y = \sqrt{3}$.

To minimize the sum of the distances, take P as the point $(0, \sqrt{3})$.

40. Maximize $A(x) = \frac{1}{2}(r + x)2\sqrt{r^2 - x^2}$



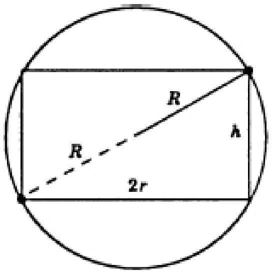
$= (r + x)\sqrt{r^2 - x^2},\ 0 \le x \le r.$

$$A'(x) = \frac{r^2 - rx - 2x^2}{\sqrt{r^2 - x^2}};$$

$$A'(x) = 0 \quad \Longrightarrow \quad x = \frac{r}{2}.$$

Since A increases on $[0, r/2]$ and decreases on $[r/2, r]$, A has an abs max at $x = r/2$;
$A(r/2) = \dfrac{3\sqrt{3}}{4}\, r^2.$

41.



Maximize V

$V = \pi r^2 h$

By the Pythagorean Theorem,

$(2r)^2 + h^2 = (2R)^2$

so

$h = 2\sqrt{R^2 - r^2}.$

$$V(r) = 2\pi r^2\sqrt{R^2 - r^2}, \quad 0 \le r \le R.$$

$$V'(r) = 2\pi\left[2r\sqrt{R^2 - r^2} - \frac{r^3}{\sqrt{R^2 - r^2}}\right] = \frac{2\pi r\left(2R^2 - 3r^2\right)}{\sqrt{R^2 - r^2}}$$

$$V'(r) = 0 \quad \Longrightarrow \quad r = \tfrac{1}{3}R\sqrt{6}.$$

Since V increases on $\left(0, \frac{1}{3}R\sqrt{6}\right]$ and decreases on $\left[\frac{1}{3}R\sqrt{6}, R\right)$, the local max at $r = \frac{1}{3}R\sqrt{6}$ is the abs max.

The cylinder of maximal volume has base radius $\frac{1}{3}R\sqrt{6}$ and height $\frac{2}{3}R\sqrt{3}$.

SECTION 4.6

21. $f'(x) = 2x + 2\cos 2x, \quad f''(x) = 2 - 4\sin 2x;$

concave up on $\left(0,\, \frac{1}{12}\pi\right)$ and on $\left(\frac{5}{12}\pi,\, \pi\right)$, concave down on $\left(\frac{1}{12}\pi,\, \frac{5}{12}\pi\right)$;

pts of inflection $\left(\dfrac{1}{12}\pi,\, \dfrac{72+\pi^2}{144}\right)$ and $\left(\dfrac{5}{12}\pi,\, \dfrac{72+25\pi^2}{144}\right)$

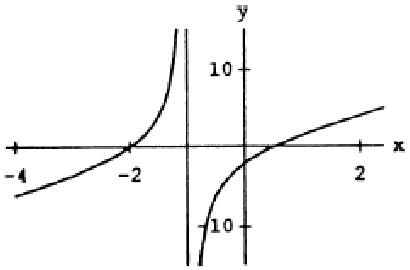
40. $f'(x) = 2cx - 2x^{-3}, \quad f''(x) = 2c + 6x^{-4}$. To have a point of inflection at 1 we need $f''(1) = 0 \implies 2c + 6 = 0 \implies c = -3$

SECTION 4.7

2. (a) d (b) c (c) $x = a,\ x = b$

(d) $y = d$ (e) p (f) q

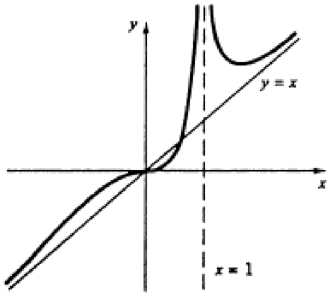
50.



vertical asymptote: $x = -1$

oblique asymptote: $y = 2x + 1$

51.



vertical asymptote: $x = 1$

oblique asymptote: $y = x$

SECTION 4.8

10. $f(x) = x - x^{-1}$,

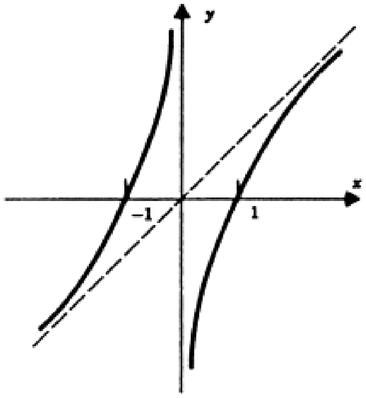
$f'(x) = 1 + x^{-2}$

$f''(x) = -x^{-3}$

f': + + +   + + + +
0

f'': + + +   − − − −
0

asymptotes: $x = 0,\ y = x$

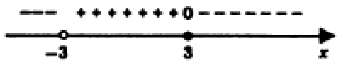


23. $f(x) = \dfrac{x}{(x+3)^2}$

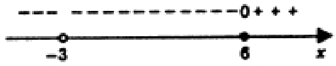
$f'(x) = \dfrac{3-x}{(x+3)^3}$

$f''(x) = \dfrac{2x-12}{(x+3)^4}$

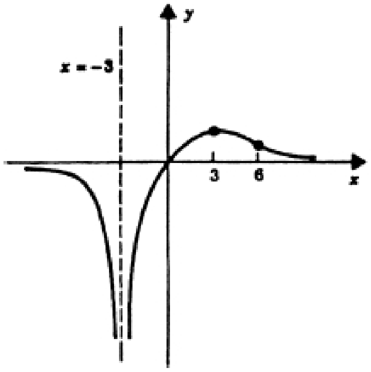
f':



f'':



asymptotes: $x = -3,\ y = 0$

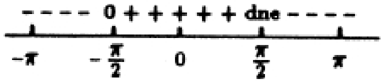


53. $f(x) = \dfrac{\sin x}{1-\sin x}, \quad x \in (-\pi, \pi)$

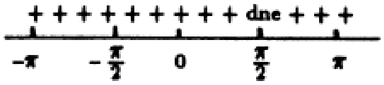
$f'(x) = \dfrac{\cos x}{(1-\sin x)^2}$

$f''(x) = \dfrac{1-\sin x + \cos^2 x}{(1-\sin x)^3}$

f':



f'':



asymptote: $x = \frac{1}{2}\pi$