Definition 1: Classically we define Fourier transform of a function $u(x) \in R$ defined by

(Eq. 1) $\hat{u}(k) = \int_{-\infty}^{\infty} e^{-ikx} u(x) dx$ and inverse Fourier transform

(Eq. 2) $u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \hat{u}(k) dk$

Theorem 1(Plancherel): One can associate to each $f \in L^2$ a function $\hat{f} \in L^2$ so that the following properties hold:

(a) If $f \in L^1 \cap L^2$, then $\hat{f}$ is the $\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x) dm(x)$, here $dm(x) = \frac{1}{\sqrt{2\pi}} dx$

(b) For every $f \in L^2$, $\|f\|_2 = \|\hat{f}\|_2$

(c) The mapping $f \to \hat{f}$ is a Hilbert space isomorphism of L^2 onto L^2

(d) The following symmetric relation exists between f and $\hat{f}$: If

$\varphi_A(k) = \int_{-A}^{A} e^{-ikx} f(x) dm(x)$ and $\psi_A(x) = \int_{-A}^{A} e^{ikx} \hat{f}(k) dm(k)$, then

$\|\varphi_A - \hat{f}\|_2 \to 0$ and $\|\psi_A - f\|_2 \to 0$

<proof> see page 186 in [1]

Semidiscrete Fourier transform

Let $x_j = hj$ over uniform grid hZ

(Eq. 3) $\hat{v}(k) = h \sum_{j=-\infty}^{\infty} e^{-ikx_j} v_j$ for $k \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]$, this form is valid if $\{v_j\} \in l^2$

(Eq. 4) $v_j = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} e^{ikx_j} \hat{v}(k) dk$

$$\hat{v}\left(k + \frac{\pi}{h}\right) = h \sum_{j=-\infty}^{\infty} e^{-ikx_j} \exp\left(-i\frac{\pi}{h} x_j\right) v_j = h \sum_{j=-\infty}^{\infty} e^{-ikx_j} \exp(-i\pi j) v_j = h \sum_{j=-\infty}^{\infty} (-1)^j e^{-ikx_j} v_j$$

Let us introduce band-limit interpolant

(Eq. 5) $p(x) = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} e^{ikx} \hat{v}(k) dk$ satisfying two properties

(1) $p(x_j) = v_j$

(2) $\hat{p}(k) = \begin{cases} \hat{v}(k) & k \in [-\pi/h, \pi/h] \\ 0 & \text{otherwise} \end{cases}$ has compact support.

This is because $\langle e^{ikx}, e^{imx} \rangle = \delta_{k-m}$

Prop 1: If we define Kronecker delta function $\delta_j = \begin{cases} 1 & j = 0 \\ 0 & j \neq 0 \end{cases}$, then we have

(1) $\hat{\delta}(k) = h$ for all $k \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]$

(2) band-limit interpolant of δ is $S_h(x) = \dfrac{\sin(\pi x/h)}{\pi x/h}$, called sinc function

(3) band-limit interpolant of δ_{j-m} is $S_h(x - x_m)$

(4) $v_j = \sum_{n=-\infty}^{\infty} v_m \delta_{j-m}$

(5) $p(x) = \sum_{n=-\infty}^{\infty} v_m S_h(x - x_m)$

<proof> (1) by definition (Eq. 3), $\hat{\delta}(k) = h\sum_{j=-\infty}^{\infty} e^{-ikx_j}\delta_j = h$

(2) $S_h(x) = \dfrac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{ikx}\hat{\delta}(k)\,dk = \dfrac{h}{2\pi}\int_{-\pi/h}^{\pi/h} e^{ikx}dk = \dfrac{h}{2\pi}\dfrac{1}{ix}e^{ikx}\Big|_{-\pi/h}^{\pi/h} = \dfrac{\sin(\pi x/h)}{\pi x/h}$ for $x \neq 0$ and for $x = 0$, we have $S_h(0) = \dfrac{1}{2\pi}\int_{-\pi/h}^{\pi/h}\hat{\delta}(k)\,dk = 1$. Moreover S_h is continuous since $\lim_{x\to 0} S_h(x) = 1$.

(3) $\left(\hat{\delta}_{j-m}\right)(k) = h\sum_{j=-\infty}^{\infty} e^{-ikx_j}\delta_{j-m} = he^{-ikx_m}$, hence band-limit interpolant of δ_{j-m} is

$$\frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} e^{ikx}e^{-ikx_m}dk = \frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} e^{ik(x-x_m)}dk = S_n(x - x_m)$$

(4) $v_j = \sum_{n=-\infty}^{\infty} v_m \delta_{j-m}$ is obvious

(5) $\hat{v}(k) = h\sum_{j=-\infty}^{\infty} e^{-ikx_j}v_j = h\sum_{j=-\infty}^{\infty} e^{-ikx_j}\sum_{m=-\infty}^{\infty} v_m\delta_{j-m} = \sum_{m=-\infty}^{\infty} v_m\left(h\sum_{j=-\infty}^{\infty} e^{-ikx_j}\delta_{j-m}\right) = \sum_{m=-\infty}^{\infty} v_m\left(\hat{\delta}_{j-m}\right)(k)$

and then

$$p(x) = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{ikx}\hat{v}(k)\,dk = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{ikx}\left(\sum_{m=-\infty}^{\infty} v_m\left(\hat{\delta}_{j-m}\right)(k)\right)dk$$

$$= \sum_{m=-\infty}^{\infty} v_m\frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{ikx}\left(\hat{\delta}_{j-m}\right)(k)\,dk = \sum_{m=-\infty}^{\infty} v_m S_n(x - x_m)$$

Definition 2: we define derivative $w_j = p'(x_j)$ to approximate derivative of grid function $\{v_j\}$ according to

(Eq. 6) $w_j = p'(x_j) = \sum_{m=-\infty}^{\infty} v_m S_n'(x_j - x_m) = \dfrac{1}{2\pi}\int_{-\pi/h}^{\pi/h} ike^{ikx_j}\hat{v}(k)\,dk$

Prop 2: sinc function $S_h(x) = \dfrac{\sin(\pi x/h)}{\pi x/h}$ is even function, which has first derivative

(Eq. 7) $S_h'(x_j) = \begin{cases} 0 & j = 0 \\ \dfrac{(-1)^j}{jh} & j \neq 0 \end{cases}$ is odd function and second derivative

(Eq. 8) $S_h''(x_j)=\begin{cases} -\dfrac{\pi^2}{3h^2} & j=0 \\ 2\dfrac{(-1)^{j+1}}{j^2h^2} & j\neq 0 \end{cases}$ is even function

<proof> for $x\neq 0$, $S_h'(x)=\dfrac{dy}{dx}|_{y=\pi x/h}\dfrac{d}{dy}\dfrac{\sin(y)}{y}=\dfrac{\pi}{h}\left[\dfrac{\cos(y)}{y}-\dfrac{\sin(y)}{y^2}\right]$ is odd function and then $S_h'(x_j)=\dfrac{\pi}{h}\left[\dfrac{\cos(j\pi)}{j\pi}-\dfrac{\sin(j\pi)}{(j\pi)^2}\right]=\dfrac{(-1)^j}{jh}$. for $x=0$, according to Taylor expansion for $\sin x = x-\dfrac{1}{3!}x^3+O(x^5)$, we have

$$S_h'(0)=\lim_{x\to 0}\frac{1}{x}\left[\frac{\sin(\pi x/h)}{\pi x/h}-1\right]=\lim_{x\to 0}\frac{1}{x}\left[-\frac{1}{3!}(\pi x/h)^3+O(\pi x/h)^5\right]=0.$$

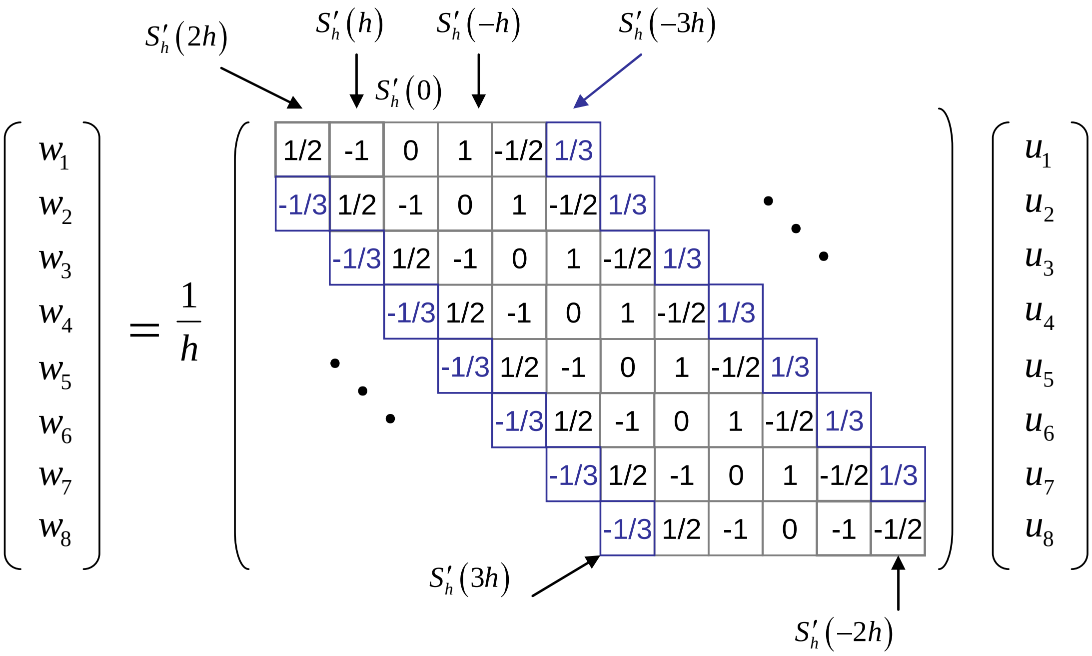

Figure 1: matrix representation of $w_j = \sum_{m=-\infty}^{\infty} v_m S_n'(x_j - x_m)$

Similarly, for $x\neq 0$, $S_h''(x)=\dfrac{dy}{dx}|_{y=\pi x/h}\dfrac{d}{dy}S_h'(x)=\left(\dfrac{\pi}{h}\right)^2\left[\dfrac{-\sin(y)}{y}-\dfrac{2\cos(y)}{y^2}+\dfrac{2\sin(y)}{y^3}\right]$ is even function , then $S_h''(x_j)=-\left(\dfrac{\pi}{h}\right)^2\dfrac{2\cos(j\pi)}{(j\pi)^2}=2\dfrac{(-1)^{j+1}}{j^2h^2}$. As for $x=0$,

$$S_h''(0)=\lim_{x\to 0}\frac{1}{x}\frac{\pi}{h}\left[\frac{\cos(y)}{y}-\frac{\sin(y)}{y^2}\right]=\left(\frac{\pi}{h}\right)^2\lim_{y\to 0}\frac{y\cos y-\sin y}{y^3}$$

$$=\left(\frac{\pi}{h}\right)^2\lim_{y\to 0}\frac{-1/3+O(y^5)}{y^3}=-\frac{\pi^2}{3h^2}$$

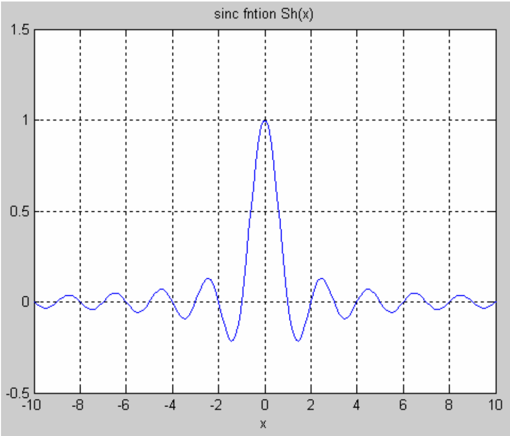

Figure 2: $S_h(x)=\dfrac{\sin(\pi x/h)}{\pi x/h}$ is even function.

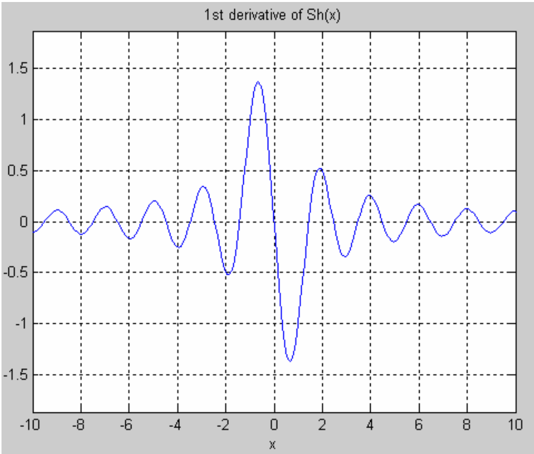

Figure 3: $S_h'(x_j)=\begin{cases}0 & j=0\\ \dfrac{(-1)^j}{jh} & j\neq 0\end{cases}$ is odd function.

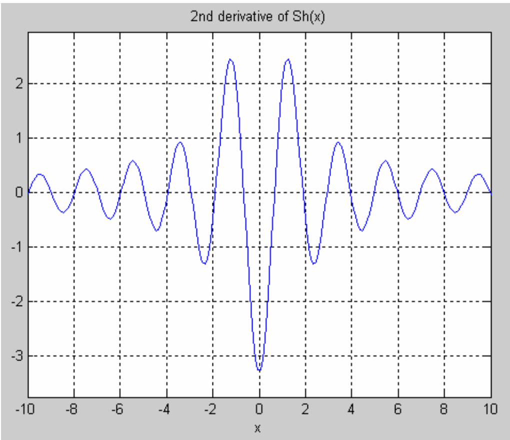

Figure 4: $S_h''(x_j)=\begin{cases}-\dfrac{\pi^2}{3h^2} & j=0\\ 2\dfrac{(-1)^{j+1}}{j^2h^2} & j\neq 0\end{cases}$ is even function.

Example 1: use band-limited interpolant $p(x)=\sum_{n=-\infty}^{\infty} v_m S_h(x-x_m)$ for 3 grid function, delta function, square function and hat function, see Figure 5.

Program: F:\course\2008spring\spectral_method\matlab\p3.m

Note that

(1) Top figure is the same as Figure 2
(2) Square function is not differentiable, so sinc function cannot resolve discontinuity, in fact, near jump discontinuity, we have Gibb phenomenon.

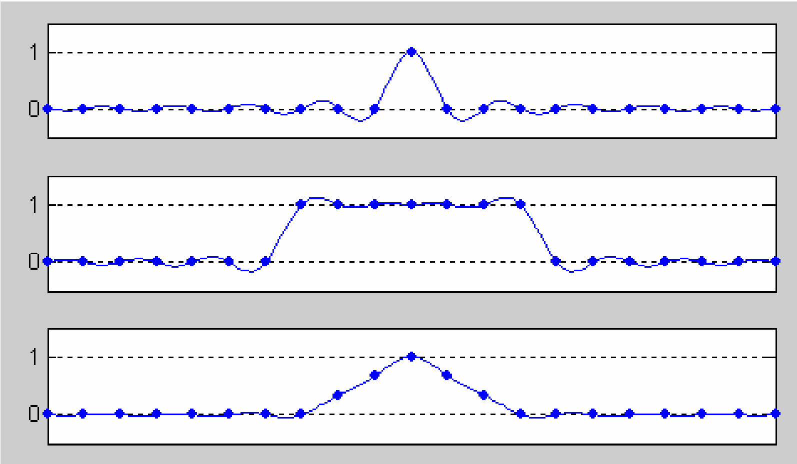

Figure 5: top is delta function, middle is square function and bottom is hat function

we measure maxima error in the sinc function interpolants of the square wave and hat function, here we choose $h = 2^{-3}, 2^{-4}, 2^{-5}, 2^{-6}$.

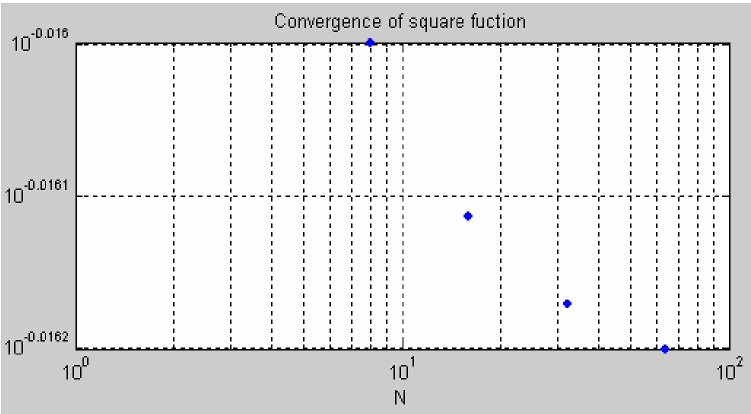

Figure 6: O(1) supnorm error between square function and its approximated sinc function, this is due to Gibb phenomenon.

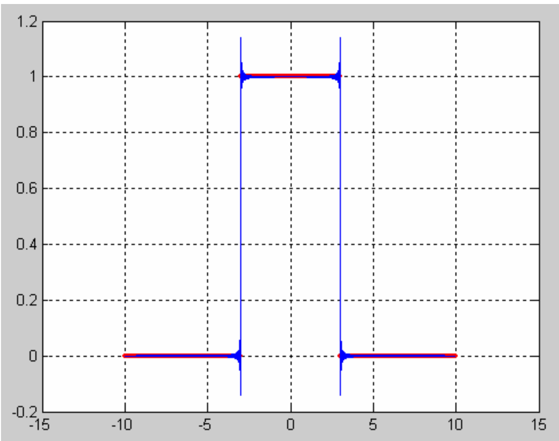

Figure 7: $h = 2^{-6}$, Gibbs phenomenon occurs at jump discontinuity.

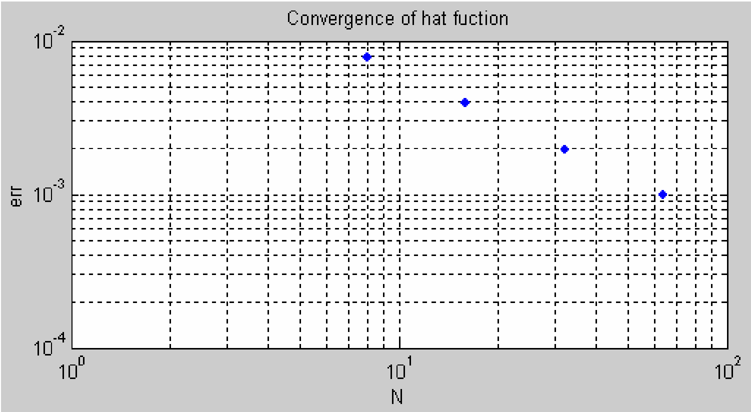

Figure 8: supnorm error between hat function and its approximated sinc function, it is good than square function.

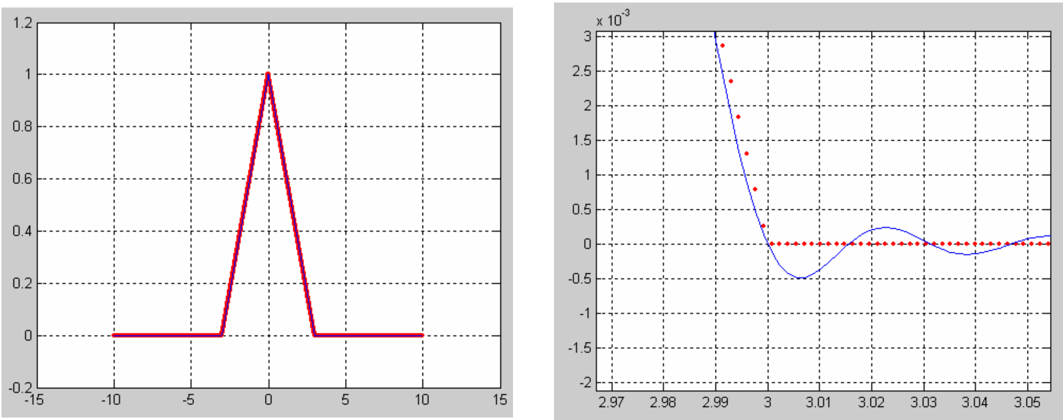

Figure 9: $h = 2^{-6}$, Gibbs phenomenon does not occur.

Table 1: find convergence factor m of $err = h^m$ where $err = \|p - hat\|_\infty$

	$h = 2^{-3}$	$h = 2^{-4}$	$h = 2^{-5}$	$h = 2^{-6}$
m	3.33	2.5	2	1.666

Example 2(Exercise 3): plot $\sin(1:3000)$ to show aliasing. How about $\sin(1:1000)$?

In Figure 10, we can follow a straight line along any edge of diamond to obtain aliasing in the plot of $\sin(1:3000)$ but such aliasing is hard to be detected in the plot of $\sin(1:1000)$, why? This is just trick of senses, in fact if we shrink the figure of $\sin(1:1000)$, then aliasing effect is apparent, see left panel in Figure 11. In fact we can try and error to find out aliasing location, note that $T = 7 \times 2\pi = 43.98 \approx 44$, hence we expect that aliasing occurs for period kT. In the right panel of Figure 11, we plot 4 lines to enclose a diamond,

Line 1 (red color): $\sin(19:T:N)$, Line 2 (green color): $\sin(3:T:N)$

Line 3 (cyan color): $\sin(44:T:N)$, Line 4 (magenta color): $\sin(28:T:N)$

In our setting $T = 7 \times 2\pi + \delta$, where $\delta = 0.0177$, hence

$\sin(x_0 + kT) = \sin(x_0 + k\delta) = \sin(x_0)\cos(k\delta) + \cos(x_0)\sin(k\delta)$, if $N\max = 1000$, then

$k_{\max} = floor\left(\frac{N_{\max}}{T}\right) = 22$ and $k_{\max}\delta = 0.3895$, $\sin(k_{\max}\delta) = 0.3797 \approx k_{\max}\delta$ and

$\cos(k_{\max}\delta) = 0.925 \approx 1$. This means that for all $k = 1 : k_{\max}$, we have $\sin(x_0 + kT) \approx \sin(x_0) + \delta\cos(x_0)k$ is a straight line with slop $\delta\cos(x_0)$, note that $\cos(3) < 0$ and $\cos(28) < 0$, hence slope of green line and magenta line is negative. This is why we have diamond shape (parallelogram) since all straight line aliasing have the same slop.

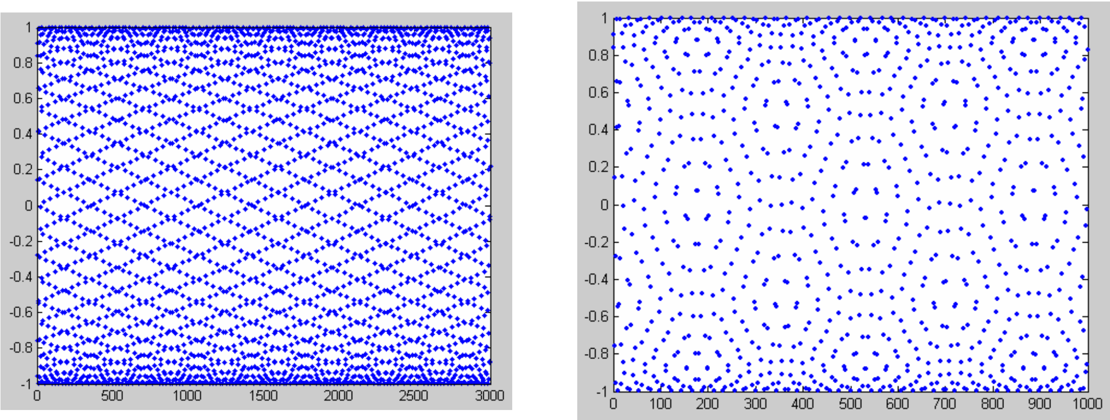

Figure 10: left panel is $\sin(1:3000)$ whereas right panel is $\sin(1:1000)$. aliasing is clear in left panel, just see the shape of diamond but aliasing is not clear in right panel, don't be depressed, this is a trick of senses (錯覺).

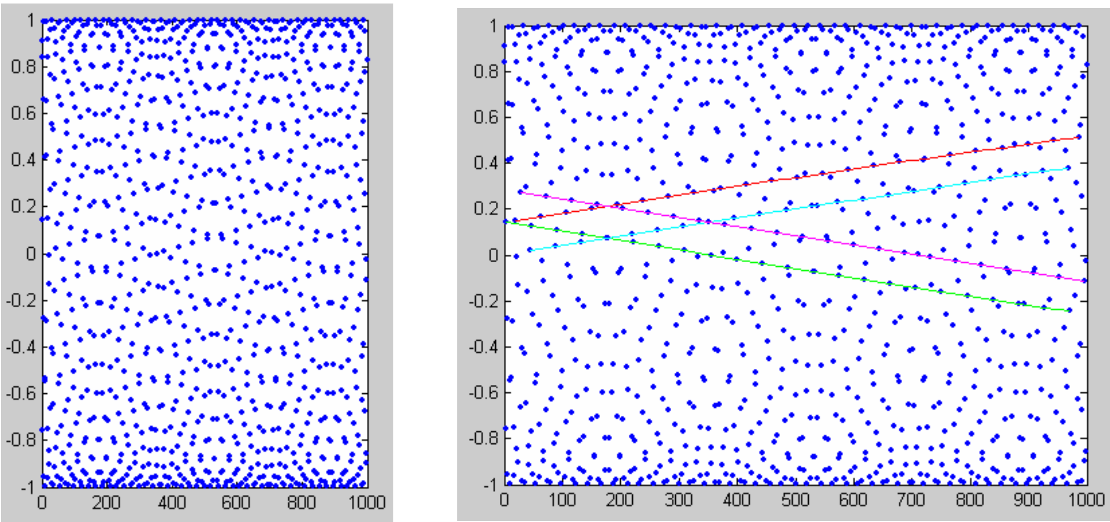

Figure 11: if we shrink $\sin(1:1000)$, then aliasing effect appears in left panel. Moreover we can find out alias location, see red, green, cyan(青綠色) and magenta(洋紅色) color line.

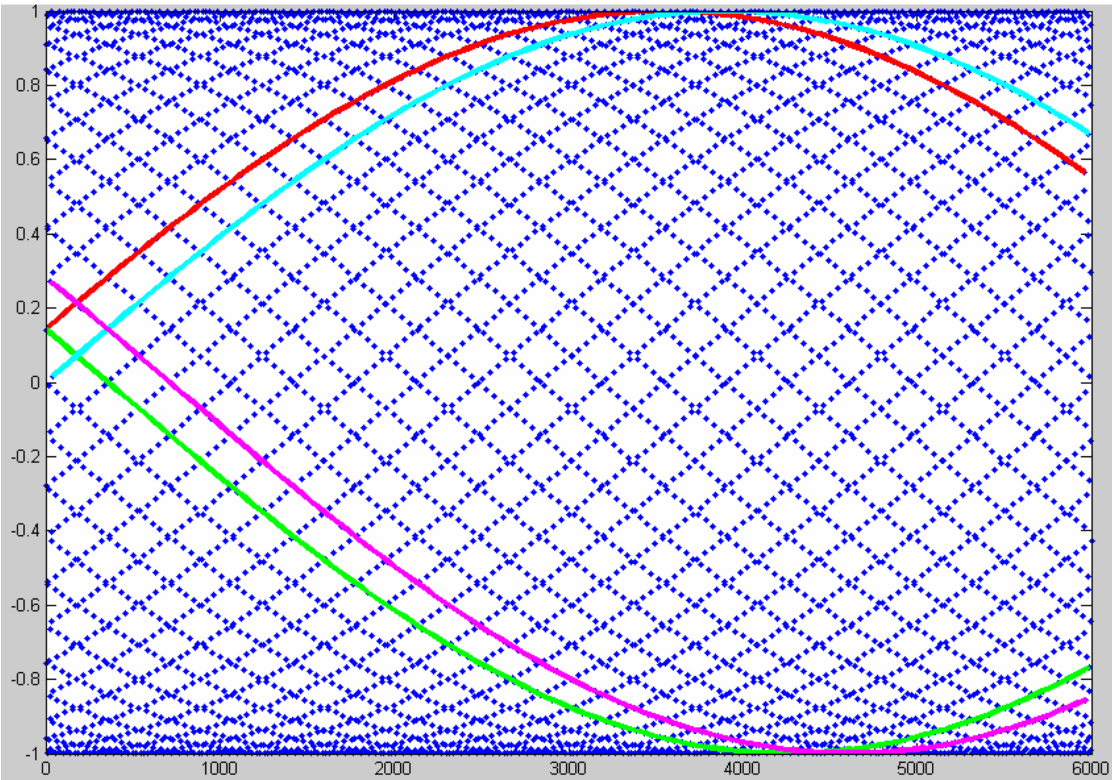

Figure 12: for $N_{\max} = 6000$, we cannot keep the straight line since first order approximation fails. You can see, line bends after $N = 2000$

Reference

[1] Rudin, Real and Complex Analysis