［廣義柯西不等式］

任給 $m \times n$ 個非負實數 $a_{11}, a_{12}, \ldots, a_{1n}$, $a_{21}, a_{22}, \ldots, a_{2n}$, ... , $a_{m1}, a_{m2}, \ldots, a_{mn}$ 其中 m,n ≥ 2 則有以下的不等式：

$$\left(a_{11}{}^m + a_{12}{}^m + \ldots + a_{1n}{}^m\right) \cdot \left(a_{21}{}^m + a_{22}{}^m + \ldots + a_{2n}{}^m\right) \cdot \ldots \cdot \left(a_{m1}{}^m + a_{m2}{}^m + \ldots + a_{mn}{}^m\right)$$
$$\geq \left(a_{11}a_{21}a_{31}\ldots a_{m1} + a_{12}a_{22}a_{32}\ldots a_{m2} + \ldots\ldots + a_{1n}a_{2n}a_{3n}\ldots a_{mn}\right)^{\mathrm{m}}$$

[證明] 令

$M = \sqrt[m]{\left(a_{11}{}^m + a_{12}{}^m + \ldots + a_{1n}{}^m\right) \cdot \left(a_{21}{}^m + a_{22}{}^m + \ldots + a_{2n}{}^m\right) \cdot \ldots \cdot \left(a_{m1}{}^m + a_{m2}{}^m + \ldots + a_{mn}{}^m\right)}$ 。利用

m 階的算幾不等式 $\dfrac{x_1 + x_2 + \ldots + x_m}{m} \geq \sqrt[m]{x_1 x_2 \ldots x_m}$ 可知，對於任何 $i = 1, 2, 3, \ldots, n$，

若取 $x_j = \dfrac{a_{ji}{}^m}{a_{j1}{}^m + a_{j2}{}^m + \ldots + a_{jn}{}^m}$ ，則有：

$$\frac{1}{m}\left(\frac{a_{1i}{}^m}{a_{11}{}^m + a_{12}{}^m + \ldots + a_{1n}{}^m} + \ldots + \frac{a_{mi}{}^m}{a_{m1}{}^m + a_{m2}{}^m + \ldots + a_{mn}{}^m}\right)$$
$$\geq \sqrt[m]{\frac{a_{1i}{}^m}{a_{11}{}^m + a_{12}{}^m + \ldots + a_{1n}{}^m} \cdots \frac{a_{mi}{}^m}{a_{m1}{}^m + a_{m2}{}^m + \ldots + a_{mn}{}^m}} = \frac{a_{1i}\ldots a_{mi}}{M}$$

故得：

$$\sum_{i=1}^{n} \frac{1}{m}\left(\frac{a_{1i}{}^m}{a_{11}{}^m + a_{12}{}^m + \ldots + a_{1n}{}^m} + \ldots + \frac{a_{mi}{}^m}{a_{m1}{}^m + a_{m2}{}^m + \ldots + a_{mn}{}^m}\right) \geq \sum_{i=1}^{n} \frac{a_{1i}\ldots a_{mi}}{M}$$

又因爲

$$\sum_{i=1}^{n} \frac{1}{m}\left(\frac{a_{1i}{}^m}{a_{11}{}^m + a_{12}{}^m + \ldots + a_{1n}{}^m} + \ldots + \frac{a_{mi}{}^m}{a_{m1}{}^m + a_{m2}{}^m + \ldots + a_{mn}{}^m}\right)$$
$$= \frac{1}{m}\left(\sum_{i=1}^{n} \frac{a_{1i}{}^m}{a_{11}{}^m + a_{12}{}^m + \ldots + a_{1n}{}^m} + \ldots + \sum_{i=1}^{n} \frac{a_{mi}{}^m}{a_{m1}{}^m + a_{m2}{}^m + \ldots + a_{mn}{}^m}\right)$$
$$= \frac{1}{m}(1 + \ldots + 1) = \frac{1}{m} \cdot m = 1$$

所以得到 $1 \geq \sum_{i=1}^{n} \dfrac{a_{1i}\ldots a_{mi}}{M}$ ，也就是 $\sum_{i=1}^{n} a_{1i}\ldots a_{mi} \leq M$ 。

由上式可立刻得到原不等式。 □