Chapter 11 polar coordinate

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Book: Lloyd N. Trefethen, Spectral Methods in MATLAB

Discretization on unit disk

Consider eigen-value problem on unit disk

$$\Delta u = -\lambda^2 u \quad \text{with boundary condition } u(r=1,\theta)=0$$

We adopt polar coordinate $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, then

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = -\lambda^2 u \quad \text{, on } r\in(0,1] \text{ and } \theta\in[0,2\pi]$$

Usually we take periodic Fourier grid in θ , and non-periodic Chebyshev grid in r

① $r\in[0,1]$

Chebyshev grid in $x\in[-1,1]$ $\xrightarrow{r=\frac{x+1}{2}}$ Chebyshev grid in $r\in[0,1]$

Observation: nodes are clustered near origin $r=0$, for time evolution problem,

we need smaller time-step to maintain numerical stability.

② $r\in[-1,1]$

$$\begin{cases} x = r\cos\theta = -r\cos(\theta+\pi) \\ y = r\sin\theta = -r\sin(\theta+\pi) \end{cases}$$

$$(x,y)\neq(0,0) \xrightarrow{\text{1 to 2 mapping}} (r,\theta)$$

Asymptotic behavior of spectrum of Chebyshev diff. matrix

In chapter 10, we have showed that spectrum of Chebyshev differential matrix $\tilde{D}_N^2$ (second order) approximates

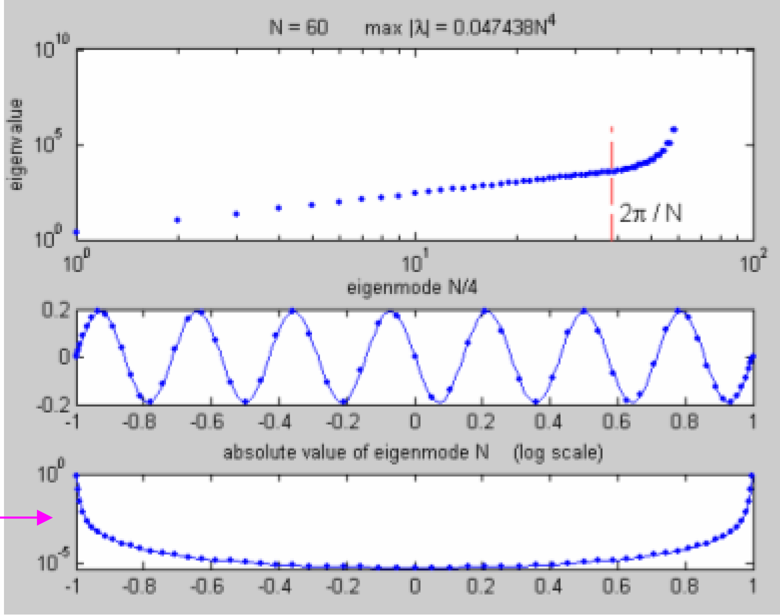
$$u_{xx} = \lambda u, \quad -1 < x < 1, \;\; \text{B.C. } u(\pm 1) = 0 \quad \text{with eigenmode } \lambda_k = -\frac{\pi^2}{4} k^2$$

① Eigenvalue of $\tilde{D}_N^2$ is negative (real number) and $\lambda_{\max} \approx -0.048 N^4$

② Large eigenmode of $\tilde{D}_N^2$ does not approximate to $\lambda_k = -\frac{\pi^2}{4} k^2$

Since ppw is too small such that

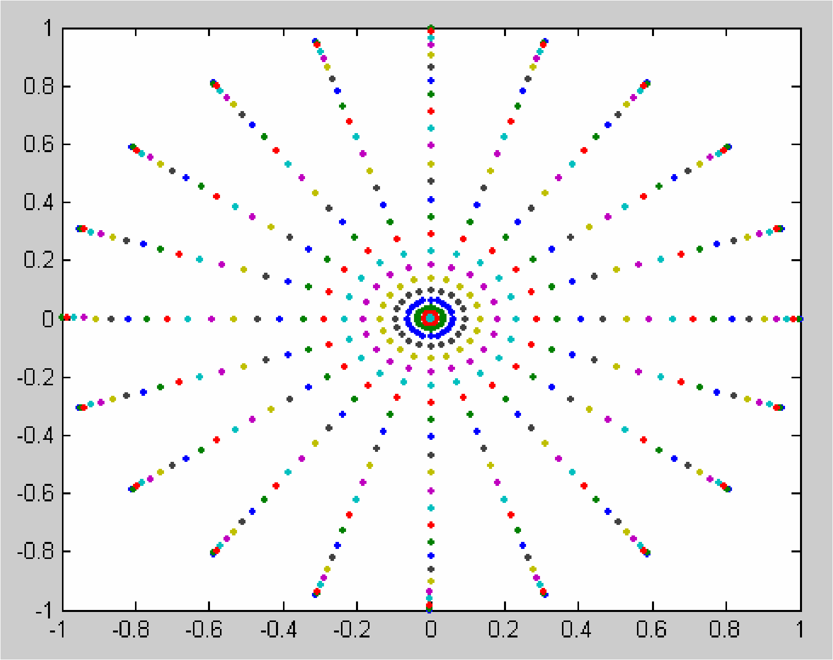
resolution is not enough



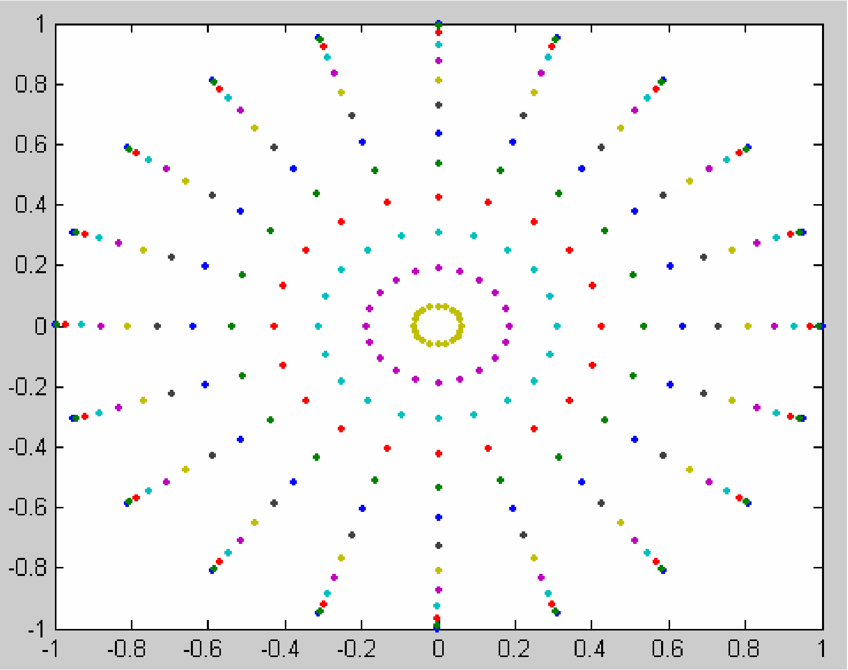
Mode *N* is spurious and localized near boundaries $x = \pm 1$

Grid distribution

① $r \in [0,1]$

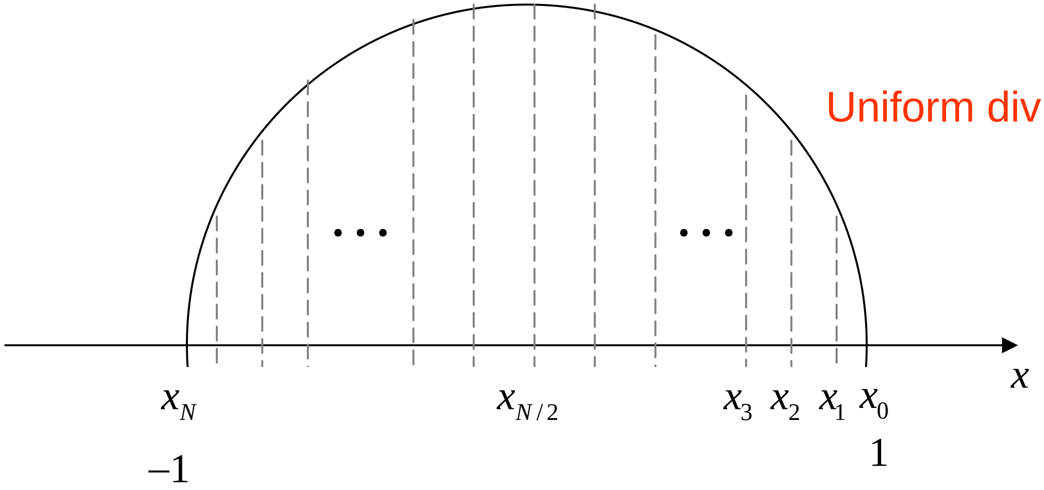


② $r \in [-1,1]$



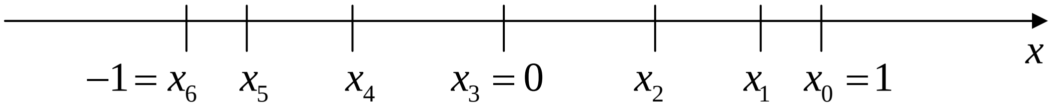
Preliminary: Chebyshev node and diff. matrix [1]

Consider $N+1$ Chebyshev node on $[-1,1]$, $x_j = \cos\left(\frac{j\pi}{N}\right)$ for $j = 0, 1, 2, \cdots, N$

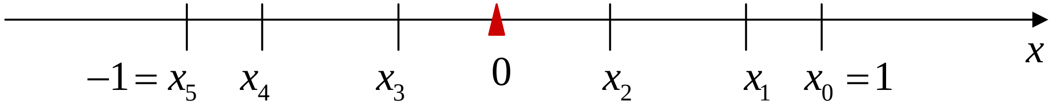


Uniform division in arc

Even case: $N = 6$



Odd case: $N = 5$



Preliminary: Chebyshev node and diff. matrix [2]

Given $N+1$ Chebyshev nodes , $x_j = \cos\left(\frac{j\pi}{N}\right)$ and corresponding function value v_j

We can construct a unique polynomial of degree N , called $p(x) = \sum_{j=0}^{N} v_j S_j(x)$

$S_j(x_k) = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}$ is a basis.

$p'(x_i) = \sum_{j=0}^{N} v_j S_j^{(1)}(x_i) = \sum_{j=0}^{N} D_{ij}^N v_j$ where differential matrix $D_N \triangleq \left(D_{ij}^N\right)$ is expressed as

$$D_{00}^N = \frac{2N^2+1}{6}, \quad D_{NN}^N = -\frac{2N^2+1}{6}, \quad D_{jj}^N = \frac{-x_j}{2\left(1-x_j^2\right)}, \quad \text{for} \quad j = 1, 2, \cdots, N-1$$

$$D_{ij}^N = \frac{c_i}{c_j}\frac{(-1)^{i+j}}{x_i - x_j}, \quad \text{for} \quad i \neq j, \quad i, j = 0, 1, 2, \cdots, N \qquad \text{where} \quad c_i = \begin{cases} 2 & i = 0, N \\ 1 & \text{otherwise} \end{cases}$$

with identity $D_{ii}^N = -\sum_{j=0,\, j\neq i}^{N} D_{ij}^N$

Second derivative matrix is $D_N^2 = D_N \cdot D_N$

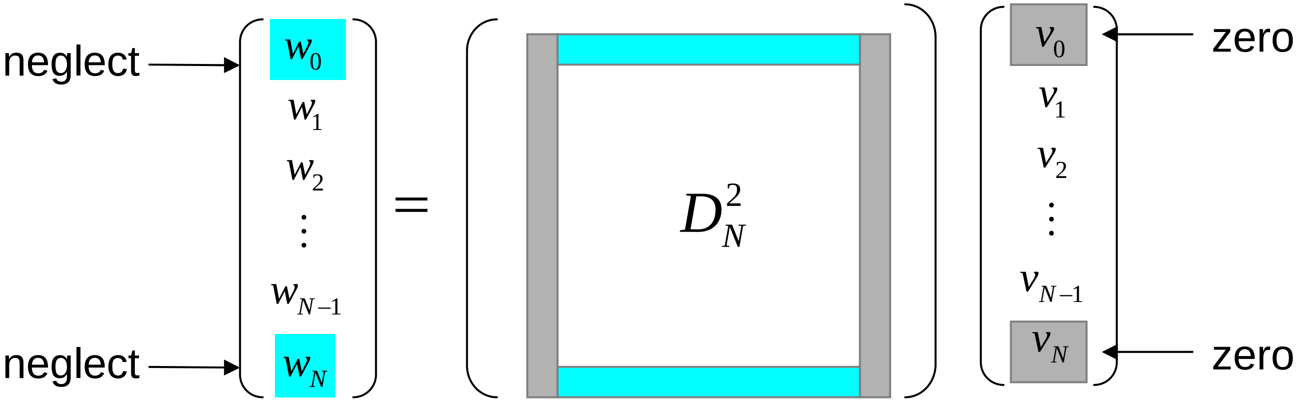
Preliminary: Chebyshev node and diff. matrix [3]

Let $p(x)$ be the unique polynomial of degree $\leq N$ with $p(\pm 1) = 0$ and $p(x_j) = v_j$

define $w_j = p''(x_j)$ and $z_j = p'(x_j)$ for $0 \leq j \leq N$

We abbreviate $w = D_N^2 \cdot v$, then impose B.C. $p(\pm 1) = 0$,that is, $v_0 = v_N = 0$

In order to keep solvability, we neglect $w_0 = w_N$,that is, $\tilde{D}_N^2 = D_N^2 (1 : N-1, 1 : N-1)$



Similarly, we also modify differential matrix as $\tilde{D}_N = D_N (1 : N-1, 1 : N-1)$

Preliminary: DFT [1]

Given a set of data point $\{v_1, v_2, \cdots, v_N\} \in R^N$ with $N = 2m$ is even, $h = \frac{2\pi}{N}$

Then *DFT* formula for $\{v_j\}$

$$\hat{v}_k = \frac{2\pi}{N} \sum_{j=1}^{N} v_j \exp\left(-ikx_j\right) \qquad \text{for} \quad k = -m, -m+1, \cdots, m-1, m$$

$$v_j = \frac{1}{2\pi} \sum_{k=-m+1}^{m} \hat{v}_k \exp\left(ikx_j\right) \qquad \text{for} \quad j = 1, 2, \cdots, N$$

Definition: band-limit interpolant of $\delta - \text{function}$, is periodic **sinc** function $S_N\left(x\right)$

$$S_N\left(x\right) \triangleq \frac{h}{2\pi} P \sum_{k=-m}^{m} e^{ikx} = \frac{\sin\left(\pi x / h\right)}{\left(2\pi / h\right) \tan\left(x/2\right)} = \frac{1}{2m} \frac{\sin\left(mx\right)}{\tan\left(x/2\right)}$$

If we write $v_j = \sum_{k=1}^{N} v_k \delta_{j-k} = \delta * v$, then $p\left(x\right) = \frac{1}{2\pi} P \sum_{k=-m}^{m} e^{ikx} \hat{v}_k = \sum_{k=1}^{N} v_k S_N\left(x - x_k\right)$

Also derivative is according to $w_j = p'\left(x_j\right) = \sum_{k=1}^{N} v_k S_N^{(1)}\left(x_j - x_k\right)$

Preliminary: DFT [2]

Direct computation of derivative of $S_N(x) = \dfrac{\sin(\pi x / h)}{(2\pi / h)\tan(x/2)}$, we have

$$S_N^{(1)}(x_j) = \begin{cases} 0 & j = 0 (\operatorname{mod} N) \\ \dfrac{1}{2}(-1)^j \cot\left(\dfrac{jh}{2}\right), & j \neq 0 (\operatorname{mod} N) \end{cases}$$

Example: $D_5 = \left(S_5^{(1)}(x_j - x_k)\right) = \begin{pmatrix} 0 & S_5^{(1)}(-h) & S_5^{(1)}(-2h) & S_5^{(1)}(-3h) & S_5^{(1)}(-4h) \\ S_5^{(1)}(h) & 0 & S_5^{(1)}(-h) & S_5^{(1)}(-2h) & S_5^{(1)}(-3h) \\ S_5^{(1)}(2h) & S_5^{(1)}(h) & 0 & S_5^{(1)}(-h) & S_5^{(1)}(-2h) \\ S_5^{(1)}(3h) & S_5^{(1)}(2h) & S_5^{(1)}(h) & 0 & S_5^{(1)}(-h) \\ S_5^{(1)}(4h) & S_5^{(1)}(3h) & S_5^{(1)}(2h) & S_5^{(1)}(h) & 0 \end{pmatrix}$

is a Toeplitz matrix.

Second derivative is $S_N^{(2)}(x_j) = \begin{cases} -\dfrac{1}{6} - \dfrac{\pi^2}{3h^2} & j = 0 (\operatorname{mod} N) \\ -\dfrac{(-1)^j}{2\sin^2(jh/2)}, & j \neq 0 (\operatorname{mod} N) \end{cases}$

Preliminary: DFT [3]

$$S_N^{(2)}(x_j) = \begin{cases} -\dfrac{1}{6} - \dfrac{\pi^2}{3h^2} & j = 0 \pmod N \\ -\dfrac{(-1)^j}{2\sin^2(jh/2)}, & j \neq 0 \pmod N \end{cases}$$

For second derivative operation $w_j = p''(x_j) = \sum_{k=1}^{N} v_k S_N^{(2)}(x_j - x_k) \equiv \sum_{k=1}^{N} D_N^2(j,k) v_k$

second diff. matrix is explicitly defined by using Toeplitz matrix (command in MATLAB)

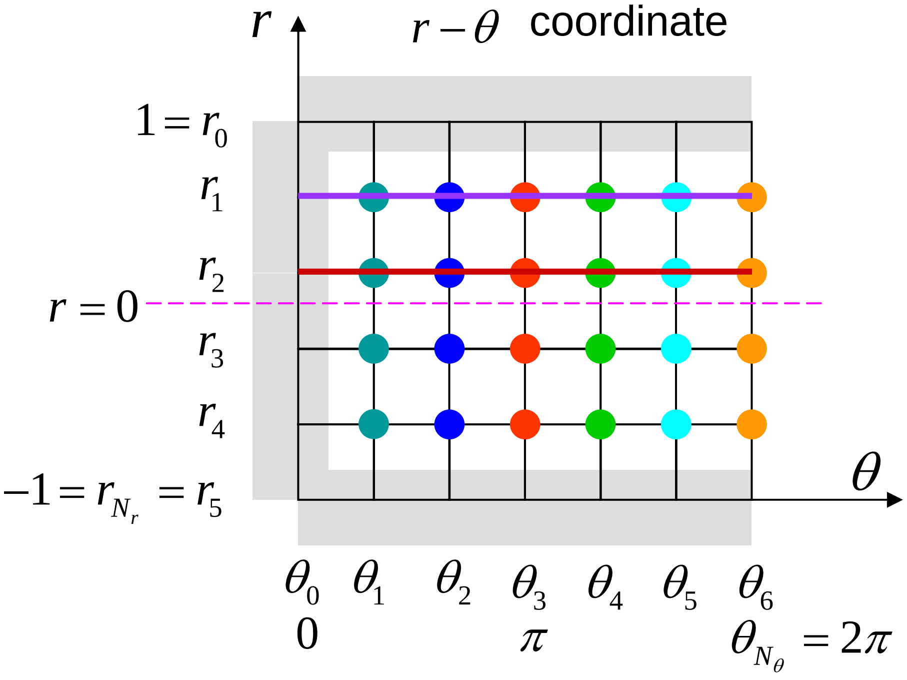
$$D_N^2 = \left(S_N^{(2)}(x_j - x_k)\right) = \begin{pmatrix} \ddots & \vdots & \\ \ddots & -\csc^2(2h/2)/2 & \\ \ddots & \csc^2(h/2)/2 & \\ \ddots & -\dfrac{1}{6} - \dfrac{\pi^2}{3h^2} & \ddots \\ & \csc^2(h/2)/2 & \ddots \\ & -\csc^2(2h/2)/2 & \ddots \end{pmatrix} = toeplitz\left(-\frac{1}{6} - \frac{\pi^2}{3h^2}, \frac{(-1)^{2:N}}{2\sin^2((1:N-1)h/2)}\right)$$

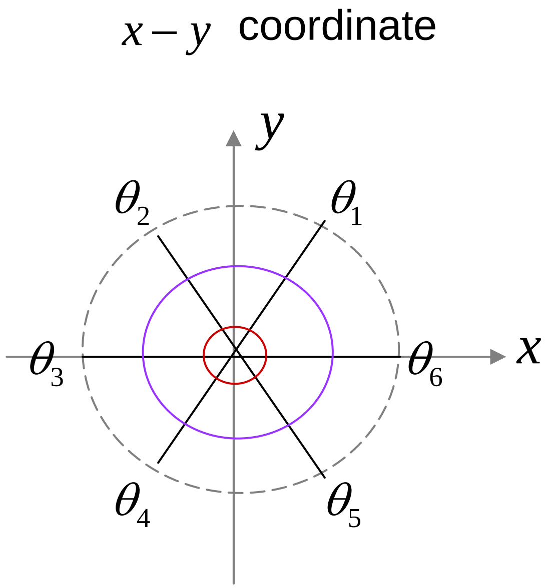
Fornberg's idea : extend radius to negative image [1]

$(x,y) \in \mathit{unit\ disk}$ ⟷ $r \in [-1,1]$ and $\theta \in [0,2\pi]$

① $N_r = 5$ (odd): to avoid singularity of coordinate transformation $r = 0$

② $N_\theta = 6$ (even): to keep symmetry condition $u(r,\theta) = u(-r,(\theta+\pi)(\mathrm{mod}\ \ 2\pi))$

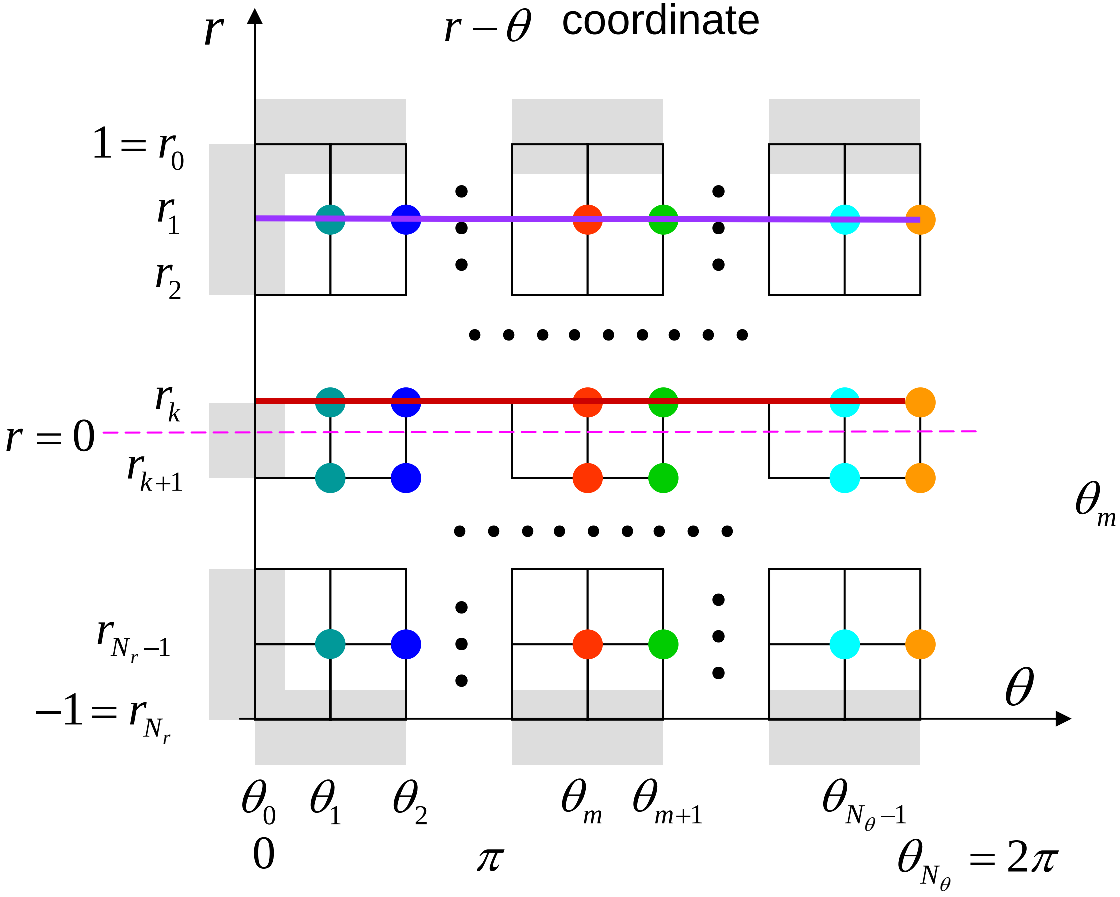




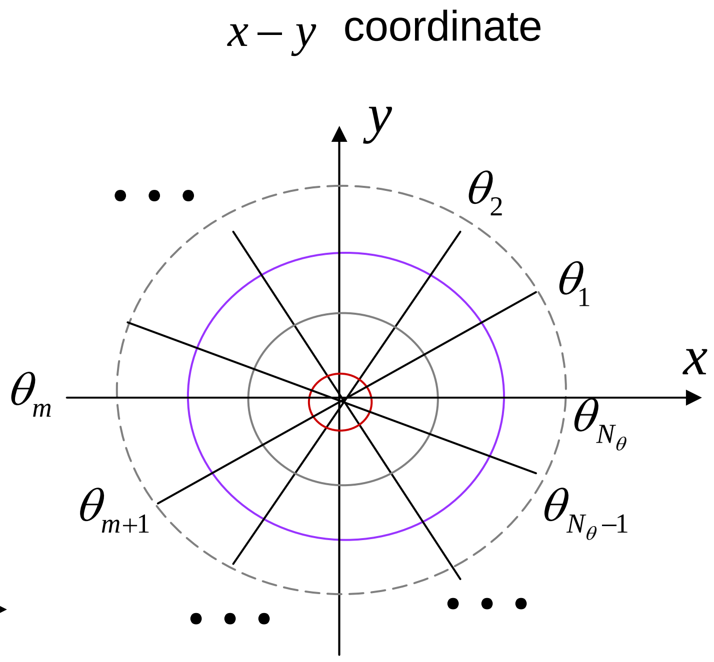
Fornberg's idea : extend radius to negative image [2]

In general $N_r = 2k+1$ is odd, and $N_\theta = 2m$ is even, then $\Delta r : \text{non-unif}$ and $\Delta\theta = \frac{2\pi}{N_\theta}$

Active variable $\begin{cases} r_i = 1 - i\Delta r : \ 1 \le i \le N_r - 1 \\ \theta_j = j\Delta\theta : \ 1 \le j \le N_\theta \end{cases}$, total number is $(N_r - 1)N_\theta$



$r - \theta$ coordinate

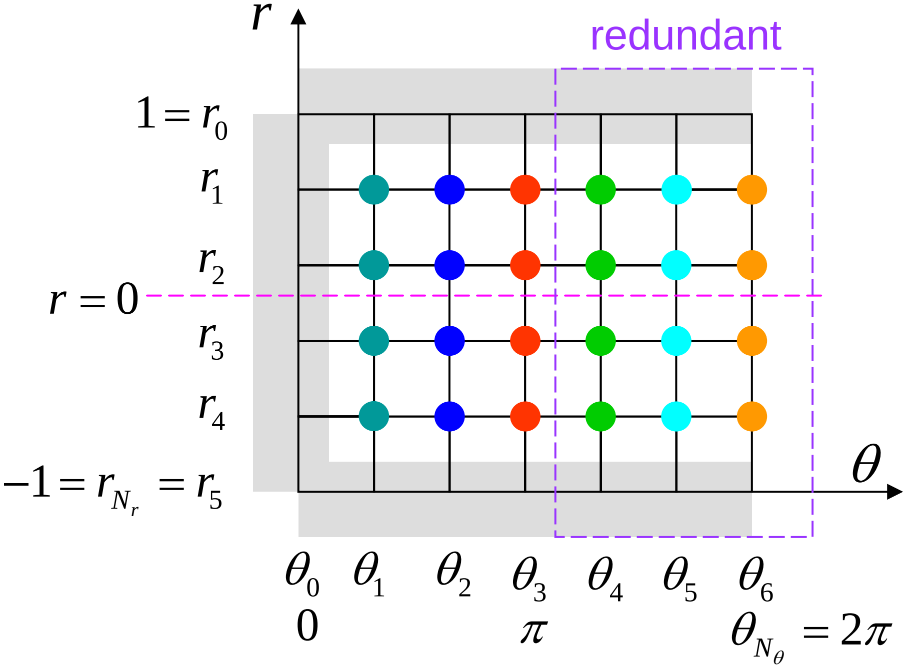


$x - y$ coordinate

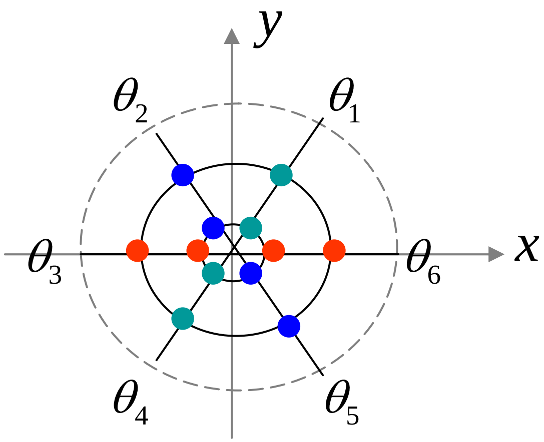
Redundancy in coordinate transformation [1]

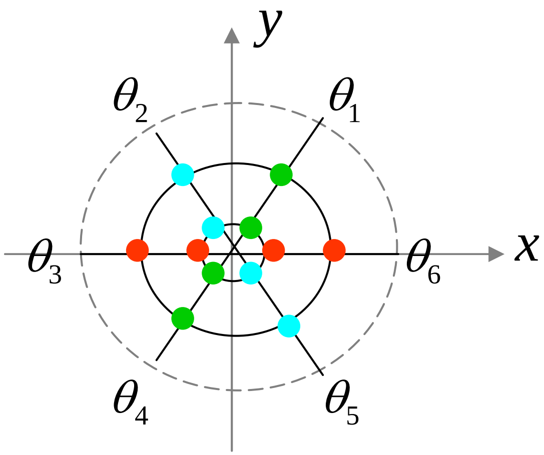
2 to 1 mapping $(r,\theta),\left(-r,(\theta+\pi)(\mathrm{mod}\ 2\pi)\right)\leftrightarrow(x,y)$

$r-\theta$ coordinate



$x-y$ coordinate

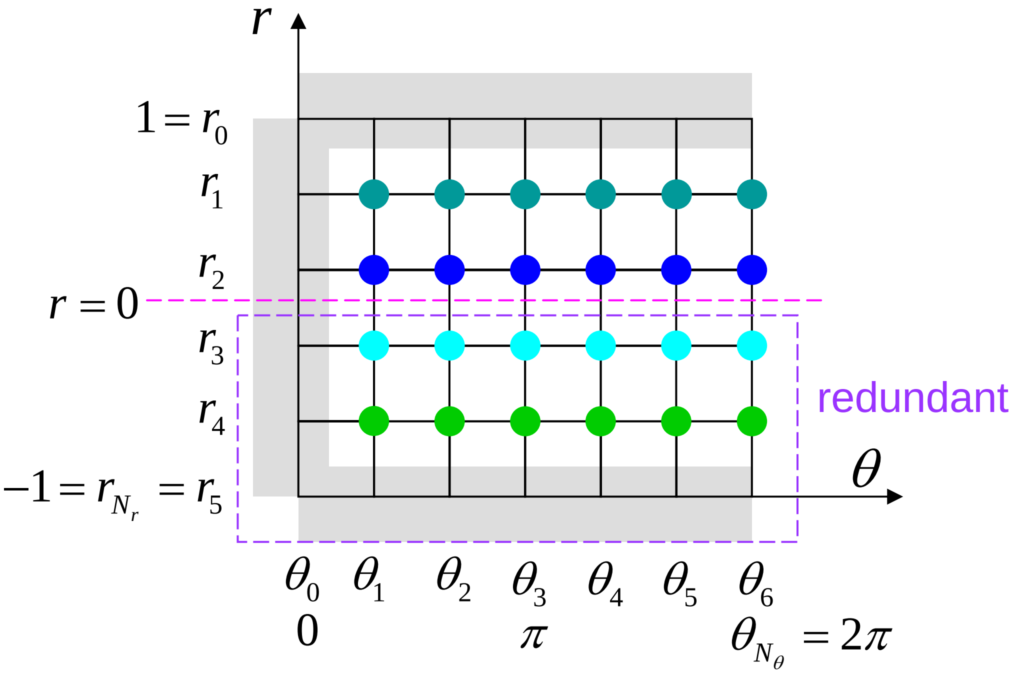




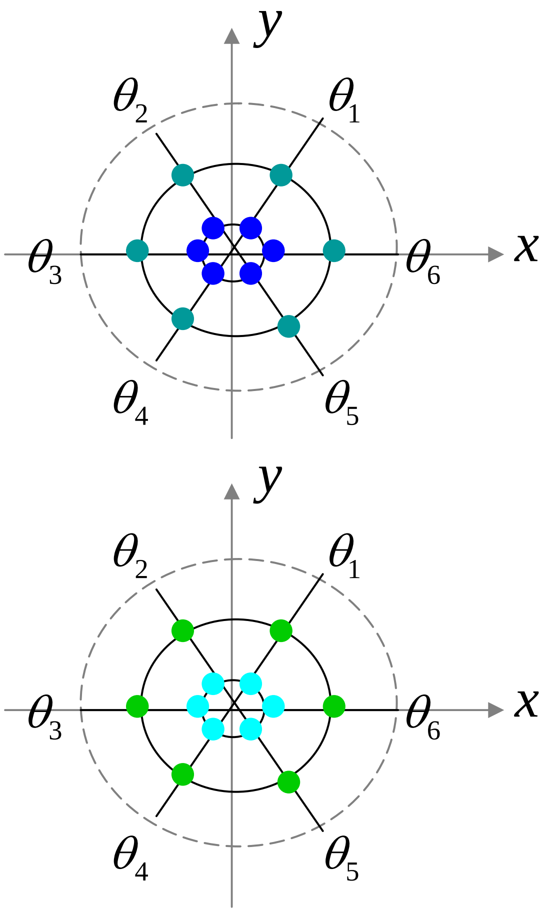
Redundancy in coordinate transformation [2]

2 to 1 mapping $(r,\theta),\left(-r,(\theta+\pi)(\text{mod }\ 2\pi)\right)\leftrightarrow(x,y)$

$r-\theta$ coordinate



$x-y$ coordinate



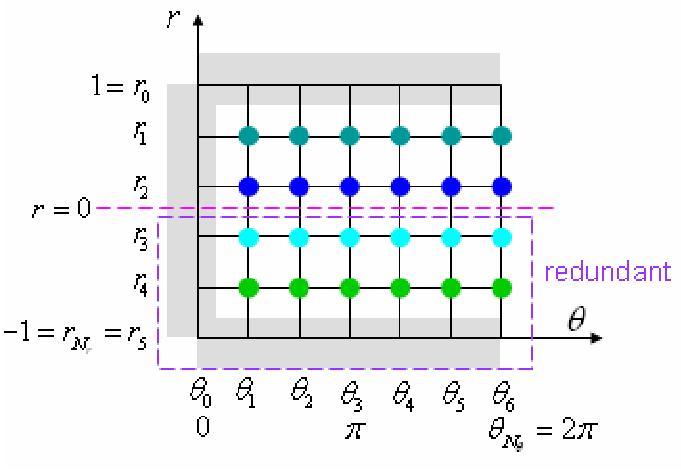
Redundancy in coordinate transformation [3]

$N_r = 2k+1 = 5$ is odd, then Chebyshev differential matrix D_{N_r} is expressed as

$$D_{N_r} = \begin{array}{cccccc|c}
r_0 & r_1 & r_2 & r_3 & r_4 & r_5 & \\
\hline
8.5 & -10.4721 & 2.8944 & -1.5279 & 1.1056 & -0.5 & r_0 \\
2.6180 & -1.1708 & -2 & 0.8944 & -0.618 & 0.2764 & r_1 \\
-0.7236 & 2 & -0.1708 & -1.6180 & 0.8944 & -0.382 & r_2 \\
0.3820 & -0.8944 & 1.618 & 0.1708 & -2 & 0.7236 & r_3 \\
-0.2764 & 0.6180 & -0.8944 & 2 & 1.1708 & -2.618 & r_4 \\
0.5 & -1.1056 & 1.5279 & -2.8944 & 10.4721 & -8.5 & r_5
\end{array}$$

$$\tilde{D}_{N_r} = \begin{pmatrix}
-1.1708 & -2 & 0.8944 & -0.618 \\
2 & -0.1708 & -1.6180 & 0.8944 \\
-0.8944 & 1.618 & 0.1708 & -2 \\
0.6180 & -0.8944 & 2 & 1.1708
\end{pmatrix}$$

neglect



Redundancy in coordinate transformation [4]

Symmetry property of Chebyshev differential matrix : $\left(D_N\right)_{i,j} = -\left(D_N\right)_{N-i,N-j}$

$$\tilde{D}_{N_r} = \begin{pmatrix} -1.1708 & -2 & 0.8944 & -0.618 \\ 2 & -0.1708 & -1.6180 & 0.8944 \\ -0.8944 & 1.618 & 0.1708 & -2 \\ 0.6180 & -0.8944 & 2 & 1.1708 \end{pmatrix}$$

Permute column by $P = (1, 2, 4, 3)$

$$\tilde{D}_{N_r} P^T = \begin{pmatrix} -1.1708 & -2 & -0.618 & 0.8944 \\ 2 & -0.1708 & 0.8944 & -1.6180 \\ -0.8944 & 1.618 & -2 & 0.1708 \\ 0.6180 & -0.8944 & 1.1708 & 2 \end{pmatrix}$$

$$E_1 = \begin{pmatrix} -1.1708 & -2 \\ 2 & -0.1708 \end{pmatrix}$$ is symmetric

$$\tilde{E}_2 = \begin{pmatrix} -0.618 & 0.8944 \\ 0.8944 & -1.6180 \end{pmatrix}$$ is NOT symmetric

$$\begin{pmatrix} E_1 & \tilde{E}_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$ is faster ?

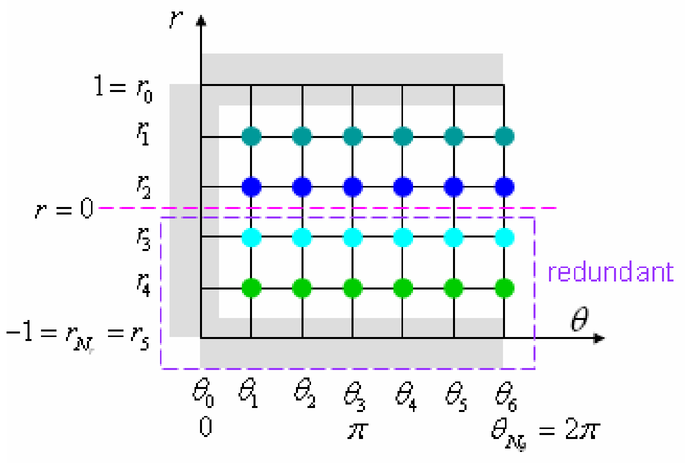
Redundancy in coordinate transformation [5]

$N_r = 2k+1 = 5$ is odd, then Chebyshev differential matrix $D_{N_r}^2$ is expressed as

$$D_{N_r}^2 = D_{N_r} \cdot D_{N_r} = \begin{array}{cccccc|c} r_0 & r_1 & r_2 & r_3 & r_4 & r_5 & \\ \hline 41.6 & -68.3607 & 40.8276 & -23.6393 & 17.5724 & -8 & r_0 \\ 21.2859 & -31.5331 & 12.6833 & -3.6944 & 2.2111 & -0.9528 & r_1 \\ -1.8472 & 7.3167 & -10.0669 & 5.7889 & -1.9056 & 0.7141 & r_2 \\ 0.7141 & -1.9056 & 5.7889 & -10.0669 & 7.3167 & -1.8472 & r_3 \\ -0.9528 & 2.2111 & -3.6944 & 12.6833 & -31.5331 & 21.2859 & r_4 \\ -8 & 17.5724 & -23.6393 & 40.8276 & -68.3607 & 41.6 & r_5 \end{array}$$

$$\tilde{D}_{N_r}^2 = \begin{pmatrix} -31.5331 & 12.6833 & -3.6944 & 2.2111 \\ 7.3167 & -10.0669 & 5.7889 & -1.9056 \\ -1.9056 & 5.7889 & -10.0669 & 7.3167 \\ 2.2111 & -3.6944 & 12.6833 & -31.5331 \end{pmatrix}$$

neglect



Redundancy in coordinate transformation [6]

Symmetry property of Chebyshev differential matrix : $\left(D_N^2\right)_{i,j} = \left(D_N^2\right)_{N-i,N-j}$

$$\tilde{D}_{N_r}^2 = \begin{pmatrix} -31.5331 & 12.6833 & -3.6944 & 2.2111 \\ 7.3167 & -10.0669 & 5.7889 & -1.9056 \\ -1.9056 & 5.7889 & -10.0669 & 7.3167 \\ 2.2111 & -3.6944 & 12.6833 & -31.5331 \end{pmatrix}$$

$\downarrow$ Permute column by $P = (1, 2, 4, 3)$

$$\tilde{D}_{N_r}^2 P^T = \begin{pmatrix} -31.5331 & 12.6833 & 2.2111 & -3.6944 \\ 7.3167 & -10.0669 & -1.9056 & 5.7889 \\ -1.9056 & 5.7889 & 7.3167 & -10.0669 \\ 2.2111 & -3.6944 & -31.5331 & 12.6833 \end{pmatrix}$$

$$D_1 = \begin{pmatrix} -31.5331 & 12.6833 \\ 7.3167 & -10.0669 \end{pmatrix}$$

is **NOT** sym.

$$\tilde{D}_2 = \begin{pmatrix} 2.2111 & -3.6944 \\ -1.9056 & 5.7889 \end{pmatrix}$$

is **NOT** sym.

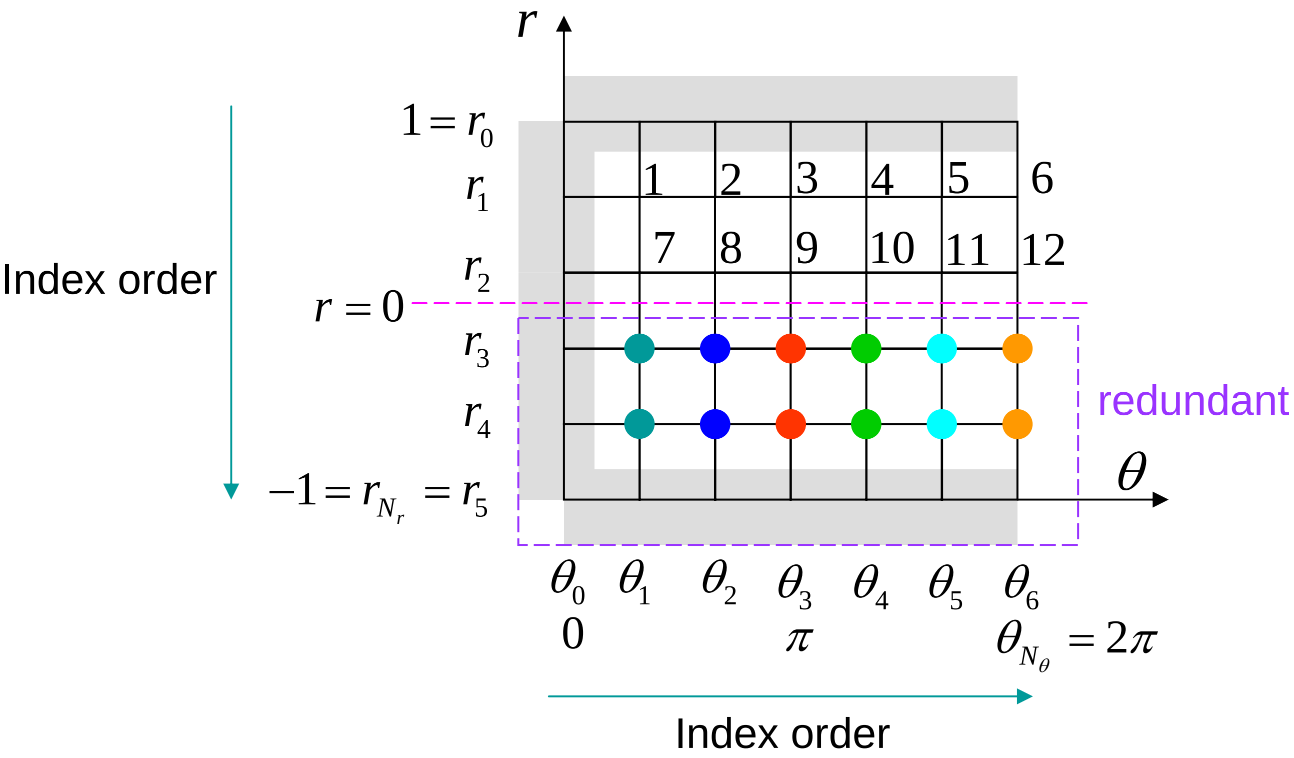
Row-major indexing: remove redundancy [1]

Define active variable $u_{ij} \triangleq u\left(r_i,\theta_j\right)$ for $1 \le i \le N_r - 1$ and $1 \le j \le N_\theta$

$$\vec{u}_i \triangleq \left(\left(u_{i,1},u_{i,2},\cdots,u_{i,m}\right),\left(u_{i,m+1},u_{i,m+2},\cdots,u_{i,N_\theta}\right)\right)^T$$

total number of active variables is $k\cdot N_\theta = 2\cdot 6 = 12$

NOT $\left(N_r - 1\right)\cdot N_\theta = 4\cdot 6 = 24$



$$\vec{u}_1 = \begin{pmatrix} \begin{pmatrix} u_{11} \\ u_{12} \\ u_{13} \end{pmatrix} \\ \begin{pmatrix} u_{14} \\ u_{15} \\ u_{16} \end{pmatrix} \end{pmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

$$\vec{u}_2 = \begin{pmatrix} \begin{pmatrix} u_{21} \\ u_{22} \\ u_{23} \end{pmatrix} \\ \begin{pmatrix} u_{24} \\ u_{25} \\ u_{26} \end{pmatrix} \end{pmatrix} \begin{matrix} 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{matrix}$$

Row-major indexing: remove redundancy [2]

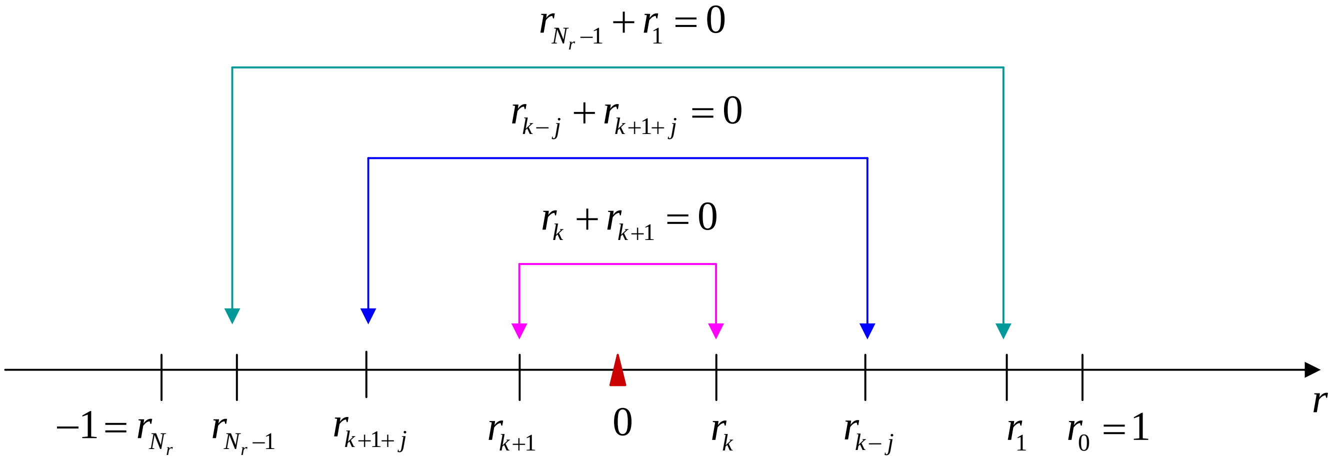
$N_r = 2k+1$ is odd, and $r_i = \cos\left(\dfrac{i\pi}{N_r}\right): \; 0 \le i \le N_r$

suppose $0 \le j \le k$, then $r_{k-j} + r_{k+1+j} = 0$ since

$$r_{k-j} = \cos\frac{(k-j)\pi}{N_r} = -\cos\left(\pi - \frac{(k-j)\pi}{N_r}\right) \xrightarrow{N_r = 2k+1} -\cos\left(\frac{(k+j+1)\pi}{N_r}\right) = -r_{k+1+j}$$

$N_\theta = 2m$ is even, and $\Delta\theta = \dfrac{2\pi}{N_\theta} = \dfrac{\pi}{m}, \quad \theta_j = j\Delta\theta: \; 1 \le j \le N_\theta$

Hence for $1 \le j \le m, \quad \theta_j + \pi = \theta_{m+j}$ and $(\theta_{m+j} + \pi)(\operatorname{mod} 2\pi) = \theta_j$



Row-major indexing: remove redundancy [3]

From symmetry condition, we have $u\left(r_{k+1+i},\theta_j\right)=u\left(-r_{k+1+i},\left(\theta_j+\pi\right)\left(\bmod 2\pi\right)\right)$

for $0\le i\le k$ and $0\le j\le m$, symmetry condition implies $\begin{cases} u\left(r_{k+1+i},\theta_j\right)=u\left(r_{k-i},\theta_{m+j}\right) \\ u\left(r_{k+1+i},\theta_{m+j}\right)=u\left(r_{k-i},\theta_j\right) \end{cases}$

Therefore, we have two important relationships

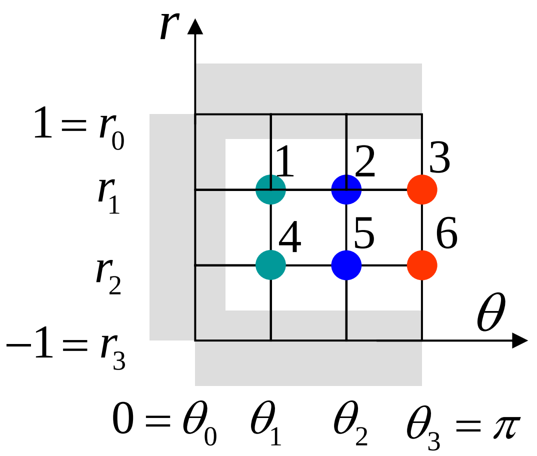
① $$\begin{pmatrix} u_{k+1,j} \\ u_{k+2,j} \\ \vdots \\ u_{N_r-1,j} \end{pmatrix} = \begin{pmatrix} u_{k,m+j} \\ u_{k-1,m+j} \\ \vdots \\ u_{1,m+j} \end{pmatrix} = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix} \begin{pmatrix} u_{1,m+j} \\ u_{2,m+j} \\ \vdots \\ u_{k,m+j} \end{pmatrix}$$

② $$\begin{pmatrix} u_{k+1,m+j} \\ u_{k+2,m+j} \\ \vdots \\ u_{N_r-1,m+j} \end{pmatrix} = \begin{pmatrix} u_{k,j} \\ u_{k-1,j} \\ \vdots \\ u_{1,j} \end{pmatrix} = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix} \begin{pmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{k,j} \end{pmatrix}$$

Kronecker product [1]

Define active variable $u_{ij} \triangleq u\left(r_i, \theta_j\right)$

for $1 \le i \le 2$ and $1 \le j \le 3$



$$\begin{array}{rcl} \vec{u}_1 = & \begin{pmatrix} u_{11} \\ u_{12} \\ u_{13} \end{pmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \\ \vec{u}_2 = & \begin{pmatrix} u_{21} \\ u_{22} \\ u_{23} \end{pmatrix} & \begin{matrix} 4 \\ 5 \\ 6 \end{matrix} \end{array}$$

Separation of variable: assume matrix ***A*** acts on *r-dir* and matrix ***B*** acts on $\theta - dir$

$$\left(Au\right)\begin{pmatrix} \left(r_1, \theta_j\right) \\ \left(r_2, \theta_j\right) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} u_{1,j} \\ u_{2,j} \end{pmatrix}$$

is independent of j

$$\left(Bu\right)\begin{pmatrix} \left(r_i, \theta_1\right) \\ \left(r_i, \theta_2\right) \\ \left(r_i, \theta_3\right) \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}\begin{pmatrix} u_{i,1} \\ u_{i,2} \\ u_{i,3} \end{pmatrix}$$

is independent of i

Let $X = \begin{pmatrix} \vec{u}_1 \\ \vec{u}_2 \end{pmatrix} \in R^6$ be row-major

index of active variable

Kronecker product [2]

Kronecker product is defined by $A \oplus B \equiv \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix} \in R^{6\times 6}$

$$(A \oplus B)\,X = \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix}\begin{pmatrix} \vec{u}_1 \\ \vec{u}_2 \end{pmatrix} = \begin{pmatrix} a_{11}B\vec{u}_1 + a_{12}B\vec{u}_2 \\ a_{21}B\vec{u}_1 + a_{22}B\vec{u}_2 \end{pmatrix}$$

Case 1: $A = I$

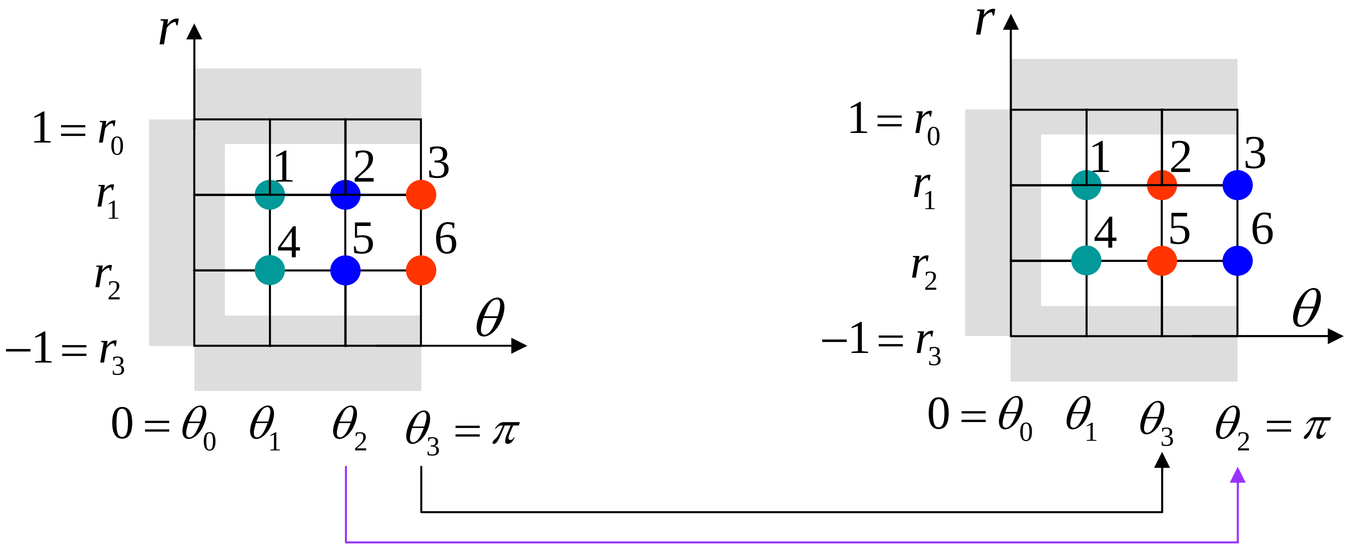
$$(I \oplus B)\,X = \begin{pmatrix} B\vec{u}_1 \\ B\vec{u}_2 \end{pmatrix} \longrightarrow (Bu)\begin{pmatrix} (r_i, \theta_1) \\ (r_i, \theta_2) \\ (r_i, \theta_3) \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}\begin{pmatrix} u_{i,1} \\ u_{i,2} \\ u_{i,3} \end{pmatrix}$$

Case 2: $B = I$

$$(A \oplus I)\,X = \begin{pmatrix} a_{11}\vec{u}_1 + a_{12}\vec{u}_2 \\ a_{21}\vec{u}_1 + a_{22}\vec{u}_2 \end{pmatrix} = \begin{pmatrix} a_{11}\begin{pmatrix} u_{11} \\ u_{12} \\ u_{13} \end{pmatrix} + a_{12}\begin{pmatrix} u_{21} \\ u_{22} \\ u_{23} \end{pmatrix} \\ a_{21}\begin{pmatrix} u_{11} \\ u_{12} \\ u_{13} \end{pmatrix} + a_{22}\begin{pmatrix} u_{21} \\ u_{22} \\ u_{23} \end{pmatrix} \end{pmatrix} \longrightarrow (Au)\begin{pmatrix} (r_1, \theta_j) \\ (r_1, \theta_j) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} u_{1,j} \\ u_{2,j} \end{pmatrix}$$

Kronecker product [3]

Case 3: permutation, if permute $\theta_2 \leftrightarrow \theta_3$



$$X = \begin{pmatrix} \vec{u}_1 \\ \vec{u}_2 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} u_{11} \\ u_{12} \\ u_{13} \end{pmatrix} \\ \begin{pmatrix} u_{21} \\ u_{22} \\ u_{23} \end{pmatrix} \end{pmatrix} \xrightarrow{P_\theta = (1,3,2)} \tilde{X} = \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} u_{11} \\ u_{13} \\ u_{12} \end{pmatrix} = P_\theta \cdot \vec{u}_1 \\ \begin{pmatrix} u_{21} \\ u_{23} \\ u_{22} \end{pmatrix} = P_\theta \cdot \vec{u}_2 \end{pmatrix}$$

$$\Rightarrow \quad (I \oplus P_\theta) X = \begin{pmatrix} P_\theta \vec{u}_1 \\ P_\theta \vec{u}_2 \end{pmatrix} = \tilde{X}$$

Kronecker product [4]

Case 4: $A = diag\left(a_1, a_2\right)$

$$\left(A \oplus B\right) X = \begin{pmatrix} a_1 B\vec{u}_1 \\ a_2 B\vec{u}_2 \end{pmatrix} \longrightarrow \begin{cases} a_1\left(Bu\right)\begin{pmatrix} \left(r_1, \theta_1\right) \\ \left(r_1, \theta_2\right) \\ \left(r_1, \theta_3\right) \end{pmatrix} = a_1 \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}\begin{pmatrix} u_{1,1} \\ u_{1,2} \\ u_{1,3} \end{pmatrix} \quad \boxed{i = 1} \\ a_2\left(Bu\right)\begin{pmatrix} \left(r_2, \theta_1\right) \\ \left(r_2, \theta_2\right) \\ \left(r_2, \theta_3\right) \end{pmatrix} = a_2 \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}\begin{pmatrix} u_{2,1} \\ u_{2,2} \\ u_{2,3} \end{pmatrix} \quad \boxed{i = 2} \end{cases}$$

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = -\lambda^2 u$$

$$diag\left(\frac{1}{r_1^2}, \frac{1}{r_2^2}, \cdots, \frac{1}{r_k^2}\right) \oplus \tilde{D}^2_{\theta, N_\theta}$$

Non-active variable → active variable [1]

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = -\lambda^2 u$$, on $r \in [-1,1]$ and $\theta \in [0, 2\pi]$

$N_r = 2k+1$ is odd, and $N_\theta = 2m$ is even, then

Active variable is $\begin{cases} r_i = \cos\left(\dfrac{i\pi}{N_r}\right): \ 1 \le i \le k \\ \theta_j = j\Delta\theta: \ 1 \le j \le N_\theta \end{cases}$, that is $r_i > 0, \ 0 < \theta_j \le 2\pi$

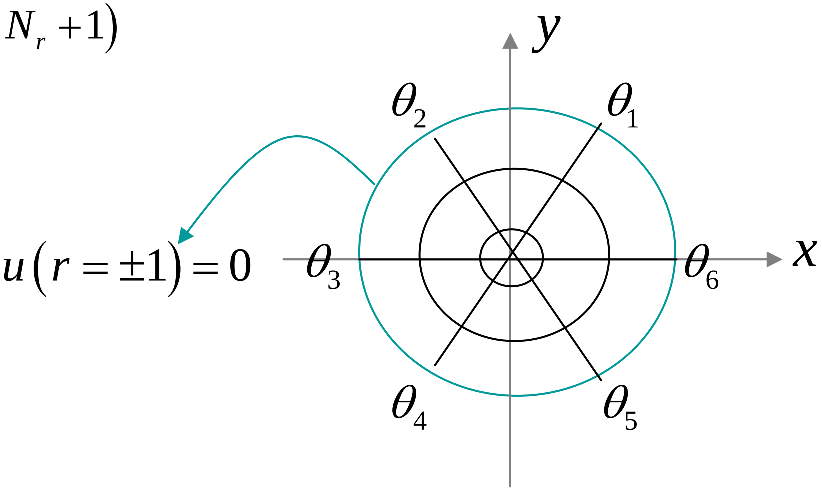
Total number is kN_θ, **NOT** $(N_r - 1) \cdot N_\theta$

$$\tilde{D}_{N_r}^2 u + \frac{1}{r} \tilde{D}_{N_r} u_r + \frac{1}{r^2} D_{N_\theta} u = -\lambda^2 u$$

Note that differential matrix D_{N_r} is of dimension $(N_r + 1)$

,acts on $\begin{pmatrix} \boxed{u_{0,j}} \\ u_{1,j} \\ \vdots \\ u_{N_r-1,j} \\ \boxed{u_{N_r,j}} \end{pmatrix}$

Neglect due to B.C.


$u(r = \pm 1) = 0$

Non-active variable → active variable [2]

However $\tilde{D}^2_{N_r}$ and $\tilde{D}_{N_r}$ act on

$$\begin{pmatrix} \begin{pmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{k,j} \end{pmatrix} \\ \boxed{\begin{pmatrix} u_{k+1,j} \\ u_{k+2,j} \\ \vdots \\ u_{N_r-1,j} \end{pmatrix}} \end{pmatrix}$$

NOT active variable, how to deal with?

From previous discussion, we have following relationships which can solve this problem

$$\begin{pmatrix} u_{k+1,j} \\ u_{k+2,j} \\ \vdots \\ u_{N_r-1,j} \end{pmatrix} = \begin{pmatrix} u_{k,m+j} \\ u_{k-1,m+j} \\ \vdots \\ u_{1,m+j} \end{pmatrix} = (k:-1:1) \begin{pmatrix} u_{1,m+j} \\ u_{2,m+j} \\ \vdots \\ u_{k,m+j} \end{pmatrix}$$

and

$$\begin{pmatrix} u_{k+1,m+j} \\ u_{k+2,m+j} \\ \vdots \\ u_{N_r-1,m+j} \end{pmatrix} = \begin{pmatrix} u_{k,j} \\ u_{k-1,j} \\ \vdots \\ u_{1,j} \end{pmatrix} = (k:-1:1) \begin{pmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{k,j} \end{pmatrix}$$

where $(k:-1:1)$ is a permutation matrix

Non-active variable → active variable [3]

Recall $u_{ij} \triangleq u\left(r_i, \theta_j\right)$ for $1 \le i \le N_r - 1$ and $1 \le j \le N_\theta$

Consider Chebyshev differential matrix $\tilde{D}_{r,N_r}$ acts on u and evaluate at $\left(r_i, \theta_j\right)$

$$\left(\tilde{D}_{r,N_r} u\right)\left(r_i, \theta_j\right) = \left(i - \text{row of } \tilde{D}_{r,N_r}\right) \cdot u\left(\bullet, \theta_j\right)$$

We write in matrix notation

$$\left(\tilde{D}_{r,N_r} u\right)\left(r_i, \theta_j\right) = \Big(\overbrace{d_{i,1} \quad d_{i,2} \quad \cdots \quad d_{i,k}}^{r>0} \;\Big|\; \overbrace{d_{i,k+1} \quad d_{i,k+2} \quad \cdots \quad d_{i,N_r-1}}^{r<0}\Big) \left(\begin{array}{c} \begin{pmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{k,j} \end{pmatrix} \\ \begin{pmatrix} u_{k+1,j} \\ u_{k+2,j} \\ \vdots \\ u_{N_r-1,j} \end{pmatrix} \end{array}\right) \begin{array}{l} \left.\vphantom{\begin{pmatrix} u \\ u \\ \vdots \\ u \end{pmatrix}}\right\} r>0 \\ \left.\vphantom{\begin{pmatrix} u \\ u \\ \vdots \\ u \end{pmatrix}}\right\} r<0 \end{array}$$

Question: How about if we arrange equations on $\left(r_i, \theta_j\right)$ $\forall\, i$ when fixed j

Non-active variable → active variable [4]

$$
\left(\tilde{D}_{N_r}u\right)\begin{pmatrix}(r_1,\theta_j)\\(r_2,\theta_j)\\\vdots\\(r_k,\theta_j)\\(r_{k+1},\theta_j)\\(r_{k+2},\theta_j)\\\vdots\\(r_{N_r-1},\theta_j)\end{pmatrix}
=
\begin{array}{cc}
\begin{array}{cc} \qquad\qquad r>0 \qquad\qquad & \qquad\qquad r<0 \qquad\qquad \end{array} & \\
\left(\begin{array}{cccc|cccc}
d_{11} & d_{12} & \cdots & d_{1,k} & d_{1,k+1} & d_{1,k+2} & \cdots & d_{1,N_r-1}\\
d_{21} & d_{22} & \cdots & d_{2,k} & d_{2,k+1} & d_{2,k+2} & \cdots & d_{2,N_r-1}\\
\vdots & \cdots & \ddots & \vdots & \cdots & \cdots & \ddots & \vdots\\
d_{k,1} & d_{k,2} & \cdots & d_{k,k} & d_{k,k+1} & d_{k,k+2} & \cdots & d_{k,N_r-1}\\
\hline
d_{k+1,1} & d_{k+1,2} & \cdots & d_{k+1,k} & d_{k+1,k+1} & d_{k+1,k+2} & \cdots & d_{k+1,N_r-1}\\
d_{k+2,1} & d_{k+2,2} & \cdots & \vdots & \vdots & \cdots & \cdots & \vdots\\
\vdots & \cdots & \ddots & \vdots & \cdots & \cdots & \ddots & \vdots\\
d_{N_r-1,1} & d_{N_r-1,2} & \cdots & d_{N_r-1,k} & d_{N_r-1,k+1} & \cdots & \cdots & d_{N_r-1,N_r-1}
\end{array}\right) & \begin{array}{c} r>0 \\[6em] r<0 \end{array}
\end{array}
\overset{u(r,\theta_j)}{\overset{\|}{\begin{pmatrix}\begin{pmatrix}u_{1,j}\\u_{2,j}\\\vdots\\u_{k,j}\end{pmatrix}\\ \boxed{\begin{pmatrix}u_{k+1,j}\\u_{k+2,j}\\\vdots\\u_{N_r-1,j}\end{pmatrix}}\end{pmatrix}}}
$$

$$
\Big\downarrow\qquad
\begin{pmatrix}u_{k,m+j}\\u_{k-1,m+j}\\\vdots\\u_{1,m+j}\end{pmatrix}
$$

abbreviate $\tilde{D}_{N_r}=\left(\begin{array}{c|c}E_1 & E_2\\\hline E_3 & E_4\end{array}\right)$

We only keep operations on active variable $u(r>0,\theta)$

That is, only consider equation $\left(\tilde{D}_{N_r}u\right)(r_i,\theta_j)$ for $r_i>0$

Later on, we use the same symbol $\tilde{D}_{N_r}=\begin{pmatrix}E_1 & E_2\end{pmatrix}$

Non-active variable → active variable [5]

Define permutation matrix $P=\left(\left(1:k\right),\left(N_r-1:-1:k+1\right)\right)=\left(\begin{array}{c|ccc} I & & & \\ \hline & & & 1 \\ & & \iddots & \\ & 1 & & \end{array}\right)\begin{array}{l} \}\,k \\ \left.\begin{array}{l} \\ \\ \end{array}\right\}\,k \end{array}$

Let $u\left(r,\theta_j\right)\equiv$

$$\begin{pmatrix} \begin{pmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{k,j} \end{pmatrix} \\ \boxed{\begin{pmatrix} u_{k+1,j} \\ u_{k+2,j} \\ \vdots \\ u_{N_r-1,j} \end{pmatrix}} \end{pmatrix} \xrightarrow{P\left(u\left(r,\theta_j\right)\right)} \begin{pmatrix} \begin{pmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{k,j} \end{pmatrix} \\ \boxed{\begin{pmatrix} u_{1,m+j} \\ u_{2,m+j} \\ \vdots \\ u_{k,m+j} \end{pmatrix}} \end{pmatrix} \longrightarrow$$

Active variable with the same indexing in ***r**-direction*

$$\boxed{\begin{pmatrix} u_{k+1,j} \\ u_{k+2,j} \\ \vdots \\ u_{N_r-1,j} \end{pmatrix}} \longrightarrow \begin{pmatrix} u_{k,m+j} \\ u_{k-1,m+j} \\ \vdots \\ u_{1,m+j} \end{pmatrix}$$

Non-active variable → active variable [6]

Moreover, we modify differential matrix according to permutation ***P*** by

$$\tilde{D}_{N_r}\left(u\left(r,\theta_j\right)\right)=\left(\tilde{D}_{N_r}P^T\right)\left(Pu\left(r,\theta_j\right)\right)$$

$$r>0 \qquad\qquad\qquad r<0$$

where $$\tilde{D}_{N_r}P^T=\left(\begin{array}{cccc|cccc} d_{11} & d_{12} & \cdots & d_{1,k} & d_{1,N_r-1} & \cdots & d_{1,k+2} & d_{1,k+1} \\ d_{21} & d_{22} & \cdots & d_{2,k} & d_{2,N_r-1} & \cdots & d_{2,k+2} & d_{2,k+1} \\ \vdots & \ldots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{k,1} & d_{k,2} & \cdots & d_{k,k} & d_{k,N_r-1} & \cdots & d_{k,k+2} & d_{k,k+1} \end{array}\right)=\begin{pmatrix} E_1 & \tilde{E}_2 \end{pmatrix}$$

such that for $1\le j\le m$

$$\left(\tilde{D}_{r,N_r}u\right)\left(r,\theta_j\right)=E_1\begin{pmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{k,j} \end{pmatrix}+\tilde{E}_2\begin{pmatrix} u_{1,m+j} \\ u_{2,m+j} \\ \vdots \\ u_{k,m+j} \end{pmatrix} \quad \text{and} \quad \left(\tilde{D}_{r,N_r}u\right)\left(r,\theta_{m+j}\right)=E_1\begin{pmatrix} u_{1,m+j} \\ u_{2,m+j} \\ \vdots \\ u_{k,m+j} \end{pmatrix}+\tilde{E}_2\begin{pmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{k,j} \end{pmatrix}$$

Evaluated at $\left(\left(r_1,\theta_j\right),\left(r_2,\theta_j\right),\cdots,\left(r_k,\theta_j\right)\right)$

Non-active variable → active variable [7]

$$\left(\tilde{D}_{r,N_r}u\right)\left(\left(r,\theta_1\right),\left(r,\theta_2\right),\cdots,\left(r,\theta_m\right)\right)=E_1\begin{pmatrix} u_{1,1} & u_{1,2} & \cdots & u_{1,m} \\ u_{2,1} & u_{2,2} & \cdots & u_{2,m} \\ \vdots & \cdots & \ddots & \vdots \\ u_{k,1} & u_{k,2} & \cdots & u_{k,m} \end{pmatrix}+\tilde{E}_2\begin{pmatrix} u_{1,m+1} & u_{1,m+2} & \cdots & u_{1,2m} \\ u_{2,m+1} & u_{2,m+2} & \cdots & u_{2,2m} \\ \vdots & \cdots & \ddots & \vdots \\ u_{k,m+1} & u_{k,m+2} & \cdots & u_{k,2m} \end{pmatrix}$$

$$\left(\tilde{D}_{r,N_r}u\right)\left(\left(r,\theta_{m+1}\right),\left(r,\theta_{m+2}\right),\cdots,\left(r,\theta_{2m}\right)\right)=E_1\begin{pmatrix} u_{1,m+1} & u_{1,m+2} & \cdots & u_{1,2m} \\ u_{2,m+1} & u_{2,m+2} & \cdots & u_{2,2m} \\ \vdots & \cdots & \ddots & \vdots \\ u_{k,m+1} & u_{k,m+2} & \cdots & u_{k,2m} \end{pmatrix}+\tilde{E}_2\begin{pmatrix} u_{1,1} & u_{1,2} & \cdots & u_{1,m} \\ u_{2,1} & u_{2,2} & \cdots & u_{2,m} \\ \vdots & \cdots & \ddots & \vdots \\ u_{k,1} & u_{k,2} & \cdots & u_{k,m} \end{pmatrix}$$

define $U_i \equiv \begin{pmatrix} u_{i,1} \\ u_{i,2} \\ \vdots \\ u_{i,m} \end{pmatrix}$, $V_i \equiv \begin{pmatrix} u_{i,m+1} \\ u_{i,m+2} \\ \vdots \\ u_{i,2m} \end{pmatrix}$ and active variable $X=\left(X_1 \quad X_2\right),\ X_1=\begin{pmatrix} U_1^T \\ U_2^T \\ \vdots \\ U_k^T \end{pmatrix},\ X_2=\begin{pmatrix} V_1^T \\ V_2^T \\ \vdots \\ V_k^T \end{pmatrix}$

To sum up $\left(\tilde{D}_{r,N_r}u\right)\left(\left(r,\theta_1\right),\cdots,\left(r,\theta_{2m}\right)\right)=E_1\left(X_1 \mid X_2\right)+E_2\left(X_2 \mid X_1\right)$

Non-active variable → active variable [8]

Note that under row-major indexing, memory storage of $X = \begin{pmatrix} X_1 & X_2 \end{pmatrix} = \begin{pmatrix} U_1^T & V_1^T \\ U_2^T & V_2^T \\ \vdots & \vdots \\ U_k^T & V_k^T \end{pmatrix}$

is $mem\left(X\right) = \left(U_1^T \mid V_1^T \mid U_2^T \mid V_2^T \mid \cdots \mid U_j^T \mid V_j^T \mid \cdots \mid U_k^T \mid V_k^T\right)^T$

but $mem\left(X_2 \mid X_1\right) = \left(V_1^T \mid U_1^T \mid V_2^T \mid U_2^T \mid \cdots \mid V_j^T \mid U_j^T \mid \cdots \mid V_k^T \mid U_k^T\right)^T$

If we adopt Kronecker products, then $mem\left(X_2 \mid X_1\right) = \left\{ I_k \oplus \begin{pmatrix} & I_m \\ I_m & \end{pmatrix} \right\} \cdot mem\left(X\right)$

$$\left(\tilde{D}_{r,N_r} u\right)\left(\left(r,\theta_1\right),\cdots,\left(r,\theta_{2m}\right)\right) = \left\{ E_1 \oplus \begin{pmatrix} I_m & \\ & I_m \end{pmatrix} + E_2 \oplus \begin{pmatrix} & I_m \\ I_m & \end{pmatrix} \right\} mem\left(X\right)$$

Definition: Kronecker product is defined by $A \oplus B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{12}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \cdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}$

Non-active variable → active variable [9]

The same reason holds for second derivative operator

$$\tilde{D}^2_{r,N_r} = \left(\begin{array}{c|c} D_1 & D_2 \\ \hline D_3 & D_4 \end{array}\right) \xrightarrow[r<0]{\text{neglect equation on}} \tilde{D}^2_{r,N_r} = \begin{pmatrix} D_1 & D_2 \end{pmatrix} \longrightarrow \tilde{D}^2_{r,N_r} P^T = \begin{pmatrix} D_1 & \tilde{D}_2 \end{pmatrix}$$

then $$\left(\tilde{D}^2_{r,N_r} u\right)\left((r,\theta_1),\cdots,(r,\theta_{2m})\right) = \left\{ D_1 \oplus \begin{pmatrix} I_m & \\ & I_m \end{pmatrix} + D_2 \oplus \begin{pmatrix} & I_m \\ I_m & \end{pmatrix} \right\} mem(X)$$

$$\left[\frac{1}{r}\left(\tilde{D}_{r,N_r} u\right)\right]\left(r_i,\theta_j\right) = \frac{1}{r_i} \times \left(i - \text{row of matrix } \tilde{D}_{r,N_r} P^T\right)$$

collect equations on each $r_i : 1 \le i \le k$

we have $$\frac{1}{r}\frac{\partial}{\partial r} \to R\left\{ E_1 \oplus \begin{pmatrix} I_m & \\ & I_m \end{pmatrix} + E_2 \oplus \begin{pmatrix} & I_m \\ I_m & \end{pmatrix} \right\}$$

where $$R = \begin{pmatrix} 1/r_1 & & & \\ & 1/r_2 & & \\ & & \ddots & \\ & & & 1/r_k \end{pmatrix} = diag\left(\frac{1}{r_1},\frac{1}{r_2},\cdots,\frac{1}{r_k}\right)$$

Non-active variable → active variable [10]

We write second derivative operator on $\theta - \text{direction}$ as

$$D^2_{\theta,N_\theta} \equiv \left(S_N^{(2)}\left(x_k - x_j\right)\right) \quad \text{where} \quad S_N^{(2)}\left(x_j\right) = \begin{cases} -\frac{1}{6} - \frac{\pi^2}{3h^2} & j = 0\left(\operatorname{mod} N\right) \\ -\frac{(-1)^j}{2\sin^2\left(jh/2\right)}, & j \neq 0\left(\operatorname{mod} N\right) \end{cases}$$

for $1 \leq i \leq k,\ 1 \leq j \leq 2m \quad \left(r_i > 0\right)$

$$\left(D^2_{\theta,N_\theta} u\right)\left(r_i, \theta_j\right) = j - \text{row of } \left\{ D^2_{\theta,N_\theta} \begin{pmatrix} u_{i,1} \\ u_{i,2} \\ \vdots \\ u_{i,2m} \end{pmatrix} \right\} = j - \text{row of } \left\{ D^2_{\theta,N_\theta} \begin{pmatrix} U_i \\ V_i \end{pmatrix} \right\}$$

so $\left(D^2_{\theta,N_\theta} u\right)\left(r_i, \theta\right) = D^2_{\theta,N_\theta} \begin{pmatrix} u_{i,1} \\ u_{i,2} \\ \vdots \\ u_{i,2m} \end{pmatrix} = D^2_{\theta,N_\theta} \begin{pmatrix} U_i \\ V_i \end{pmatrix}$ implies $\left(\frac{1}{r^2} D^2_{\theta,N_\theta} u\right)\left(r_i, \theta\right) = \frac{1}{r_i^2} D^2_{\theta,N_\theta} \begin{pmatrix} U_i \\ V_i \end{pmatrix}$

Non-active variable → active variable [11]

$$\left(\frac{1}{r^2}D^2_{\theta,N_\theta}u\right)\begin{pmatrix}(r_1,\theta)\\(r_2,\theta)\\\vdots\\(r_k,\theta)\end{pmatrix}=\begin{pmatrix}\frac{1}{r_1^2}D^2_{\theta,N_\theta}\begin{pmatrix}U_1\\V_1\end{pmatrix}\\\frac{1}{r_2^2}D^2_{\theta,N_\theta}\begin{pmatrix}U_2\\V_2\end{pmatrix}\\\vdots\\\frac{1}{r_k^2}D^2_{\theta,N_\theta}\begin{pmatrix}U_k\\V_k\end{pmatrix}\end{pmatrix}=\begin{pmatrix}\frac{1}{r_1^2}D^2_{\theta,N_\theta} & & & \\ & \frac{1}{r_2^2}D^2_{\theta,N_\theta} & & \\ & & \ddots & \\ & & & \frac{1}{r_k^2}D^2_{\theta,N_\theta}\end{pmatrix}\begin{pmatrix}\begin{pmatrix}U_1\\V_1\end{pmatrix}\\\begin{pmatrix}U_2\\V_2\end{pmatrix}\\\vdots\\\begin{pmatrix}U_k\\V_k\end{pmatrix}\end{pmatrix}$$

If we adopt Kronecker products, then

$$\left(\frac{1}{r^2}D^2_{\theta,N_\theta}u\right)\begin{pmatrix}(r_1,\theta)\\(r_2,\theta)\\\vdots\\(r_k,\theta)\end{pmatrix}=\left(R^2\oplus D^2_{\theta,N_\theta}\right)mem\left(X\right)$$

Non-active variable → active variable [12]

summary

① $$\left(\tilde{D}^2_{r,N_r} u\right)\left((r,\theta_1),\cdots,(r,\theta_{2m})\right) = \left\{ D_1 \oplus \begin{pmatrix} I_m & \\ & I_m \end{pmatrix} + D_2 \oplus \begin{pmatrix} & I_m \\ I_m & \end{pmatrix} \right\} mem(X)$$

② $$\frac{1}{r}\frac{\partial}{\partial r} \to \left(\frac{1}{r} D_{r,N_r} u\right)\left((r,\theta_1),\cdots,(r,\theta_{2m})\right) = R\left\{ E_1 \oplus \begin{pmatrix} I_m & \\ & I_m \end{pmatrix} + E_2 \oplus \begin{pmatrix} & I_m \\ I_m & \end{pmatrix} \right\} mem(X)$$

③ $$\left(\frac{1}{r^2} D^2_{\theta,N_\theta} u\right)\begin{pmatrix}(r_1,\theta)\\(r_2,\theta)\\\vdots\\(r_k,\theta)\end{pmatrix} = \left(R^2 \oplus D^2_{\theta,N_\theta}\right) mem(X)$$

Note that

$$\left((r,\theta_1) \quad (r,\theta_2) \quad \cdots \quad (r,\theta_{2m})\right) = \begin{pmatrix}(r_1,\theta)\\(r_2,\theta)\\\vdots\\(r_k,\theta)\end{pmatrix} = \begin{pmatrix}(r_1,\theta_1) & (r_1,\theta_2) & \cdots & (r_1,\theta_{2m})\\(r_2,\theta_1) & (r_2,\theta_2) & \cdots & (r_2,\theta_{2m})\\\vdots & \cdots & \ddots & \vdots\\(r_k,\theta_1) & (r_k,\theta_2) & \cdots & (r_k,\theta_{2m})\end{pmatrix}$$

so that all three system of equations are of the same order.

Non-active variable → active variable [13]

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=-\lambda^2 u \quad \text{, on } r\in[-1,1] \text{ and } \theta\in[0,2\pi]$$

$$\Big\downarrow \quad \text{Discretization on } mem(X)$$

$$L\cdot mem(X)=-\lambda^2 mem(X)$$

then

$$L=\left\{D_1\oplus\begin{pmatrix} I_m & \\ & I_m \end{pmatrix}+D_2\oplus\begin{pmatrix} & I_m \\ I_m & \end{pmatrix}\right\}+R\left\{E_1\oplus\begin{pmatrix} I_m & \\ & I_m \end{pmatrix}+E_2\oplus\begin{pmatrix} & I_m \\ I_m & \end{pmatrix}\right\}$$

$$+\left(R^2\oplus D^2_{\theta,N_\theta}\right)$$

$$=\left(D_1+RE_1\right)\oplus\begin{pmatrix} I_m & \\ & I_m \end{pmatrix}+\left(D_2+RE_2\right)\oplus\begin{pmatrix} & I_m \\ I_m & \end{pmatrix}+\left(R^2\oplus D^2_{\theta,N_\theta}\right)$$

where

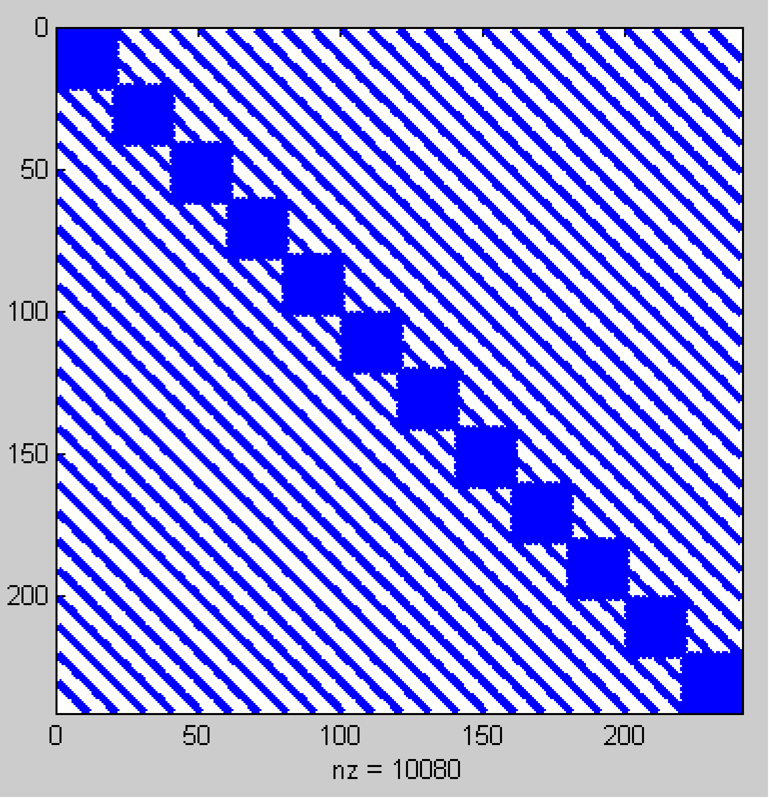
$$R=\begin{pmatrix} 1/r_1 & & & \\ & 1/r_2 & & \\ & & \ddots & \\ & & & 1/r_k \end{pmatrix}=diag\left(\frac{1}{r_1},\frac{1}{r_2},\cdots,\frac{1}{r_k}\right)$$

Example: program 28

$\Delta u = -\lambda^2 u$ with boundary condition $u(r=1,\theta)=0$

$N_r = 25$ is odd and $N_\theta = 20$ is even, let eigen-pair be (λ_k, V_k)

sparse structure of $L = (D_1 + RE_1) \oplus \begin{pmatrix} I_m & \\ & I_m \end{pmatrix} + (D_2 + RE_2) \oplus \begin{pmatrix} & I_m \\ I_m & \end{pmatrix} + (R^2 \oplus D^2_{\theta,N_\theta})$



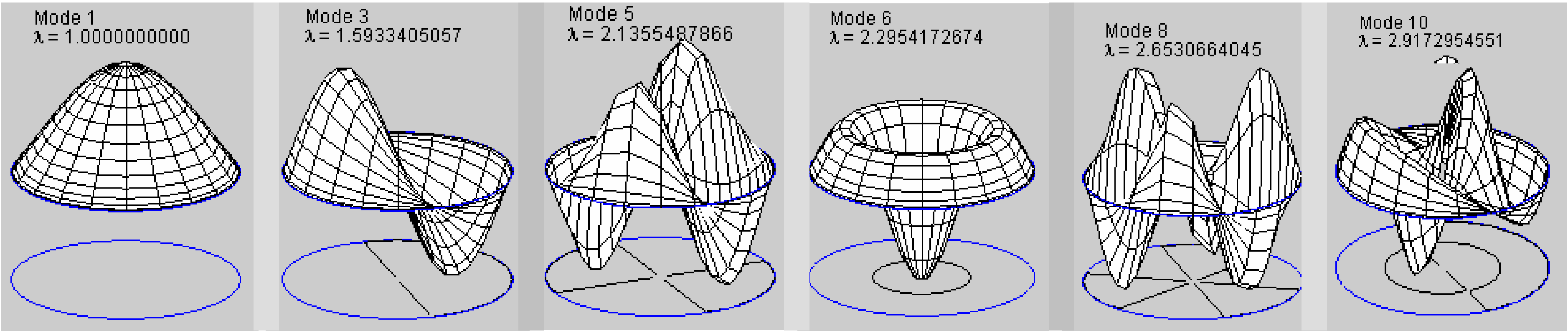
Dimension : $\frac{N_r - 1}{2} N_\theta = 12 \times 20 = 240$

Example: program 28 (mash plot of eigenvector)

① Eigenvalue is sorted, monotone increasing and normalized to first eigenvalue

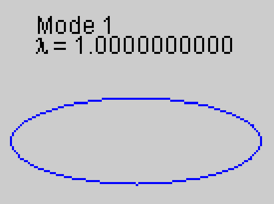
$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$$

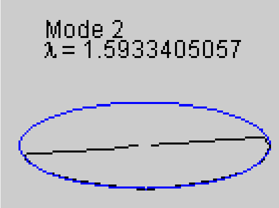
② Eigenvector is normalized by supremum norm, $V_k \leftarrow \dfrac{V_k}{\left\|V_k\right\|_\infty}$

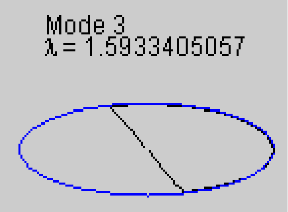


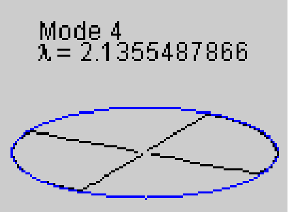
$\lambda_1 = 5.7832$ $\lambda_2 = \lambda_3 = 14.682$ $\lambda_4 = \lambda_5 = 26.3746$ $\lambda_6 = 30.4713$ $\lambda_7 = \lambda_8 = 40.7065$ $\lambda_9 = \lambda_{10} = 42.2185$

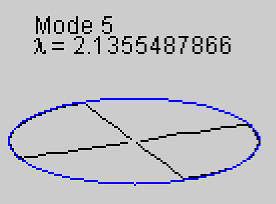
Example: program 28 (nodal set)

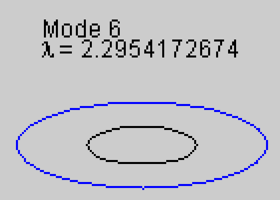

Mode 1
λ = 1.0000000000

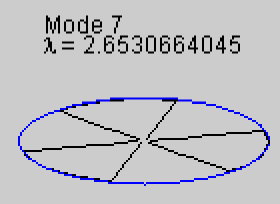

Mode 2
λ = 1.5933405057

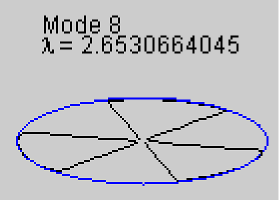

Mode 3
λ = 1.5933405057

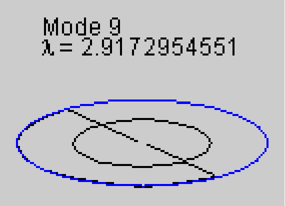

Mode 4
λ = 2.1355487866

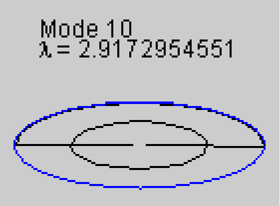

Mode 5
λ = 2.1355487866

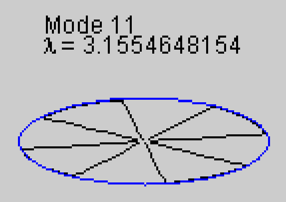

Mode 6
λ = 2.2954172674

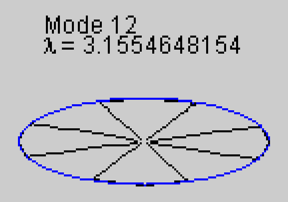

Mode 7
λ = 2.6530664045

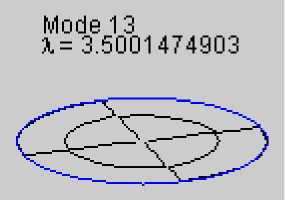

Mode 8
λ = 2.6530664045

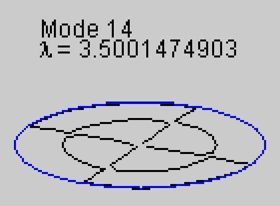

Mode 9
λ = 2.9172954551

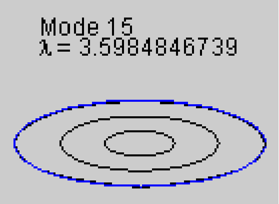

Mode 10
λ = 2.9172954551

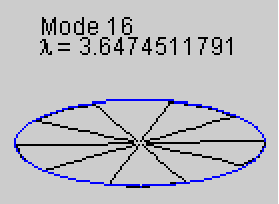

Mode 11
λ = 3.1554648154

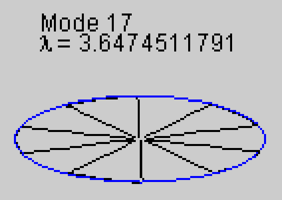

Mode 12
λ = 3.1554648154

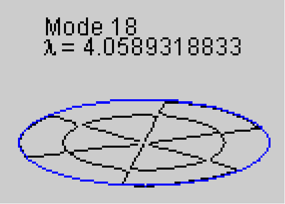

Mode 13
λ = 3.5001474903

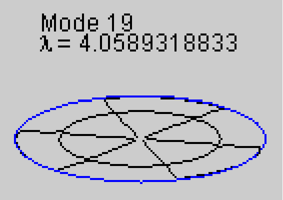

Mode 14
λ = 3.5001474903

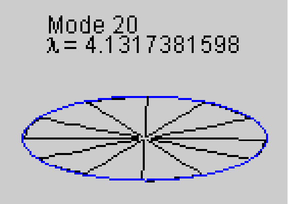

Mode 15
λ = 3.5984846739

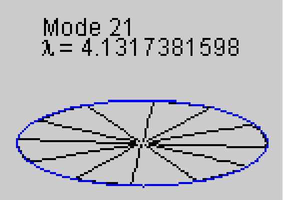

Mode 16
λ = 3.6474511791

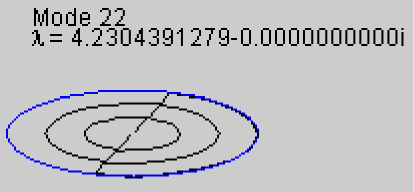

Mode 17
λ = 3.6474511791

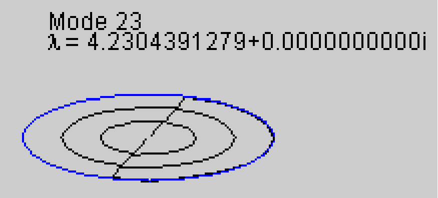

Mode 18
λ = 4.0589318833

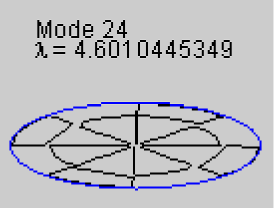

Mode 19
λ = 4.0589318833


Mode 20
λ = 4.1317381598


Mode 21
λ = 4.1317381598


Mode 22
λ = 4.2304391279-0.0000000000i


Mode 23
λ = 4.2304391279+0.0000000000i


Mode 24
λ = 4.6010445349

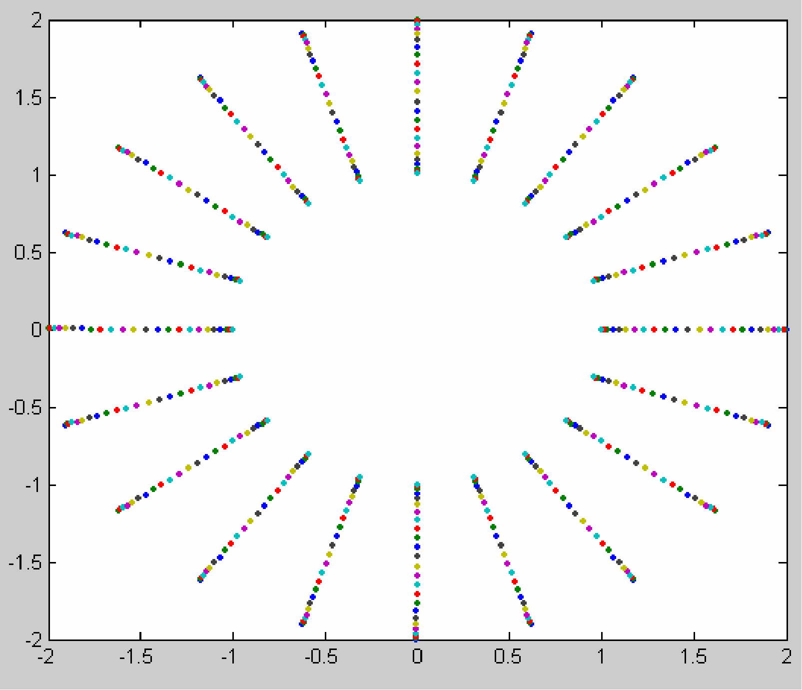
Exercise 1

$\Delta u = -\lambda^2 u$ on annulus $\{1 < r < 2\}$ with boundary condition $u(r = 1, 2) = 0$

$N_r = 25$ is odd and $N_\theta = 20$ is even, let eigen-pair be (λ_k, V_k)

Chebyshev node on $x \in [-1, 1] \xrightarrow{r = 1 + \frac{x+1}{2}}$ Chebyshev node on $\{1 < r < 2\}$

$$L = \left(\tilde{D}^2_{r,N_r} + R\tilde{D}_{r,N_r}\right) \oplus I_{N_\theta} + \left(R^2 \oplus D^2_{\theta,N_\theta}\right)$$

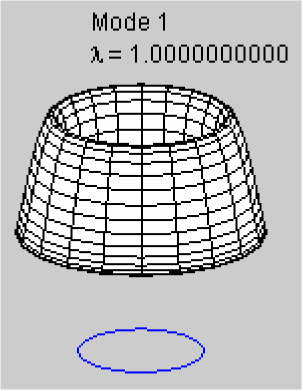


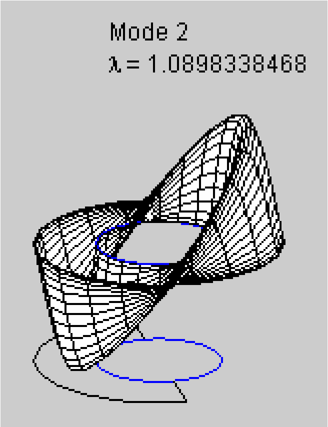
Exercise 1 (mash plot of eigenvector)

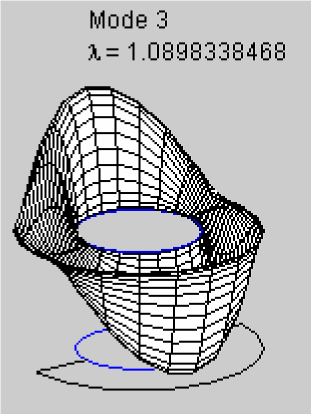
① Eigenvalue is sorted, monotone increasing and normalized to first eigenvalue

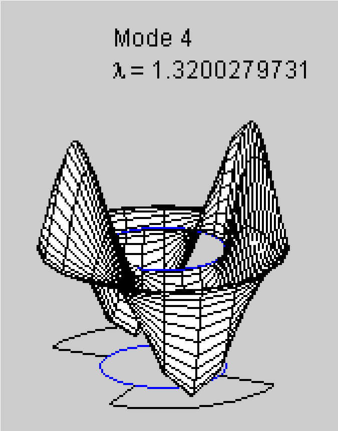
$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$$

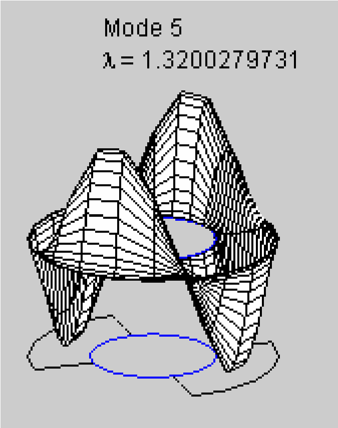
② Eigenvector is normalized by supremum norm, $V_k \leftarrow \dfrac{V_k}{\|V_k\|_\infty}$

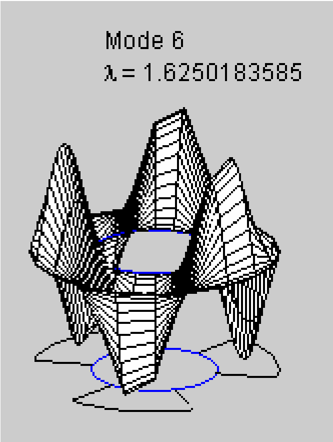

Mode 1
λ = 1.0000000000

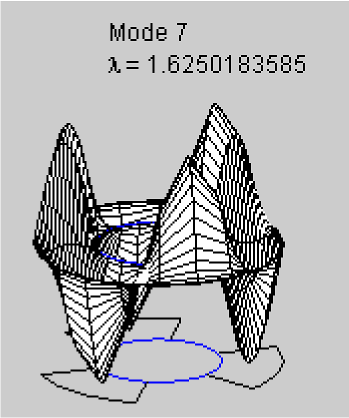

Mode 2
λ = 1.0898338468

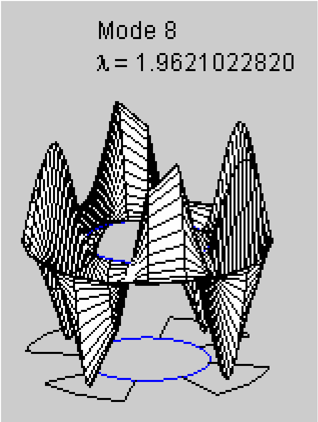

Mode 3
λ = 1.0898338468

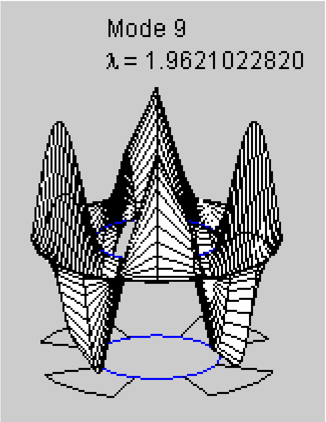

Mode 4
λ = 1.3200279731

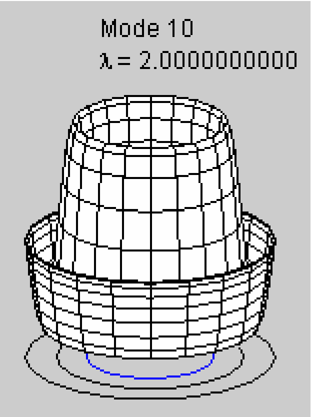

Mode 5
λ = 1.3200279731


Mode 6
λ = 1.6250183585


Mode 7
λ = 1.6250183585


Mode 8
λ = 1.9621022820


Mode 9
λ = 1.9621022820


Mode 10
λ = 2.0000000000