Chapter 3

Given a set of data point $\{v_1, v_2, \cdots, v_N\} \in R^N$ with $N = 2m$ is even, $h = \frac{2\pi}{N}$, then the DFT formula for $\{v_j\}$ is

(Eq. 1) $\hat{v}_k = h\sum_{j=1}^{N} e^{-ikx_j} v_j = \frac{2\pi}{N}\sum_{j=1}^{N} e^{-ikx_j} v_j$ for $k = -m+1, \cdots, m$

(Eq. 2) $v_j = \frac{1}{2\pi}\sum_{k=-m+1}^{m} e^{ikx_j}\hat{v}_k$ for $j = 1, 2, \cdots, N$

Remark 1: In fact we can write $v_j = \frac{1}{2\pi}\sum_{k=1}^{N} e^{ikx_j}\hat{v}_k$ since $\{\varphi_k(x) = e^{ikx} : k = 1, 2, \cdots, N\}$ is orthogonal basis. However it is easy to show that $\varphi_{N+k}(x_j) = e^{i(N+k)x_j} = e^{ikx_j}e^{i2\pi j} = e^{ikx_j} = \varphi_k(x_j)$, this means that $\varphi_{N+k}(x_j) = \varphi_k(x_j)$ for all $j = 1, 2, \cdots, N$. In other words, in discrete set $I = \{1, 2, \cdots, N\}$, $\varphi_{k\pm N}(I) = \varphi_k(I)$. We can use another index set $k = -m+1, \cdots, m$ by $k \to k - N$ for $k = N, N-1, \cdots, m+1$.

Prop 1: (1) $\varphi_k(I)$ forms orthogonal basis $\langle \varphi_k, \varphi_l \rangle = N\delta_{k,l}$

(2) $\hat{v}_{-k} = \hat{v}_k^*$ and $\hat{v}_{N+k} = \hat{v}_k$ for $\{v_j\} \in R^N$ and then $\hat{v}_m \in R$

(3) $v_j = \frac{1}{2\pi}\left(\hat{v}_0 + (-1)^j \hat{v}_m\right) + \frac{1}{\pi}\sum_{k=1}^{m-1}\left(\cos(kx_j)\operatorname{Re}\hat{v}_k - \sin(kx_j)\operatorname{Im}\hat{v}_k\right)$

<proof> $\langle \varphi_k, \varphi_k \rangle = \sum_{j=1}^{N} e^{ikx_j}\left(e^{ikx_j}\right)^* = \sum_{j=1}^{N} 1 = N$

$\langle \varphi_k, \varphi_l \rangle = \sum_{j=1}^{N} e^{ikx_j}\left(e^{ilx_j}\right)^* = \sum_{j=1}^{N} e^{i(k-l)hj} = \sum_{j=1}^{N} z^j = z\frac{z^N - 1}{z - 1} = 0$ where $z = e^{i(k-l)h}$ and $z^N = 1$

$\hat{v}_{-k} = h\sum_{j=1}^{N} e^{ikx_j} v_j = \left(h\sum_{j=1}^{N} e^{-ikx_j} v_j\right)^* = \hat{v}_k^*$

$\hat{v}_{N+k} = \hat{v}_k$ comes from $\varphi_{k\pm N}(I) = \varphi_k(I)$.

$\hat{v}_m = h\sum_{j=1}^{N} e^{-imx_j} v_j = h\sum_{j=1}^{N} (-1)^j v_j \in R$

In fact $\hat{v}_m \in R$ due to $\varphi_m(I) = (-1|1|-1|1|\cdots|-1|1)^T$

$$v_j = \frac{1}{2\pi}\sum_{k=-m+1}^{m} e^{ikx_j}\hat{v}_k = \frac{1}{2\pi}\left(\hat{v}_0 + e^{imx_j}\hat{v}_m\right) + \frac{1}{2\pi}\sum_{k=1}^{m-1}\left(e^{ikx_j}\hat{v}_k + e^{-ikx_j}\hat{v}_{-k}\right)$$

$$= \frac{1}{2\pi}\left(\hat{v}_0 + (-1)^j \hat{v}_m\right) + \frac{1}{\pi}\sum_{k=1}^{m-1}\left(\cos(kx_j)\operatorname{Re}\hat{v}_k - \sin(kx_j)\operatorname{Im}\hat{v}_k\right)$$

From **Prop 1**, we have $\hat{v}_m \in R$, then we can re-write (Eq. 2) to be a symmetric form.

Consider $v_j = \frac{1}{2\pi}\sum_{k=-m+1}^{m} e^{ikx_j}\hat{v}_k = \frac{1}{2\pi}\sum_{k=-m+1}^{m-1} e^{ikx_j}\hat{v}_k + \frac{1}{2\pi}e^{imx_j}\hat{v}_m$, if we define $\hat{v}_{-m} = \hat{v}_m = \hat{v}_{m-N}$, then $e^{imx_j}\hat{v}_m = (-1)^j \hat{v}_m = (-1)^j \frac{1}{2}(\hat{v}_m + \hat{v}_{-m}) = e^{imx_j}\hat{v}_m + e^{-imx_j}\hat{v}_{-m}$. Hence

(Eq. 3) $$v_j = \frac{1}{2\pi}\mathrm{P}\sum_{k=-m}^{m} e^{ikx_j}\hat{v}_k = \frac{1}{2\pi}\sum_{k=-(m-1)}^{m-1} e^{ikx_j}\hat{v}_k + \frac{1}{2\pi}\frac{1}{2}e^{imx_j}\hat{v}_m + \frac{1}{2\pi}\frac{1}{2}e^{-imx_j}\hat{v}_{-m}$$

Where $\mathrm{P}\sum_{k=-m}^{m} e^{ikx_j}\hat{v}_k$ borrow the name of principal value, means symmetry operation.

Remark 2: $\hat{v}_m \in R$ and $e^{imx_j} = (-1)^j \in R$ implies $e^{imx_j}\hat{v}_m = \mathrm{Re}\left(e^{imx_j}\hat{v}_m\right) = \cos\left(mx_j\right)\hat{v}_m$ but $\frac{d}{dx}\left(e^{imx}\hat{v}_m\right)|_{x_j} = ime^{imx_j}\hat{v}_m = im(-1)^j \hat{v}_m \notin R$, this is because we ask band-limit interpolant must be real-valued function, then $\frac{d}{dx}\frac{1}{2}\left(e^{imx}\hat{v}_m + e^{-imx}\hat{v}_{-m}\right) = -\sin(mx)\hat{v}_m \in R$. An alternative is to take real part of $\frac{d}{dx}\left(e^{imx}\hat{v}_m\right)|_{x_j}$. However if we do so, then $\mathrm{Re}\frac{d}{dx}\left(e^{imx}\hat{v}_m\right)|_{x_j} = 0$ holds for any data set v since $\hat{v}_m \in R$, this is equivalent to set $\hat{v}_m = 0$. Later we will discuss an alternative method to find out v' by using FFT, at this time, we set $\hat{v}_m = 0$.

We develop a band-limited interpolant from (Eq. 3) which takes the form

(Eq. 4) $$p(x) = \frac{1}{2\pi}\mathrm{P}\sum_{k=-m}^{m} e^{ikx}\hat{v}_k \quad \text{for} \quad x \in [0, 2\pi]$$

It is clear that $v_j = p(x_j)$. Moreover From **Prop 1**, we show trigonometric combination of v_j, written as $v_j = \frac{1}{2\pi}\left(\hat{v}_0 + (-1)^j \hat{v}_m\right) + \frac{1}{\pi}\sum_{k=1}^{m-1}\left(\cos\left(kx_j\right)\mathrm{Re}\,\hat{v}_k - \sin\left(kx_j\right)\mathrm{Im}\,\hat{v}_k\right)$.If we write $(-1)^j = \cos\left(mx_j\right)$, then $v_j \in sp\{1, \cos(mx)\} \cup sp\{\cos(kx), \sin(kx) : k = 1, 2, \cdots, m-1\}$.

Define discrete delta function $\delta_j = \begin{cases} 1, & j = 0 \pmod N \\ 0, & j \neq 0 \pmod N \end{cases}$, then from (Eq. 1), $\left(\hat{\delta}_j\right)_k = h$ and green function $G(x)$ corresponding to δ_j is defined as $G(x) = \frac{1}{2\pi}\mathrm{P}\sum_{k=-m}^{m} e^{ikx}\left(\hat{\delta}_j\right)_k$, then

$$G(x)=\frac{h}{2\pi}\mathrm{P}\sum_{k=-m}^{m}e^{ikx}=\frac{h}{2\pi}\left[\frac{1}{2}\left(e^{imx}+e^{-imx}\right)+1+\sum_{k=1}^{m-1}\left(e^{ikx}+e^{-ikx}\right)\right]$$

$$=\frac{h}{2\pi}\left(\cos mx+1+2\sum_{k=1}^{m-1}\cos kx\right)$$

$$\sum_{k=1}^{m-1}\cos kx=\mathrm{Re}\sum_{k=1}^{m-1}e^{ikx}=\mathrm{Re}\sum_{k=1}^{m-1}z^{k}=\mathrm{Re}\left(z\frac{z^{m-1}-1}{z-1}\right)\quad \text{where } z=e^{ix}$$

$$=\mathrm{Re}\left[e^{ix}\left(e^{ix(m-1)}-1\right)\left(e^{ix/2}-e^{-ix/2}\right)^{-1}\right]=\frac{1}{2\sin(x/2)}\left[\sin(m-1/2)x-\sin(x/2)\right]$$

Hence $G(x)=\dfrac{h}{2\pi}\dfrac{\sin(mx)}{\tan(x/2)}=\dfrac{\sin(\pi x/h)}{(2\pi/h)\tan(x/2)}$ for $x\neq 0$.

Of course, $\lim_{x\to 0}G(x)=G(0)=\dfrac{h}{2\pi}\mathrm{P}\sum_{k=-m}^{m}e^{ikx}|_{x=0}=1$.

Definition 1: band-limit interpolant of δ is periodic sinc function $S_N(x)$, defined by

(Eq. 5) $S_N(x)=\dfrac{\sin(\pi x/h)}{(2\pi/h)\tan(x/2)}=\dfrac{1}{2m}\dfrac{\sin(mx)}{\tan(x/2)}$ $\left(S_N(x)=\dfrac{h}{2\pi}\mathrm{P}\sum_{k=-m}^{m}e^{ikx}\right)$

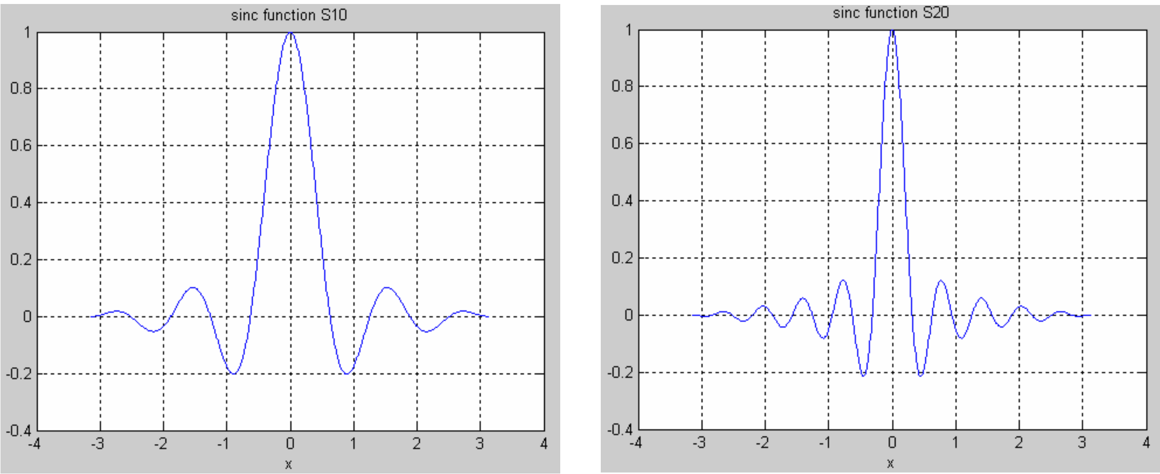

Figure 1: left panel is sinc function with $N=10$ and right panel is $N=20$. It is clear that support becomes narrower when N inceases.

Remark 3: in the classical theory of Fourier series, Dirichlet kernel is define $D_m(x)=\sum_{k=-m}^{m}e^{ikx}$,

we have $\dfrac{1}{2m+1}D_m(x)=\dfrac{1}{2m+1}\dfrac{\sin[(m+1/2)x]}{\sin(x/2)}$.

(Eq. 6) $D_m(x)-S_N(x)=D_m(x)\left(\dfrac{1}{2m+1}-\dfrac{1}{2m}\right)+\dfrac{1}{2m+1}\cos(mx)$

due to $D_m(x)=\dfrac{1}{\sin(x/2)}\left(\sin mx\cos\dfrac{x}{2}+\cos mx\sin\dfrac{x}{2}\right)=\dfrac{\sin mx}{\tan(x/2)}+\cos mx$

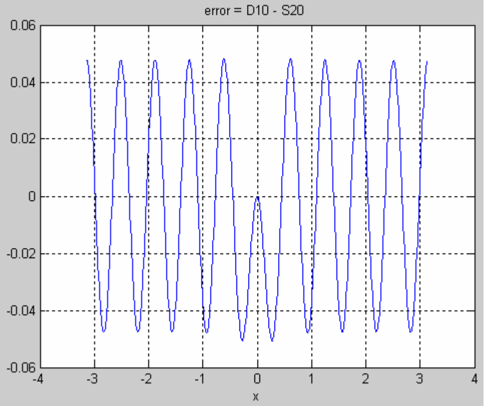

Figure 2: $D_m(x)-S_N(x)$ for $m=10$, in fact wiggle occurs.

Table 1: measure $D_m(x)-S_N(x)$ with L^2-norm and $L^\infty-norm$

	$m=10$	$m=20$	$m=40$	$m=80$
$L^2-error$	3.368E-2	1.727E-2	8.723E-3	4.399E-3
$L^\infty-error$	5.06E-2	2.589E-2	1.31E-2	6.492E-3

From (Eq. 6), we can estimate $L_\infty-norm$ of $D_m(x)-S_N(x)$

(Eq. 7) $$\|D_m(x)-S_N(x)\|_{L^\infty} \le \frac{1}{2m}\left\|\frac{D_m(x)}{2m+1}\right\|_\infty + \frac{1}{2m+1} \le \frac{1}{m}$$

If we define inner product $\langle g,f\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} g^*(x)f(x)dx$, then

(Eq. 8) $$\|D_m(x)-S_N(x)\|_{L^\infty} \le \frac{1}{2m}\left\|\frac{D_m(x)}{2m+1}\right\|_2 + \frac{1}{2m+1}\|\cos(mx)\|_2 \le \frac{3}{4m}$$

These two estimates provide first order convergence as you see in **Table 1**.

If we write $v_j = \sum_{m=1}^{N} v_m\delta_{j-m} = \delta * v$, then

(Eq. 9) $$p(x) = \frac{1}{2\pi}\mathrm{P}\sum_{k=-m}^{m} e^{ikx}\hat{v}_k = \sum_{k=1}^{N} v_k S_N(x-x_k)$$ and

the derivative is according to

(Eq. 10) $$w_j = p'(x_j) = \sum_{k=1}^{N} v_k S_N'(x_j-x_k)$$

Prop 2: direct compute directive of $S_N(x) = \dfrac{\sin(\pi x/h)}{(2\pi/h)\tan(x/2)}$ on $[0,2\pi]$

(Eq. 11) $$S_N'(x_j) = \begin{cases} 0, & j=0(\mathrm{mod}\, N) \\ \frac{1}{2}(-1)^j\cot\left(\frac{jh}{2}\right), & j\ne 0(\mathrm{mod}\, N)\end{cases}$$

<**proof**> for $x \neq 0$, then $S_N'(x) = \frac{1}{2}\cot\left(\frac{x}{2}\right)\cos\left(\frac{\pi}{h}x\right) - \frac{h}{4\pi}\sin^{-2}\left(\frac{x}{2}\right)\sin\left(\frac{\pi}{h}x\right)$.

Using Taylor expansion to show $S_N(x) = 1 + O(x^2)$, hence $S_N'(0) = 0$.

For simplicity we write $(w_j) = (D_{jk})(v_k)$ where $D_{jk} = S_N'(x_j - x_k)$, for example,

$$D_5 = \left(S_5'(x_j - x_k)\right) = \begin{bmatrix} 0 & S_5'(-h) & S_5'(-2h) & S_5'(-3h) & S_5'(-4h) \\ S_5'(h) & 0 & S_5'(-h) & S_5'(-2h) & S_5'(-3h) \\ S_5'(2h) & S_5'(h) & 0 & S_5'(-h) & S_5'(-2h) \\ S_5'(3h) & S_5'(2h) & S_5'(h) & 0 & S_5'(-h) \\ S_5'(4h) & S_5'(3h) & S_5'(2h) & S_5'(h) & 0 \end{bmatrix}$$

Since $S_N'(x_{j\pm N}) = S_N'(x_j)$, formally we write

$$D_5 = \left(S_5'(x_j - x_k)\right) = \begin{bmatrix} 0 & S_5'(N-h) & S_5'(N-2h) & S_5'(N-3h) & S_5'(N-4h) \\ S_5'(h) & 0 & S_5'(N-h) & S_5'(N-2h) & S_5'(N-3h) \\ S_5'(2h) & S_5'(h) & 0 & S_5'(N-h) & S_5'(N-2h) \\ S_5'(3h) & S_5'(2h) & S_5'(h) & 0 & S_5'(N-h) \\ S_5'(4h) & S_5'(3h) & S_5'(2h) & S_5'(h) & 0 \end{bmatrix}$$

We can use built-in function "toeplitz" to built toeplitz matrix

Let $column = \left[0 \quad S_5'(h) \quad S_5'(2h) \quad S_5'(3h) \quad S_5'(4h)\right]$ and

$row = \left[0 \quad S_5'(4h) \quad S_5'(3h) \quad S_5'(2h) \quad S_5'(h)\right] = column\left[1, (N:-1:2)\right]$, then

toeplitz(column, row) ;

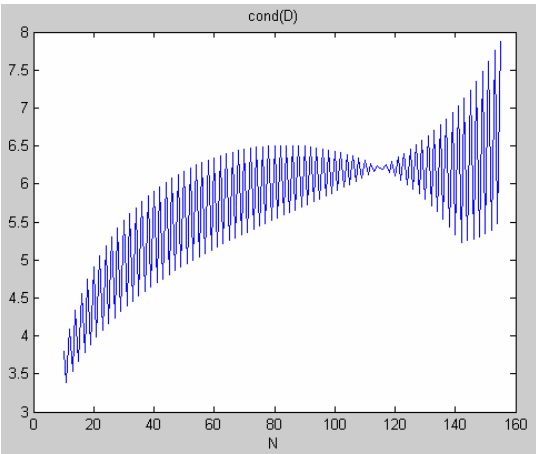

Figure 3: condition number of D_N for $N = 10:155$

But $N = 162:200$, $cond(D_N) = 1$ and $cond(D_{160}) = 1.1580\text{e}+015$, why?

Prop 3: direct compute 2nd directive of $S_N(x) = \frac{\sin(\pi x/h)}{(2\pi/h)\tan(x/2)}$ on $[0, 2\pi]$

(Eq. 12) $$S_N''(x_j) = \begin{cases} -\frac{1}{6} - \frac{\pi^2}{3h^2}, & j = 0 \pmod N \\ -\frac{(-1)^j}{2\sin^2(jh/2)}, & j \neq 0 \pmod N \end{cases}$$

<**proof**> we have shown $\frac{2\pi}{h} S_N'(x) = \frac{1}{\sin^2(x/2)}\left[\frac{\pi}{2h}\sin x \cos\left(\frac{\pi}{h}x\right) - \frac{1}{2}\sin\left(\frac{\pi}{h}x\right)\right]$ in **Prop 1**,

then using Taylor expansion,

$$\frac{1}{x}\left[S_N'(x) - S_N'(0)\right] = \frac{h/(2\pi)}{x\sin^2(x/2)}\left\{\left[-\frac{\pi}{12h} - \frac{1}{6}\left(\frac{\pi}{h}\right)^3\right]x^3 + O(x^5)\right\}$$

then divide x^3 on denominator and nominator, and take limit $x \to 0$.
Second, directly compute second derivative for $x \neq 0$

$$S_N''(x) = -\frac{1}{2}\frac{\cos(\pi x/h)}{\sin^2(x/2)} - \frac{\pi}{2h}\cot(x/2)\sin\left(\frac{\pi}{h}x\right) + \frac{h}{4\pi}\frac{\cot(x/2)}{\sin^2(x/2)}\sin\left(\frac{\pi}{h}x\right)$$

Prop 4: direct compute 3rd directive of $S_N(x) = \frac{\sin(\pi x/h)}{(2\pi/h)\tan(x/2)}$ on $[0, 2\pi]$

(Eq. 13) $$S_N^{(3)}(x_j) = \begin{cases} 0, & j = 0 \pmod N \\ (-1)^j\left[-\frac{\pi^2}{2h^2}\frac{1}{\tan(hj/2)} + \frac{3}{2}\frac{1}{\sin(hj)}\cot^2(hj/2)\right], & j \neq 0 \pmod N \end{cases}$$

<**proof**> we use symbolic toolbox in Matlab to obtain

$$S_N^{(3)}(x) = -\frac{\pi^2}{2h^2}\frac{\cos(\pi x/h)}{\tan(x/2)} + \frac{3\pi\xi}{2h}\frac{\sin(\pi x/h)}{\tan^2(x/2)} + 3\xi^2\frac{\cos(\pi x/h)}{\tan^3(x/2)} - \frac{3\xi}{2}\frac{\cos(\pi x/h)}{\tan(x/2)}$$
$$-\frac{3h\xi^3}{\pi}\frac{\sin(\pi x/h)}{\tan^4(x/2)} + \frac{5h\xi^2}{2\pi}\frac{\sin(\pi x/h)}{\tan^2(x/2)} - \frac{h\xi}{2\pi}\sin(\pi x/h)$$

where $\xi = \frac{1}{2}\sec^2(x/2)$

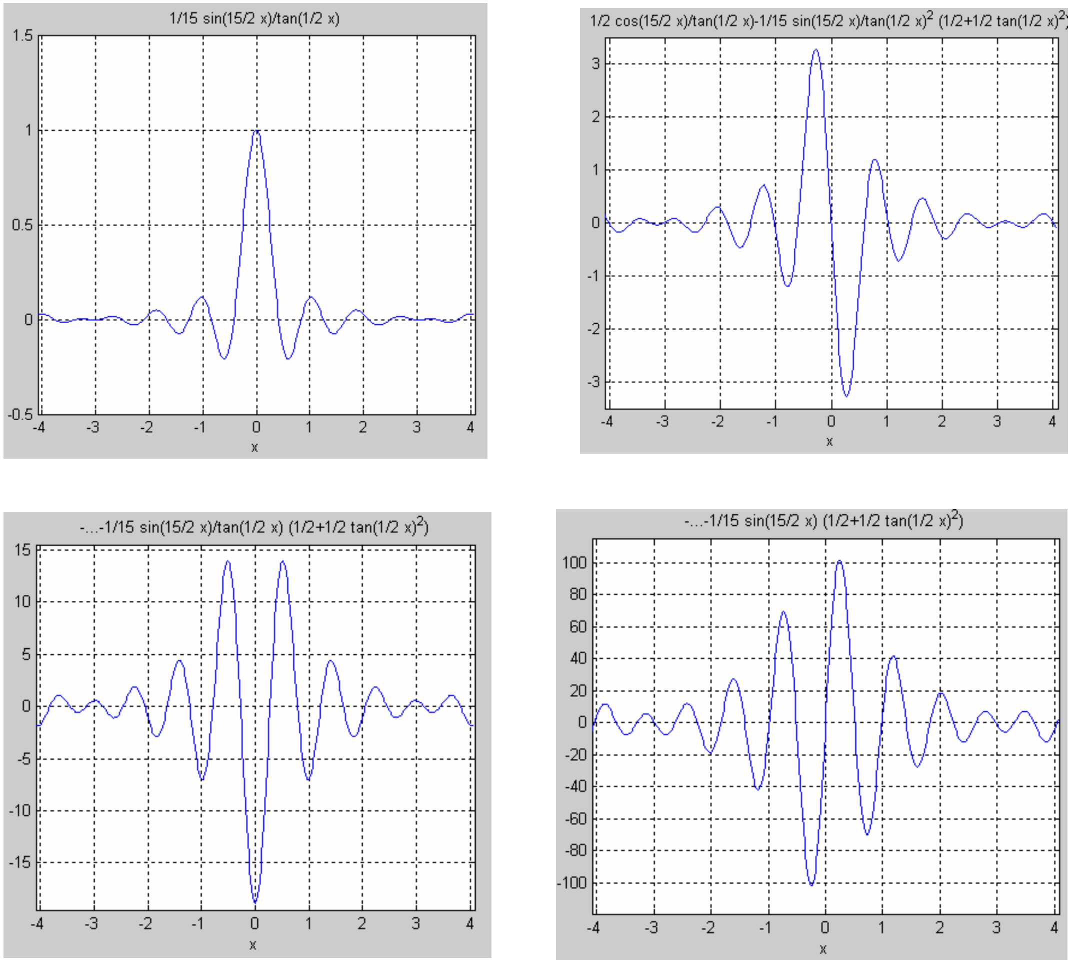

Figure 4: top-left: S_{15} , top-right: S_{15}' ; bottom-left: S_{15}'' , bottom-right: $S_{15}^{(3)}$

Example 1: we try two function, one is rough hat function

$$V_{hat} = \max\left(1, \frac{1}{2}|x-\pi|\right) = \begin{cases} 0, & \text{otherwise} \\ \frac{-1}{2}(x-\pi), & \pi - 2 < x < \pi , \\ \frac{1}{2}(x-\pi), & \pi < x < \pi + 2 \end{cases} \quad \text{with}$$

$$V_{hat}' = \begin{cases} 0, & 0 < x < \pi - 2, \pi + 2 < x < 2\pi \\ \frac{-1}{2}, & \pi - 2 < x < \pi \\ \frac{1}{2}, & \pi < x < \pi + 2 \end{cases}$$

. The other one is smooth function $V = \exp(\sin x)$.

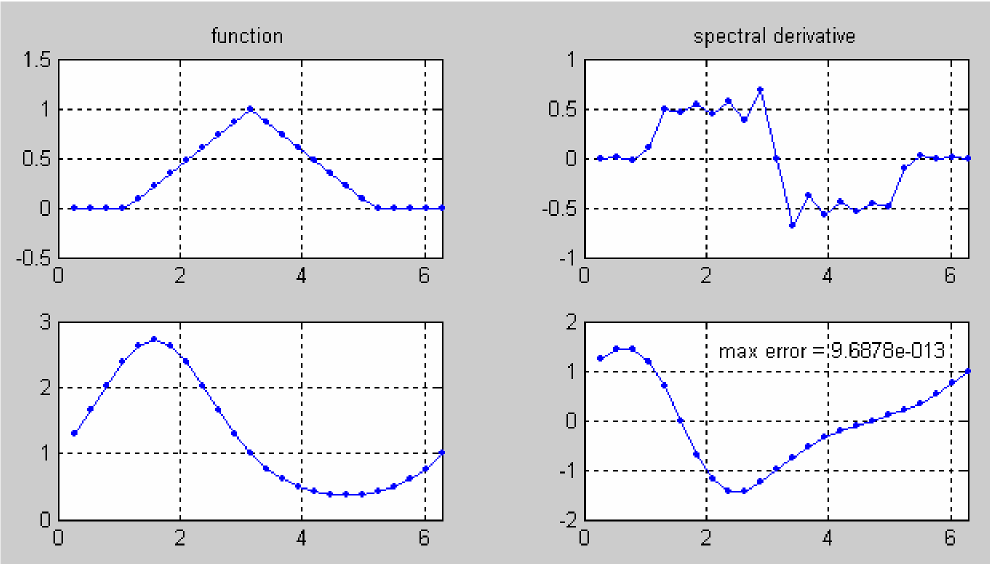

Figure 5: spectral differentiation of a rough function and a smooth one. The smooth function gives 12-digit accuracy.

If we use FFT to find v_j', then we have very close result (see Figure 6) same as Figure 5, expect that when $V = \exp(\sin x)$ in Figure 5, we have error 9.6878×10^{-13} but in Figure 6, error is 9.5679×10^{-13}, differ by 1.2×10^{-14}. In fact we can plot $\left|v'_{Conv} - v'_{FFT}\right|$, see Figure 7, and $\left|v'_{Conv} - v'_{FFT}\right|_\infty = 2.94\times10^{-14}$.

Question 1: why different?

<ans> This is due to rounding error, since theoretically speaking, both methods produce the same results.

Case 1: kernel method

(Eq. 14)
$$p'(x_j) = \frac{1}{2\pi} \mathrm{P}\sum_{k=-m}^{m} ike^{ikx_j}\hat{v}_k = \frac{1}{2\pi}\sum_{k=-(m-1)}^{m-1} ike^{ikx_j}\hat{v}_k + \frac{1}{4\pi}\left(ime^{imx_j}\hat{v}_m - ime^{-imx_j}\hat{v}_{-m}\right)$$
$$= \frac{1}{2\pi}\sum_{k=-(m-1)}^{m-1} ike^{ikx_j}\hat{v}_k$$

Since $\frac{1}{4\pi}\left(ime^{imx_j}\hat{v}_m - ime^{-imx_j}\hat{v}_{-m}\right) = \hat{v}_m\frac{1}{2\pi}\sin(mhj) = \hat{v}_m\frac{1}{2\pi}\sin(\pi j) = 0$.

Case 2: FFT method

(Eq. 15) $\hat{v}_k = h\sum_{j=1}^{N} e^{-ikx_j}v_j = \frac{2\pi}{N}\sum_{j=1}^{N} e^{-ikx_j}v_j$ for $k = -m+1,\cdots,m$

(Eq. 16) $w_j \triangleq \frac{1}{2\pi}\sum_{k=-m+1}^{m} ike^{ikx_j}\hat{v}_k$ with $\hat{v}_m = 0$

It is clear that (Eq. 14) is the same as (Eq. 16).

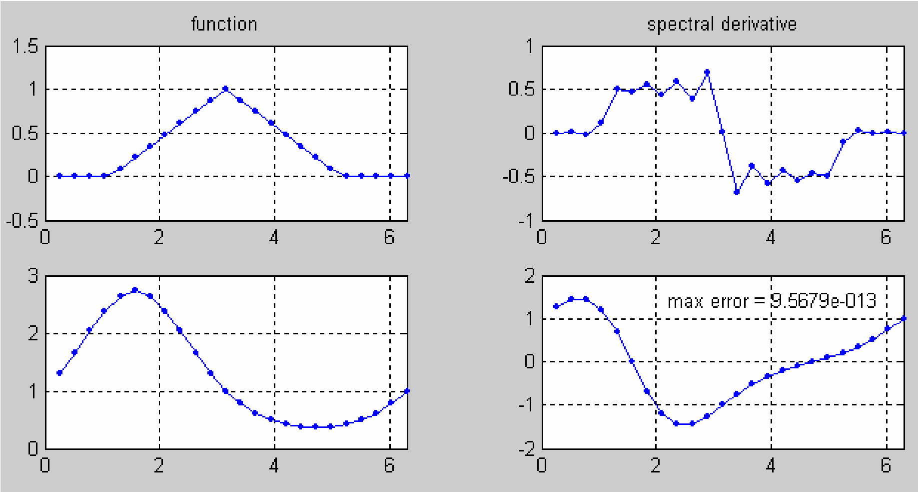

Figure 6: use FFT to find v_j'

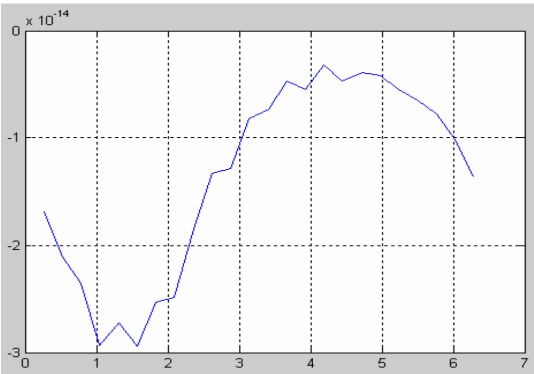

Figure 7: error $\left|v'_{Conv} - v'_{FFT}\right|$.

Table 2: we use Matlab to show $\left|v'_{Conv} - v'_{FFT}\right|_\infty$ is up to machine zero

n	8	10	20	30
$\left\|v'_{Conv} - v'_{FFT}\right\|_\infty$	8.88E-16	2.33E-15	6.21E-15	2.45E-14

Example 2: consider variable coefficient wave equation

(Eq. 17) $u_t + c(x)u_x = 0$, $c(x) = 0.2 + \sin^2(x-1)$ for $x \in [0, 2\pi], t > 0$

with initial condition $u(x,0) = \exp\left(-100(x-1)^2\right)$

Here we adopt leap-frog scheme for temporal discretization,

(Eq. 18) $\dfrac{u_j^{(n+1)} - u_j^{(n-1)}}{2\Delta t} = -c(x_j)\left(Du^{(n)}\right)_j$ for $j = 1, 2, \cdots, N$

and extrapolate another initial condition, $u(x,0) = \exp\left(-100(x - 0.2\Delta t - 1)^2\right)$, which is wave backward with constant speed of 0.2. The result of spectral method is shown in Figure 8, note that

(1) $u(x,0)$ has compact support, we may regard it as periodic function before wave touch the boundary.

(2) From Figure 8, wave move faster at $x \in (2,4)$ and slower at $x \in (4,5)$, this is reasonable since $c(1+\pi/2) = c(2.57) = 1.2$ is maxima and $c(1+\pi) = c(4.14) = 0.2$ is minima.

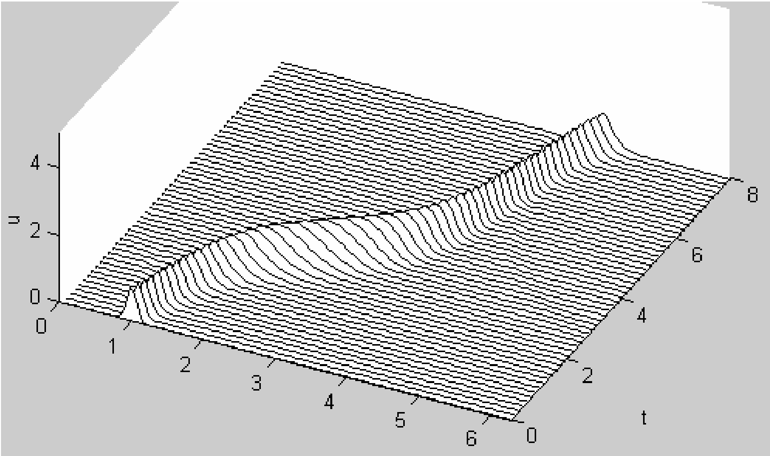

Figure 8: solution of wave equation under spectral method.

Second we use finite difference leap frog scheme

(Eq. 19) $$\frac{u_j^{(n+1)} - u_j^{(n-1)}}{2\Delta t} = -c(x_j)\frac{u_{j+1}^{(n)} - u_{j-1}^{(n)}}{2\Delta x} \quad \text{for} \quad j = 1, 2, \cdots, N$$

with Dirichlet boundary data.

Since our data set is $x_1, x_2, \cdots, x_N = 2\pi$, we set

(Eq. 20) $$D_h u_j = \frac{1}{(2\Delta x)}\begin{cases} u_2^{(n)}, & j = 1 \\ \left(u_{j+1}^{(n)} - u_{j-1}^{(n)}\right), & 1 < j < N \\ 0, & j = N \end{cases}$$

This setting guarantees that $u(2\pi, t) = u(2\pi, 0)$.

The result of (Eq. 19) is shown in Figure 9, spurious wave occurs.

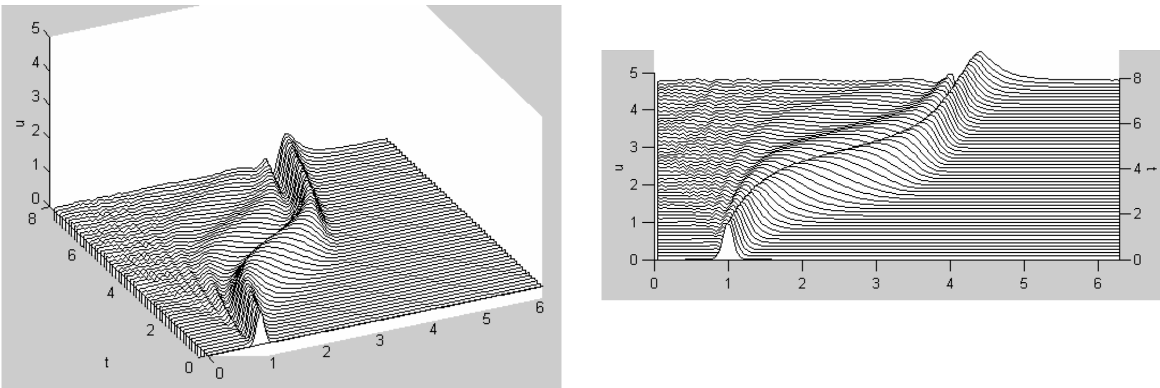

Figure 9: solution of standard finite difference leap frog scheme, (Eq. 19)

In fact, $\frac{du}{dx}(x_j) = \frac{u_{j+1} - u_{j-1}}{2\Delta x} - \frac{h^2}{6}\frac{d^3u}{dx^3}(x_j) - \frac{h^4}{120}\frac{d^5u}{dx^5}(x_j) + O(h^6)$, we can correct high order derivative to improve finite difference (this may interpret artificial dispersion)

(Eq. 21) $$\frac{u_j^{(n+1)}-u_j^{(n-1)}}{2\Delta t} = -c(x_j)\left[\frac{u_{j+1}^{(n)}-u_{j-1}^{(n)}}{2\Delta x} - \frac{h^2}{6}\frac{d^3u^{(n)}}{dx^3}(x_j)\right] \quad \text{for} \quad j=1,2,\cdots,N$$

Here we can use spectral method to find $\frac{d^3u^{(n)}}{dx^3}(x_j)$. However this does not work, the solution blows up. We correct one more term

(Eq. 22) $$\frac{u_j^{(n+1)}-u_j^{(n-1)}}{2\Delta t} = -c(x_j)\left[\frac{u_{j+1}^{(n)}-u_{j-1}^{(n)}}{2\Delta x} - \frac{h^2}{6}\frac{d^3u^{(n)}}{dx^3}(x_j) - \frac{h^4}{120}\frac{d^5u^{(n)}}{dx^5}(x_j)\right]$$

The result is shown in Figure 10, but one leading-edge curve (maybe dispersion) occurs, why? (source code: F:\course\2008spring\spectral_method\matlab\p6_3.m)

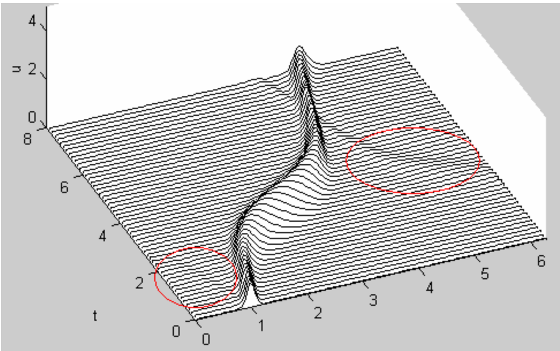

Figure 10: leap frog with high order correction in spatial derivative, (Eq. 22). As you see, there are two extra curve (maybe dispersion), one is leading to main stream, one is behind.

Example 3: we use spectral method to compute $D_x u$ and then we have

(Eq. 23) $$\frac{d}{dt}(u_j) = -diag(c_j)Du_j \quad \text{where} \quad c_j = c(x_j)$$

This is O.D.E, we use Matlab command expm to compute fundamental solution $\exp(At)$ where $A = -diag(c_j)D$. Then modify (Eq. 23) as

(Eq. 24) $$u^{(n+1)} = \exp(A\Delta t)u^{(n)}$$

We compute $\exp(A\Delta t)$ only once since $c(x_j)$ is independent of time.

Table 3: compare leap-frog scheme (Eq. 18) and expm scheme (Eq. 24). Time unit: second

N	128	256	512	1024
Time of leap frog	0.2960	0.1570	0.4530	1.9680
Timing of expm	0.4540	1.9840	13.9220	164.3590

As we expect, $\exp(A\Delta t)$ will consume large amount of time, though it is analytic formula in ODE.

Example 4: claim solution of $u_t + c(x)u_x = 0$ with periodic condition is periodic in time: for certain $T \approx 13$, $u(x,T) = u(x,0)$ for all $x \in (0,2\pi)$.

<sol> First due to periodic of $c(x)$, we have

(Eq. 25) $u_t(t,x+2\pi)+c(x+2\pi)u_x(t,x+2\pi)=u_t(t,x+2\pi)+c(x)u_x(t,x+2\pi)$

If initial condition $u(0,x)=u(0,x+2\pi)$, then

$\begin{cases} u_t+c(x)u_x=0 \\ u(x,0)=g(x) \end{cases}$ and $\begin{cases} u_t+c(x+2\pi)u_x=0 \\ u(x+2\pi,0)=g(x) \end{cases}$ have the same solution.

Second consider characteristic curve $x(t,s)$ of $u_t+c(x)u_x=0$, starting from position $s\in(0,2\pi)$, satisfying $\frac{dx}{dt}=c(x), x(0)=s$ (from $u(t,x(t))=const$, then $u_t+\frac{dx}{dt}u_x=0$), since $c(x)=\frac{1}{5}+\sin^2(x-1)>\frac{1}{5}$, we know $x(t,s)$ is monotone increasing, so $\frac{dt}{dx}=\frac{1}{c(x)}$, after $x(T)=s+2\pi$, we have $c(x(T,s))=\frac{1}{5}+\sin^2(s+2\pi-1)=c(x(0,s))$.

$T=\int_s^{s+2\pi}\frac{dt}{dx}dx=\int_s^{s+2\pi}\frac{1}{c(x)}dx$ is independent of s (this is due to $2\pi-period$ of $c(x)$).

We choose $s=0$ and use Symbolic integration in Matlab to solve

(Eq. 26) $T=\int_0^{2\pi}\frac{1}{1/5+\sin^2(x-1)}dx=\frac{5\pi}{3}\sqrt{6}\approx 12.8255$.

Table 4: compare $u(x,T)$ and $u(x,0)$

(source code: F:\course\2008spring\spectral_method\matlab\p6_5.m)

n	32	64	128	256	512	1024
$\|u(x,T)-u(x,0)\|_\infty$	3.86E-1	7.52E-2	9.16E-3	2.36E-2	1.12E-2	1.56E-4

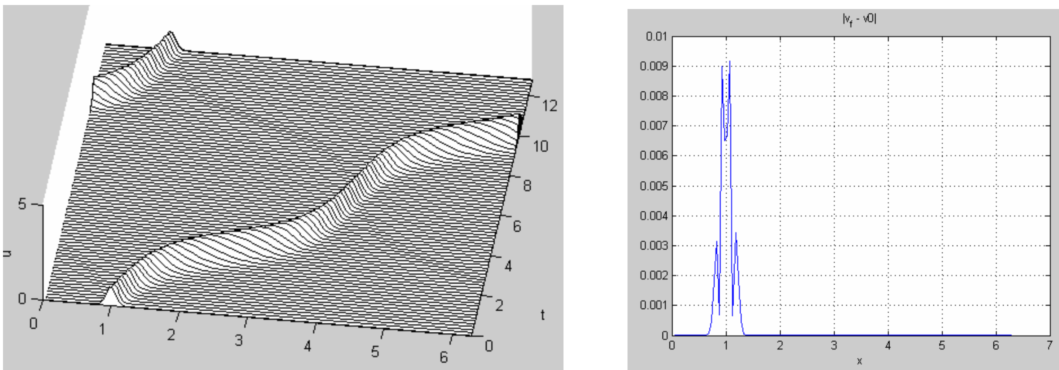

Figure 11: left panel is waterfall graph and right panel is $u(x,T)-u(x,0)$

Question 2: When $v_j=f(x_j)$ for some smooth periodic function $f(x)=f(x+2\pi)$, what is difference between f and p, that is $\|f-p\|<?$ in what norm sense (here we are interested in $\|\cdot\|_{I,\infty}$ and $\|\cdot\|_{I,l^2}$). Second, can we expect that $p'(x_j)$ is a good approximation to $f'(x_j)$, what is $\|f'-p'\|_I$?

<Ans> From $\lim_{m\to\infty}\left\|D_m(x)-S_N(x)\right\|_{L^\infty}=0$, we have

$S_N(x_0)\to f(x_0)$ if and only if $D_m * f(f,x_0)\to f(x_0)$

In order to estimate accuracy of spectral method, we consider infinite modes. Assume $V\in C^\infty$ is real-valued function, periodic with period 2π, then

(Eq. 27) $V(x)=\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}e^{ikx}\hat{V}_k$ with $\|V\|_{L^2}^2=\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\left|\hat{V}_k\right|^2$

(Eq. 28) $\hat{V}_k=\int_0^{2\pi}V(x)e^{-ikx}dx$

Because V is real, we have $\hat{V}_{-k}=\hat{V}_k^*$ and

(Eq. 29) $V(x)=\frac{1}{2\pi}\hat{V}_0+\frac{1}{\pi}\sum_{k=1}^{\infty}\mathrm{Re}\left(e^{ikx}\hat{V}_k\right)=\frac{1}{2\pi}\hat{V}_0+\frac{1}{\pi}\sum_{k=1}^{\infty}\left(\cos(kx)\mathrm{Re}\hat{V}_k-\sin(kx)\mathrm{Im}\hat{V}_k\right)$

If we sample N points $v_j=V(x_j)$, with $h=\frac{2\pi}{N}$, then (Eq. 1) $\hat{V}_k\approx\hat{v}_k=h\sum_{j=1}^{N}e^{-ikx_j}v_j$ is nothing but a numerical integration to approximate $\hat{V}_k=\int_0^{2\pi}V(x)e^{-ikx}dx$.

However under such $\{v_j\}$, we have interpolant $p_v(x)=\frac{1}{2\pi}\mathrm{P}\sum_{k=-m}^{m}e^{ikx}\hat{v}_k=\sum_{k=1}^{N}v_kS_N(x-x_k)$

The different between $V(x)$ and $p_v(x)$ would be

$V(x)-p_v(x)=\frac{1}{2\pi}\mathrm{P}\sum_{k=-m}^{m}e^{ikx}\left(\hat{V}_k-\hat{v}_k\right)+\frac{1}{4\pi}\left(e^{imx}\hat{V}_m+e^{-imx}\hat{V}_{-m}\right)+\frac{1}{2\pi}\sum_{|k|>m}^{\infty}e^{ikx}\hat{V}_k$ for $x\in\{x_j\}$

and $V(x_j)-p_v(x_j)=0$.

(Eq. 30) $\|V-p_v\|_{L^2}^2=\frac{1}{2\pi}\sum_{|k|<m}\left|\hat{V}_k-\hat{v}_k\right|^2+\frac{1}{2\pi}\left|\hat{V}_m-\frac{\hat{v}_m}{2}\right|^2+\frac{1}{2\pi}\left|\hat{V}_{-m}-\frac{\hat{v}_{-m}}{2}\right|^2+\frac{1}{2\pi}\sum_{|k|>m}^{\infty}\left|\hat{V}_k\right|^2$

We have assume $V\in C^\infty$, then we expect that

(Eq. 31) $V'(x)=\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}ike^{ikx}\hat{V}_k$

$p'(x)=\frac{1}{2\pi}\mathrm{P}\sum_{k=-m}^{m}ike^{ikx}\hat{v}_k=\sum_{k=1}^{N}v_kS_N'(x-x_k)$

$V'(x)-p_v'(x)=\frac{1}{2\pi}\mathrm{P}\sum_{k=-m}^{m}ike^{ikx}\left(\hat{V}_k-\hat{v}_k\right)+\frac{1}{4\pi}\left(ime^{imx}\hat{V}_m-ime^{-imx}\hat{V}_{-m}\right)+\frac{1}{2\pi}\sum_{|k|>m}^{\infty}ike^{ikx}\hat{V}_k$

$\|V'-p_v'\|_{L^2}^2=\frac{1}{2\pi}\sum_{|k|<m}k^2\left|\hat{V}_k-\hat{v}_k\right|^2+\frac{m^2}{2\pi}\left|\hat{V}_m-\frac{\hat{v}_m}{2}\right|^2+\frac{m^2}{2\pi}\left|\hat{V}_{-m}-\frac{\hat{v}_{-m}}{2}\right|^2+\frac{1}{2\pi}\sum_{|k|>m}^{\infty}k^2\left|\hat{V}_k\right|^2$

Lemma 1: If V is periodic with period 2π and absolutely continuous, then

(Eq. 32) $\hat{V}_k = \int_0^{2\pi} V(x) e^{-ikx} dx = \frac{1}{ik}\int_0^{2\pi} V^{(1)} e^{-ikx} dx$

means that $\left|\hat{V}_k\right| \le \frac{1}{k}\left\|V^{(1)}\right\|_{L^1}$

<proof> $\hat{V}_k = \int_0^{2\pi} V(x) e^{-ikx} dx = \frac{-1}{ik} V(x) e^{-ikx} |_0^{2\pi} + \frac{1}{ik}\int_0^{2\pi} V^{(1)} e^{-ikx} dx = \frac{1}{ik}\int_0^{2\pi} V^{(1)} e^{-ikx} dx$

Another derivation

$$\hat{V}_k = \int_0^{2\pi} V(x) e^{-ikx} dx = \int_0^{2\pi} \left[V(0) + \int_0^x V^{(1)}(t) dt\right] e^{-ikx} dx = \int_0^{2\pi} \left[\int_0^x V^{(1)}(t) dt\right] e^{-ikx} dx$$
$$= \int_0^{2\pi} V^{(1)}(t) \left[\int_t^{2\pi} e^{-ikx} dx\right] dt = \frac{-1}{ik}\int_0^{2\pi} V^{(1)}(t)\left[1 - e^{-ikt}\right] dt = \frac{1}{ik}\int_0^{2\pi} V^{(1)}(t) e^{-ikt} dt$$

Remark 4: above formula is result of Riemann-Lebesque theorem

$$\hat{f}_k = \int_0^{2\pi} f(x) e^{-ikx} dx = \int_{-(\pi/k)}^{2\pi-(\pi/k)} f\left(x + \frac{\pi}{k}\right) e^{-ik(x+(\pi/k))} dx = -\int_0^{2\pi} f\left(x + \frac{\pi}{k}\right) e^{-ikx} dx$$

So $\hat{f}_k = \frac{1}{2}\int_0^{2\pi} \left[f(x) - f(x + \pi/k)\right] e^{-ikx} dx = \frac{\pi}{2k}\int_0^{2\pi} f^{(1)}(c(x)) e^{-ikx} dx$

Question 3: Can we have other representation for $\hat{V}_k$ except (Eq. 32)?

Corollary 1: If $V^{(k)}$ is periodic with period 2π for $k = 0,1,2,\cdots,m$, then

(Eq. 33) $\hat{V}_k = \int_0^{2\pi} V(x) e^{-ikx} dx = \frac{1}{(ik)^m}\int_0^{2\pi} V^{(m)} e^{-ikx} dx$

<proof> Apply **Lemma 1** m times

Corollary 2:if V is not continuous at $x = x_0$ (i.e. $V\left(x_0^+\right) \ne V\left(x_0^-\right)$), then

(Eq. 34) $\hat{V}_k = \frac{1}{ik}[V]_0 + \frac{e^{-ikx_0}}{ik}[V]_{x_0} + \frac{1}{ik}\int_0^{2\pi} V^{(1)} e^{-ikx} dx$ for $k \ne 0$

where $[V]_{x_0} = V\left(x_0^+\right) - V\left(x_0^-\right)$ is jump at $x = x_0$, and

$[V]_0 = V\left(0^+\right) - V\left(0^-\right) = V\left(0^+\right) - V\left(2\pi^-\right)$. If V is periodic, then

(Eq. 35) $\hat{V}_k = \frac{e^{-ikx_0}}{ik}[V]_{x_0} + \frac{1}{ik}\int_0^{2\pi} V^{(1)} e^{-ikx} dx$ for $k \ne 0$

<proof> decompose $\int_0^{2\pi} V(x) e^{-ikx} dx = \int_0^{x_0} V(x) e^{-ikx} dx + \int_{x_0}^{2\pi} V(x) e^{-ikx} dx$ and do integration by parts twice for two integral.

Example 5: consider Heaviside function $H(x) = \begin{cases} 1 & \pi < x < 2\pi \\ 0 & 0 < x < \pi \end{cases}$ with periodic extension has two discontinuous point $x = 0, \pi$, then

$$\hat{H}_k = \frac{1}{ik}[H]_0 + \frac{e^{-ik\pi}}{ik}[H]_{x=\pi} = \frac{1}{ik}\left[(-1)^k - 1\right] \text{ since}$$

(1) $H'(x) = 0$ for $x \neq 0, \pi$

(2) $[H]_0 = H(0^+) - H(2\pi^-) = 0 - 1 = -1$ and $[H]_{x=\pi} = H(\pi^+) - H(\pi^-) = 1$

From (Eq. 29), we have $V(x) = \frac{1}{2} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left((-1)^k - 1\right)\sin(kx)$

Question 4: What is geometrical interpretation of (Eq. 35)?

<ans> In Figure 11, we compute $\int_{-\pi}^{\pi} V\sin(kx)dx$, since V is discontinuous at $x = 0$, as you see in left panel of Figure 11, $V([-\delta,0]) < 0$ and $V([0,\delta]) > 0$, after multiply $\sin(kx)$, then $V\sin(kx)|_{[-\delta,0]} > 0$ and $V\sin(kx)|_{[0,\delta]} > 0$ in right panel of Figure 11, when we do integration, area of blue triangle $\sim hV(0^+)$ cannot cancel area of red triangle $\sim h|V(0^-)|$, hence $\int_{-\pi}^{\pi} V\sin(kx)dx$ has at least $h[V]_0 = hV(0^+) + h|V(0^-)|$.

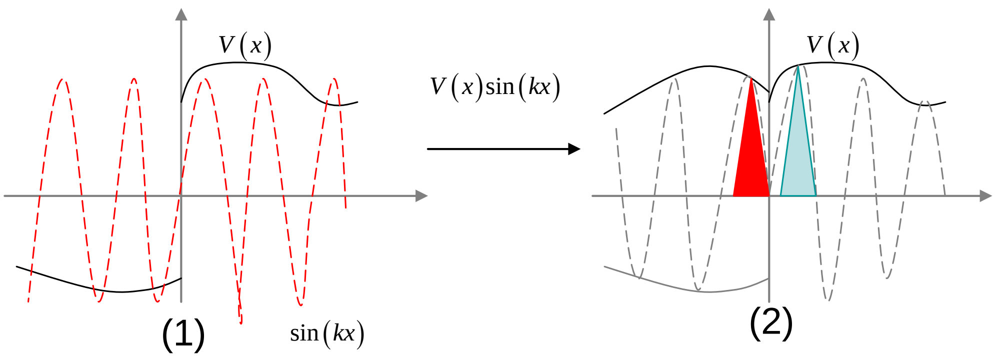

Figure 12: compute $\mathrm{Im}\hat{V}_k \propto \int_{-\pi}^{\pi} V\sin(kx)dx$ where V is discontinuous at $x = 0$

Remark 5: graph in Figure 11 is spirit of Riemann-Lebesque theorem

$$\hat{f}_k = \int_0^{2\pi} f(x)e^{-ikx}dx = \int_{-(\pi/k)}^{2\pi-(\pi/k)} f\left(x+\frac{\pi}{k}\right)e^{-ik(x+(\pi/k))}dx = -\int_0^{2\pi} f\left(x+\frac{\pi}{k}\right)e^{-ikx}dx$$

So $\hat{f}_k = \frac{1}{2}\int_0^{2\pi}\left[f(x) - f(x+\pi/k)\right]e^{-ikx}dx$

Example 6: Consider $V = |\sin x|$ on $[-\pi,\pi]$ and periodic extension, then V is continuous but does not differentiable at $x = 0$, in fact $V' = \begin{cases} \cos x & 0 < x < \pi \\ -\cos x & -\pi < x < 0 \end{cases}$, $V' \in L^1$ is piece-wise smooth but does not continuous at $x = 0, \pi$.

(1) From (Eq. 35), we have $\hat{V}_k = \frac{1}{ik}\int_0^{2\pi} V^{(1)} e^{-ikx} dx = \int_0^{2\pi} \frac{1}{ik} V^{(1)} e^{-ikx} dx$

(2) Again from (Eq. 34) for $\frac{1}{ik}V^{(1)}$, we have

$$\hat{V}_k = \frac{1}{ik}\left[\frac{1}{ik}V^{(1)}\right]_0 + \frac{e^{-ik\pi}}{ik}\left[\frac{1}{ik}V^{(1)}\right]_\pi + \frac{1}{ik}\int_0^{2\pi}\frac{1}{ik}V^{(2)}e^{-ikx}dx$$

$$= \frac{2}{k^2}\left(\left(-1\right)^k - 1\right) - \frac{1}{k^2}\int_0^{2\pi} V^{(2)} e^{-ikx} dx$$

where $V^{(2)} = \begin{cases} -\sin x & 0 < x < \pi \\ \sin x & -\pi < x < 0 \end{cases}$

Assume given smooth periodic function V, then from Fourier theory we have

$V(x) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{ikx}\hat{V}_k$, now if we sample N point value $\left\{ v_j = V\left(x_j\right) : x_j = hj, h = \frac{2\pi}{N} \right\}$ over $\left[0, 2\pi\right]$, then we obtain $\hat{v}_k = h\sum_{j=1}^{N} e^{-ikx_j} v_j$ and band-limited interpolant

$$p_v\left(x\right) = \frac{1}{2\pi}\,\mathrm{P}\sum_{k=-m}^{m} e^{ikx}\hat{v}_k = \sum_{k=1}^{N} v_k S_N\left(x - x_k\right)$$

First question is ”how large $\hat{v}_k - \hat{V}_k$ is?

Note from linearity of summation $\sum_{j=1}^{N}$, we have

(Eq. 36) $\hat{v}_k = h\sum_{j=1}^{N} e^{-ikx_j}\left(\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{imx_j}\hat{V}_m\right)$

Here we assume summation can be interchanged (for example, Dini’s test), then

(Eq. 37) $\hat{v}_k = \sum_{m=-\infty}^{\infty}\hat{V}_m\left(\frac{1}{N}\sum_{j=1}^{N} e^{-ikx_j} e^{imx_j}\right)$

Prop 5: $\frac{1}{N}\sum_{j=1}^{N} e^{imx_j} = \begin{cases} 1 & \text{if } m = Np \text{ for some integer } p \\ 0 & \text{otherwise} \end{cases}$

<**proof**> Case 1: $m = Np$ for some integer p, then $mh = 2\pi p$

$$\sum_{j=1}^{N} e^{imx_j} = \sum_{j=1}^{N} e^{i2\pi pj} = N$$

Case 2: $m \neq Np$ for any integer p

$\sum_{j=1}^{N} e^{imx_j} = \sum_{j=1}^{N} z^j = z\frac{z^N - 1}{z - 1} = 0$ where $z = e^{imh} \neq 1$ and $z^N = e^{imhN} = e^{i2\pi m} = 1$

We take $\hat{v}_0 = \sum_{m=-\infty}^{\infty} \hat{V}_m \left(\frac{1}{N} \sum_{j=1}^{N} e^{imx_j} \right)$ for example, from **Prop 5**

$\hat{v}_0 = \sum_{p=-\infty}^{\infty} \hat{V}_{Np} = \hat{V}_0 + \sum_{p=1}^{\infty} \left(\hat{V}_{Np} + \hat{V}_{-Np} \right) = \hat{V}_0 + 2\sum_{p=1}^{\infty} \operatorname{Re}\left(\hat{V}_{Np} \right)$, in other words,

$\left| \hat{v}_0 - \hat{V}_0 \right| \le 2\sum_{p=1}^{\infty} \left| \operatorname{Re}\left(\hat{V}_{Np} \right) \right| \le 2\sum_{p=1}^{\infty} \left| \hat{V}_{Np} \right|$. The largest dominant error is $\hat{V}_N$.

Similarly we can apply the same argument to all $\hat{v}_k$, then we have **Lemma 2**

Lemma 2: $\left| \hat{v}_k - \hat{V}_k \right| \le 2\sum_{p=1}^{\infty} \left| \hat{V}_{k+Np} \right|$ for $k = 0,1,2,\cdots,N-1$

<proof> $\hat{v}_k = \sum_{m=-\infty}^{\infty} \hat{V}_m \left(\frac{1}{N} \sum_{j=1}^{N} e^{i(m-k)x_j} \right) = \sum_{m=-\infty}^{\infty} \hat{V}_m \delta_{m-k=Np} = \sum_{p=-\infty}^{\infty} \hat{V}_{k+Np} = \hat{V}_k + \sum_{p=1}^{\infty} \left(\hat{V}_{k+Np} + \hat{V}_{k-Np} \right)$

Corollary 3: when If $V^{(k)}$ is periodic with period 2π for $k = 0,1,2,\cdots,m$, then Trapezoid integration of $\int_0^{2\pi} V(x)\,dx$ with uniform grid is sprectral accuracy.

<proof> $Trapezoid(V) = \frac{h}{2} v_0 + h\sum_{j=1}^{N-1} v_j + \frac{h}{2} v_N = h\sum_{j=1}^{N} v_j = \hat{v}_0$, second equality comes from periodicity of V, i.e. $V(0) = V(2\pi)$. Then $\left| Trapezoid(V) - \int_0^{2\pi} V dx \right| = \left| \hat{v}_0 - \hat{V}_0 \right|$.

Table 5: Dominant error of $\left| \hat{v}_k - \hat{V}_k \right|$, leading term $\hat{V}_{k\pm N}$

k	0	1	2	3	$m-1$	m
$\left\| \hat{v}_k - \hat{V}_k \right\|$	$\hat{V}_N$	$\hat{V}_{N+1}$, $\hat{V}_{1-N} = \hat{V}_{N-1}^*$	$\hat{V}_{N+2}$, $\hat{V}_{2-N} = \hat{V}_{N-2}^*$	$\hat{V}_{N+3}$, $\hat{V}_{3-N} = \hat{V}_{N-3}^*$	$\hat{V}_{N+m-1}$, $\hat{V}_{m+1}^*$	$\hat{V}_{N+m}$, $\hat{V}_m^*$

From above table, we know error of $\left| \hat{v}_k - \hat{V}_k \right|$ increases with respect with k.

Appendix A Fourier series

Definition 2: for $f \in L^1[-\pi,\pi]$, then

(1) $\hat{f}_n = \langle e^{inx}, f \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx$

(2) $f \sim \sum_{n=-\infty}^{\infty} \hat{f}_n e^{inx}$ is called Fourier series of f

(3) $S_n(f,x) = \sum_{k=-n}^{n} \hat{f}_k e^{ikx}$ is called n-th partial sum of Fourier series of f

(4) $D_n(x) = \sum_{k=-n}^{n} e^{ikx}$ is called n-th Dirichlet kernel

(5) $f,g \in L^1[-\pi,\pi]$, periodic in 2π, $f * g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) g(x-y) dy = g * f(x)$, say convolution of f,g

Prop 6: properties for Dirichlet kernel $D_n(x) = \sum_{k=-n}^{n} e^{ikx}$

(1) $|D_n(x)| \le \sum_{k=-n}^{n} |e^{ikx}| \le 2n+1$ and $D_n(0) = 2n+1$

(2) $1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(x) dx$

(3) $\|D_n(x)\|_{L^2} = 2m+1$

(4) $D_n(x) = \frac{\sin[(n+1/2)x]}{\sin(x/2)}$ for $0 < |x| < \pi$ and $\lim_{x \to 0} D_n(x) = D_n(0) = 2n+1$

Prop 7: (1) $f * e^{inx} = e^{inx} \hat{f}_n$ (2) $f * D_n(x) = S_n(f,x)$

Theorem 1(Dini's test): $f \in L^1[-\pi,\pi]$, $S_n(f,x_0) \to \breve{f}(x_0)$ for some constant $\breve{f}(x_0)$ if $\int_0^{\delta} \frac{|\phi_{x_0}(x)|}{x} dx < \infty$ for some $0 < \delta < \pi$ where $\phi_{x_0}(t) = f(x_0+t) + f(x_0-t) - 2\breve{f}(x_0)$

Corollary 4: $f \in C^1$, then $|f(x_0+t) - f(x_0)| \le \|f'\|_{\infty} t$. Hence $S_n(f,x_0) \to f(x_0)$

Corollary 5: f is piece-wise smooth with finite jump discontinuities. If f has jump at x_0, then set $\breve{f}(x_0) = \frac{1}{2}\left(f(x_0^+) + f(x_0^-)\right)$, then $\phi_{x_0}(t) = f'(x_1)t - f'(x_2)t$ for $x_1 \in (x_0, x_0+t)$, $x_2 \in (x_0 - t, x_0)$, then $S_n(f,x_0) \to \breve{f}(x_0)$.

Corollary 6(localization principle): $f = 0$ on $(x_0-\delta, x_0+\delta)$, then $S_n(f,x_0) \to 0$