Chapter 1 Spectral method in Matlab

Centered second-order and 4-th order finite difference on uniform grid

(Eq. 1) $\frac{du}{dx}\left(x_j\right)=\frac{u_{j+1}-u_{j-1}}{2h}-\frac{h^2}{3!}u^{(3)}\left(c\right)$

(Eq. 2) $\frac{du}{dx}\left(x_j\right)=\frac{-u_{j+2}+8u_{j+1}-8u_{j-1}+u_{j-2}}{12h}+\frac{h^4}{30}u^{(5)}\left(c\right)$

We can use another representation to derive this formula, consider polynomial interpolation $p_j\left(x\right)$ with degree ≤ 2 for smooth u over $\left[x_{j-1},x_{j+1}\right]$.

$$p_j\left(x\right)=u_{j-1}a_{-1}\left(x\right)+u_ja_0\left(x\right)+u_{j+1}a_1\left(x\right)$$

satisfying $p_j\left(x_{j-1}\right)=u_{j-1}$, $p_j\left(x_j\right)=u_j$ and $p_j\left(x_{j+1}\right)=u_{j+1}$. In other words,

$a_{-1}\left(x\right),a_0\left(x\right),a_1\left(x\right)$ are Lagrange polynomial satisfying

(Eq. 3) $\begin{cases} a_{-1}\left(x_{j-1}\right)=1 \\ a_{-1}\left(x_j\right)=0 \\ a_{-1}\left(x_{j+1}\right)=0 \end{cases}$, $\begin{cases} a_0\left(x_{j-1}\right)=0 \\ a_0\left(x_j\right)=1 \\ a_0\left(x_{j+1}\right)=0 \end{cases}$ and $\begin{cases} a_1\left(x_{j-1}\right)=0 \\ a_1\left(x_j\right)=0 \\ a_1\left(x_{j+1}\right)=1 \end{cases}$

we know

$$a_{-1}\left(x\right)=\frac{\left(x-x_j\right)\left(x-x_{j+1}\right)}{\left(x_{j-1}-x_j\right)\left(x_{j-1}-x_{j+1}\right)}=\frac{\left(x-x_j\right)\left(x-x_{j+1}\right)}{2h^2},\quad \dot{a}_{-1}\left(x_j\right)=\frac{-1}{2h}$$

$$a_0\left(x\right)=\frac{\left(x-x_{j-1}\right)\left(x-x_{j+1}\right)}{\left(x_j-x_{j-1}\right)\left(x_j-x_{j+1}\right)}=\frac{\left(x-x_{j-1}\right)\left(x-x_{j+1}\right)}{-h^2},\quad \dot{a}_0\left(x_j\right)=0$$

$$a_1\left(x\right)=\frac{\left(x-x_{j-1}\right)\left(x-x_j\right)}{\left(x_{j+1}-x_{j-1}\right)\left(x_{j+1}-x_j\right)}=\frac{\left(x-x_{j-1}\right)\left(x-x_j\right)}{2h^2},\quad \dot{a}_1\left(x_j\right)=\frac{1}{2h}$$

$$\frac{d}{dx}p_j\left(x\right)|_{x=x_j}=u_{j-1}\dot{a}_{-1}\left(x_j\right)+u_j\dot{a}_0\left(x_j\right)+u_{j+1}\dot{a}_1\left(x_j\right)=\frac{u_{j+1}-u_{j-1}}{2h}$$

If we write $w_j=\frac{u_{j+1}-u_{j-1}}{2h}$ or $w_j=\frac{-u_{j+2}+8u_{j+1}-8u_{j-1}+u_{j-2}}{12h}$, then general matrix form can be represented as $w=Au$, see Figure 1 and Figure 2. In fact matrix A is skew symmetry, say $A=-A^T$, this can be obtained from antiHermitian of operator $\frac{d}{dx}$ under periodic boundary condition through integration by parts.

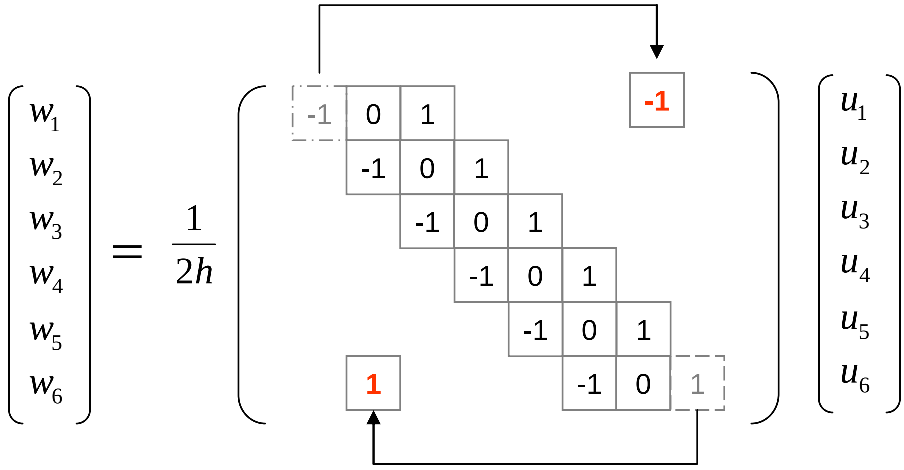

Figure 1: matrix representation of $w_j = \frac{u_{j+1} - u_{j-1}}{2h}$ with periodic boundary condition

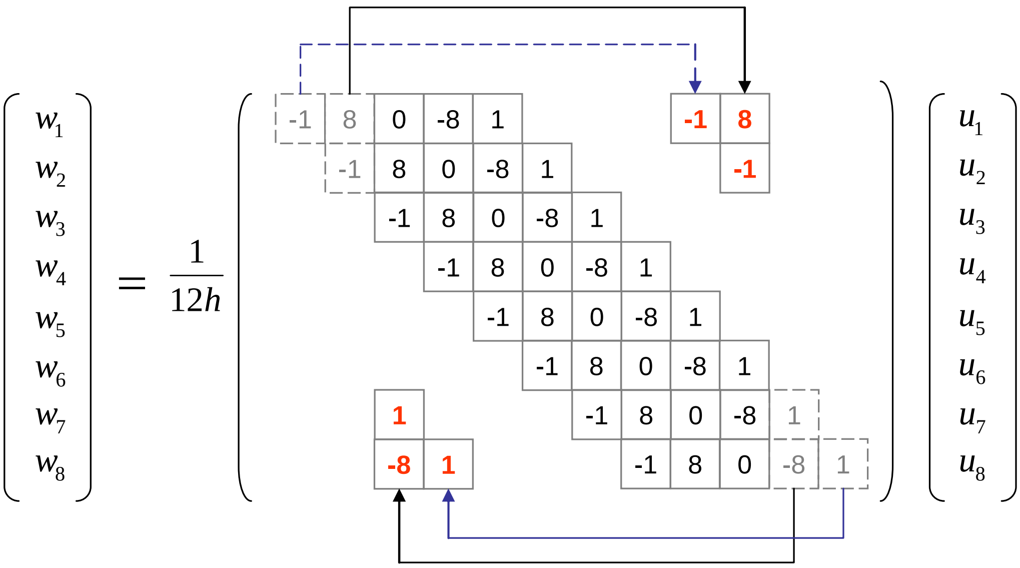

Figure 2: $w_j = \frac{-u_{j+2} + 8u_{j+1} - 8u_{j-1} + u_{j-2}}{12h}$ with periodic boundary condition.

From $w_j = \frac{u_{j+1} - u_{j-1}}{2h}$, we have $A(j, j+1) = \frac{1}{2h}$ and $A(j, j-1) = \frac{-1}{2h}$. In fact if we want to use Matlab as platform, what we only do is to construct triangle part R of A, then $A = R - R^T$. Moreover we can use ”sparse matrix representation” in Matlab to construct matrix R. (how to construct R: please see source code p1.m in **Example 1**)

```
S = SPARSE(i,j,s,m,n,nzmax) uses the rows of [i,j,s] to generate an
m-by-n sparse matrix with space allocated for nzmax nonzeros.  The
two integer index vectors, i and j, and the real or complex entries
vector, s, all have the same length, nnz, which is the number of
nonzeros in the resulting sparse matrix S .  Any elements of s
which have duplicate values of i and j are added together.
```

Figure 3: document of sparse matrix representation in Matlab.

Example 1: try $u(x)=\exp(\sin(x))$ over $[-\pi,\pi]$ by using 4-th order finite difference scheme. Matlab script: F:\course\2008spring\spectral_method\matlab\p1.m.
In this example we use $N=8,16,32,\cdots,4048$, test convergence rate of $\|D_h u-u'\|_\infty$, result is shown in Figure 4.

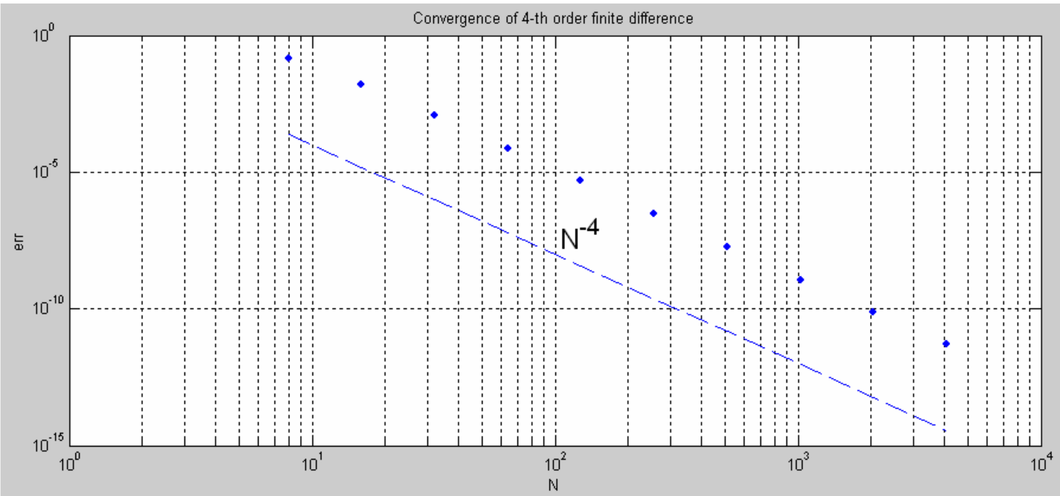

Figure 4: log-log plot for $\|D_h u-u'\|_\infty$, note that it is clear $O(h^4)$

Example 2: try $u(x)=\exp(\sin(x))$ over $[-\pi,\pi]$ by using spectral method.

We plot error for $\|D_h u-u'\|_\infty$ and $\|D_h u-u'\|_2$.

Question 1: dots in Figure 5 and Figure 6 lies in pair, means that error of $n=8,10$, $n=12,14$ … are almost the same, why? (Note that this pair phenomenon occurs even we use $\|D_h u-u'\|_2$).

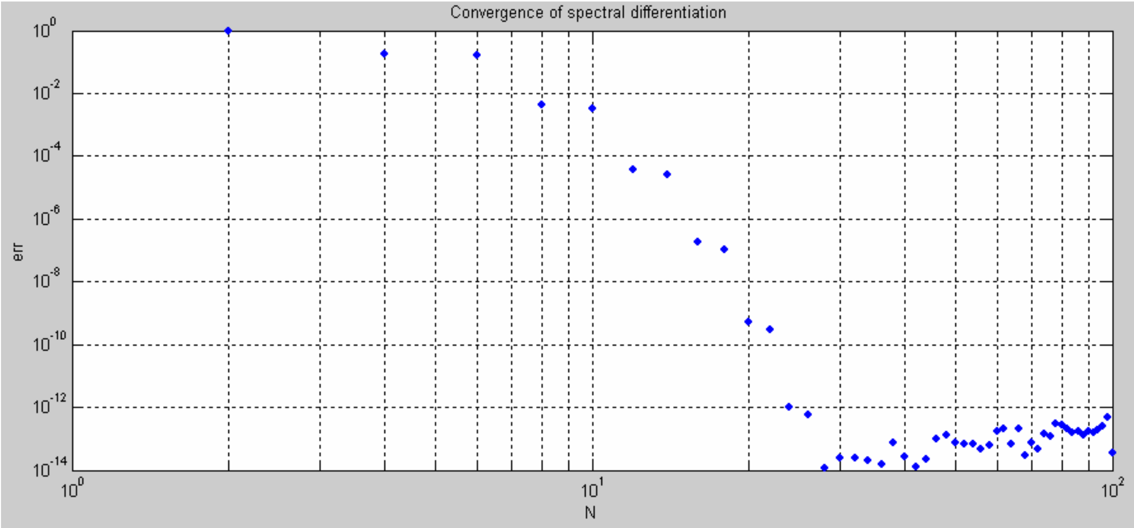

Figure 5: "spectral accuracy" of the spectral method, it achieve machine accuracy. Sup-norm

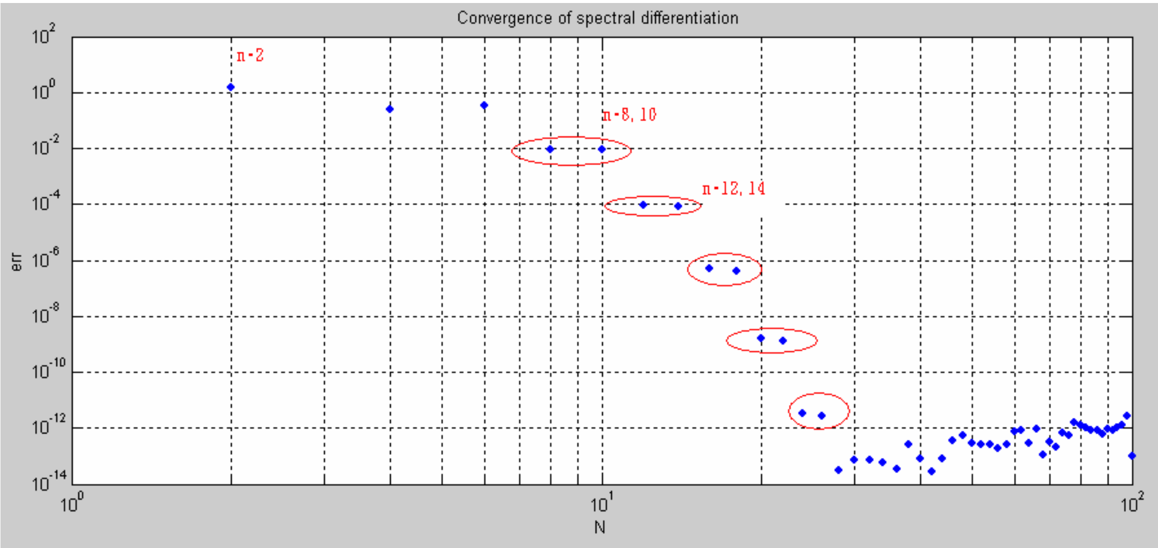

Figure 6: "spectral accuracy" of the spectral method, it achieve machine accuracy. 2-norm

Table 1: we list the value of pair ($L^2 - err$)

	$n=4,6$	$n=8,10$	$n=12,14$
$L^2 - err$	0.24777190419502 0.33078448076569	0.00871672632422 0.00864342870238	0.94197188122419E-4 0.84097522689018E-4

	$n=16,18$	$n=20,22$	$n=24,26$
$L^2 - err$	0.50008967910710E-6 0.42247516569957E-6	0.15812146492563E-8 0.12910423145203E-8	0.33254794093191E-11 0.26512109052603E-11

Table 2: we list the value of pair ($L^{\infty} - err$)

	$n = 4, 6$	$n = 8, 10$	$n = 12, 14$
$L^{\infty} - err$	0.17520119364380 0.16307358056713	0.00431791109859 0.00316947217836	0.38249095565268E-4 0.25253568612049E-4

	$n = 16, 18$	$n = 20, 22$	$n = 24, 26$
$L^{\infty} - err$	0.17618931913432E-6 0.10953028789507E-6	0.49879544938847E-9 0.29827984526776E-9	0.97200025805932E-12 0.59258153939368E-12

Table 3: we list the value of pair ($L^{\infty} - err$) for arprec 128 digits

$n = 28, 30$	1.32624221983015620680098434362948e-15 7.56612859983412967818296086049246e-16
$n = 32, 34$	1.39095872807168126193852953110517e-18 7.81451569905447385343012380339833e-19
$n = 36, 38$	1.14244909925972473070099230882981e-21 6.34086538799698880852642854985210e-22
$n = 40, 42$	7.54806757417172375166373940244838e-25 4.14826359601086676154821757222175e-25
$n = 44, 46$	4.09864701833461299055736676547760e-28 2.23423056735975763167625559605640e-28
$n = 48, 50$	1.86164252456817011022120353206289e-31 1.00786673375963438339376505661880e-31
$n = 52, 54$	7.17767982003012719000648270914229e-35 3.86321334725399173651104700894601e-35
$n = 56, 58$	2.37854454643974045195143562239834e-38 1.27374376983182194482918312198058e-38
$n = 60, 62$	6.84730032444346812021198287838704e-42 3.65071328394049314224342346654448e-42
$n = 64, 66$	1.72836808647069444865147310660496e-45 9.17936617551766050908476538695563e-46
$n = 68, 70$	3.85647559072931642402400673411656e-49 2.04115689771413325003291669779317e-49
$n = 72, 74$	7.66124774385363632507038345929950e-53 4.04254667547222108076077515483580e-53
$n = 76, 78$	1.36375587422112860698500475832273e-56 7.17625294246753762277314925189095e-57
$n = 80, 82$	2.18769132299786362708446911624823e-60 1.14833541792638712588483843196297e-60
$n = 84, 86$	3.17896250168716103543670487976991e-64 1.66490575382791587484894062926081e-64
$n = 88, 90$	4.20398304881730575304253164828023e-68 2.19721922822932053740570006526006e-68
$n = 92, 94$	5.08108143833400125413169287191248e-72 2.65065013672443604039455837009901e-72
$n = 96, 98$	5.63449984243390032676388653577986e-76 2.93428197672848600468452855610906e-76

Table 4: we compute Fourier component $\hat{v}_k = h\sum_{j=1}^{N} e^{-ikx_j} v_j$ for $k = 0,1,2,\cdots,m$, $N = 2m$ since $\hat{v}_{-k} = \hat{v}_k^* = \hat{v}_{N-k}$. Since $\hat{v}_k \in C$, we represent it as two value, real part (top value) and imaginary part (bottom value)

	$n=8$	$n=10$	$n=12$	$n=14$
$\hat{V}_0$	7.95492777270178 0	7.95492651755339 0	7.95492652101937 0	7.95492652101284 0
$\hat{V}_1$	0 -3.55100946128190	0 -3.55099934367285	0 -3.55099937850296	0 -3.55099937842424
$\hat{V}_2$	-0.85306906632123 0	-0.85292713831639 0	-0.85292776589386 0	-0.85292776416086 0
$\hat{V}_3$	0 0.14099397515943	0 0.13927827358326	0 0.13928835644092	0 0.13928832168940
$\hat{V}_4$	0.03439566712026 0	0.01705653312949 0	0.01719845940133 0	0.01719783182714 0
$\hat{V}_5$		eps eps	0 -0.00171570149744	0 -0.00170561863991
$\hat{V}_6$			-0.00028260085474 0	-0.00014067458291 0
$\hat{V}_7$				eps eps

	$n=16$	$n=18$	$n=20$	$n=22$
$\hat{V}_0$	7.95492652101284 0	7.95492652101284 0	7.95492652101284	7.95492652101285
$\hat{V}_1$	0 -3.55099937842436	0 -3.55099937842436	0 -3.55099937842436	0 - 3.55099937842436
$\hat{V}_2$	-0.85292776416412 0	-0.85292776416412 0	-0.85292776416412 0	-0.85292776416412 0
$\hat{V}_3$	0 0.13928832176800	0 0.13928832176787	0 0.13928832176788	0 0.13928832176788
$\hat{V}_4$	0.01719783356013 0	0.01719783355686 0	0.01719783355687 0	0.01719783355687 0
$\hat{V}_5$	0 -0.00170565339142	0 -0.00170565331282	0 -0.00170565331295	0 - 0.00170565331295
$\hat{V}_6$	-0.00014130215710 0	-0.00014130042411 0	-0.00014130042738 0	-0.00014130042737 0

$\hat{V}_7$	0 **0.00001008**285753	0 **0.00001004**810602	0 **0.000010048184**62	0 **0.000010048184**49
$\hat{V}_8$	0.00000125168893 0	0.00000062411474 0	**0.00000062584**773 0	**0.00000062584**446 0
$\hat{V}_9$		Eps Eps	0 **- 0.00000003**475152	0 **-0.00000003**467292
$\hat{V}_{10}$			-0.00000000345946 0	-0.00000000172646 0
$\hat{V}_{11}$				Eps Eps

In order to avoid confusion in discussion, we define $\hat{V}_k^{(n)}$ as Fourier component $\hat{V}_k$ for degree n. Then we have some observations

(O1) $\mathrm{Re}\hat{V}_{2k+1}^{(n)} = 0$ and $\mathrm{Im}\hat{V}_{2k}^{(n)} = 0$ up to machine accuracy, this matches continuous counterpart, see **Lemma 2**. Note that in the proof of **Lemma 2**, we use $\sin(\pi - x) = \sin(x)$ and claim $\int_{\pi/2}^{\pi} e^{\sin x}\sin(2kx)\,dx = -\int_0^{\pi/2} e^{\sin x}\sin(2kx)\,dx$ and $\int_{\pi/2}^{\pi} e^{\sin x}\cos((2k+1)x)\,dx = -\int_0^{\pi/2} e^{\sin x}\cos((2k+1)x)\,dx$.

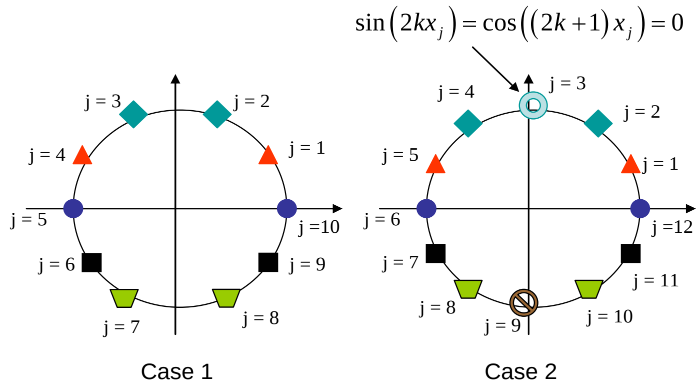

Figure 7: There are two cases in discrete version of $\mathrm{Re}\hat{V}_{2k+1}^{(n)} = 0$ and $\mathrm{Im}\hat{V}_{2k}^{(n)} = 0$.

Since we use uniform grids and N is even, we have symmetry over half pane, we just need to take care one case that $x_j = \frac{\pi}{2}$ for some j (see Figure 7) and $x_j = 0, \pi$ in the discrete sum. However $\sin(2kx_j) = \sin(k\pi) = 0$ and $\cos((2k+1)x_j) = \cos\left(k\pi + \frac{\pi}{2}\right) = 0$, also $\sin(2k\cdot 0) = \sin(2k\pi) = 0$, $\cos((2k+1)\pi) = \cos(\pi) = -1$ (this means $e^{\sin x}\cos((2k+1)x)$ cancels each other at $x = 0, \pi$), hence function value at point $x_j = \frac{\pi}{2}, 0, \pi$ does not affect the summation, hence $\mathrm{Re}\hat{V}_{2k+1}^{(n)} = 0$ and $\mathrm{Im}\hat{V}_{2k}^{(n)} = 0$ hold in discrete sum.

(O2) $\hat{V}_5^{(10)} = \hat{V}_7^{(14)} = \hat{V}_9^{(18)} = \hat{V}_{11}^{(22)} = 0$ up to machine accuracy, in fact they are exact zero, we would show this and verify this with high precision package. Note this is key point for pair phenomenon, we would discuss this later. First let us show why $\hat{V}_5^{(10)} = 0$ but $\hat{V}_5^{(12)} \neq 0$ and $\hat{V}_6^{(12)} \neq 0$. First it is easy to show that $\hat{V}_5^{(10)} = h\sum_{j=1}^{10}(-1)^j v_j$, where $v_j = \exp(\sin(hj))$. We would demo $\hat{V}_5^{(10)} = 0$ is due to (1) $(-1)^j = \exp(i5x_j)$ and (2) symmetry of $v_j = \exp(\sin(hj))$.

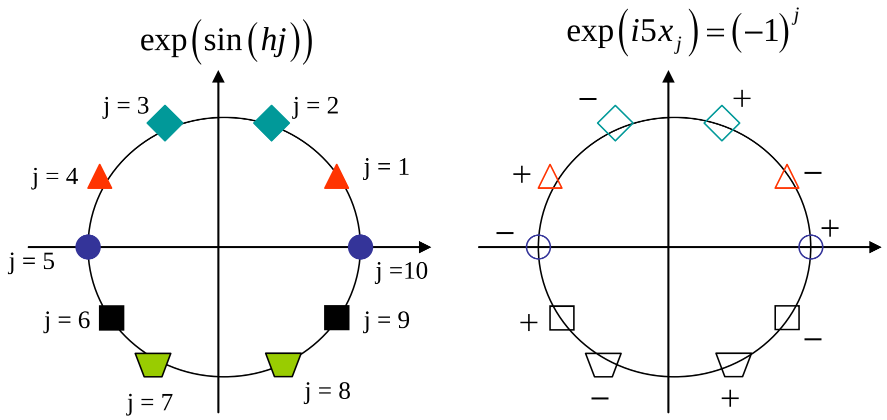

Figure 8: distribution of $v_j = \exp(\sin(hj))$ (left panel) and $(-1)^j = \exp(i5x_j)$ (right panel).

In left panel of Figure 8, we plot distribution of $v_j = \exp(\sin(hj))$, please note the locations with same color and shape have the same value, that is, $v_1 = v_4$, $v_2 = v_3$, $v_5 = v_{10}$, $v_6 = v_9$ and $v_7 = v_8$, this is consequence of $\sin(\pi - x) = \sin(x)$ and we use uniform grid.

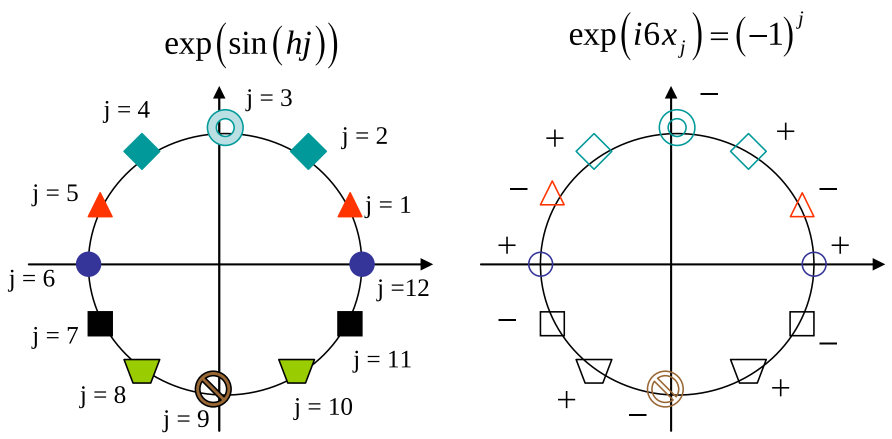

Figure 9: no cancellation for $\hat{V}_6^{(12)}$

In right panel of Figure 8, we plot distribution of $(-1)^j = \exp(i5x_j)$. From the graph, one can show cancellation in summation $\hat{V}_5^{(10)} = h\sum_{j=1}^{10}(-1)^j v_j$, so $\hat{V}_5^{(10)} = 0$.

However when $n=12$, $(-1)^j \neq \exp(i5x_j) = \exp\left(i\frac{5}{12}j\right)$, hence no cancellation occurs, that is $\hat{V}_5^{(12)} \neq 0$. Moreover $\hat{V}_6^{(12)} \neq 0$ is result of no cancellation, see Figure 9.

similarly, $\hat{V}_7^{(14)} = \hat{V}_9^{(18)} = \hat{V}_{11}^{(22)} = 0$.

Remark 1: From above argument, (O2) is valid when $V(x) = V(\pi - x)$, for example, $V = V(\sin(x))$ on $[0, 2\pi]$. However $\cos(x)$ is not permitted though $\cos(x) = \sqrt{1-\sin^2 x}$, why? Since in general $\cos(x) = \pm\sqrt{1-\sin^2 x}$, the sign depends on branch, that is $\cos(\pi - x) \neq \cos(x)$.

(O3) $\hat{V}_4^{(8)} \approx 2\hat{V}_4^{(10)}$, $\hat{V}_6^{(12)} \approx 2\hat{V}_6^{(14)}$, $\hat{V}_8^{(16)} \approx 2\hat{V}_8^{(18)}$ and $\hat{V}_{10}^{(20)} \approx 2\hat{V}_{10}^{(22)}$, although we cannot interpret this phenomenon, but such fact is also a key to pair phenomenon. Now we explain this. First, in chapter 3 (see chap3.doc), we split $\hat{V}_m^{(N)}$ into 2 parts, (say $\hat{V}_{-m}^{(N)} = \hat{V}_m^{(N)} \leftarrow \frac{1}{2}\hat{V}_m^{(N)}$), for symmetry inteepolant $p(x)$ defined by

(Eq. 4) $$p_N(x) = \frac{1}{2\pi} \mathrm{P} \sum_{k=-m}^{m} e^{ikx} \hat{V}_k^{(N)} \quad \text{for} \quad x \in [0, 2\pi], \quad N = 2m$$

where P (principal value) indicates that the terms $k = \pm m$ are multiplied by $\frac{1}{2}$.

Hence $\frac{1}{2}\hat{V}_4^{(8)} = 0.01719783356013$ and then $\left|\frac{1}{2}\hat{V}_4^{(8)} - \hat{V}_4^{(10)}\right| = 1.413004306399998\text{e-}004$.

Now we estimate the difference between $p_8(x)$ and $p_{10}(x)$.

$$p_{10}(x) - p_8(x) = \frac{1}{2\pi}\sum_{k=-3}^{3} e^{ikx}\left(\hat{V}_k^{(10)} - \hat{V}_k^{(8)}\right) + \frac{1}{2\pi}\sum_{k=\pm 4} e^{ikx}\left(\hat{V}_k^{(10)} - \frac{1}{2}\hat{V}_k^{(8)}\right) + \frac{1}{2\pi}\sum_{k=\pm 5} e^{ikx}\hat{V}_k^{(10)}$$

However we have shown $\hat{V}_5^{(10)} = 0$, so

$$\left|p_{10}(x) - p_8(x)\right| \leq \frac{1}{2\pi}\sum_{k=-3}^{3}\left|\hat{V}_k^{(10)} - \hat{V}_k^{(8)}\right| + \frac{1}{2\pi}\sum_{k=\pm 4}\left|\hat{V}_k^{(10)} - \frac{1}{2}\hat{V}_k^{(8)}\right|$$

$\left|\hat{V}_0^{(10)} - \hat{V}_0^{(8)}\right| = 1.255148390555405\text{e-}006$

$\left|\hat{V}_1^{(10)} - \hat{V}_1^{(8)}\right| = 1.011760904967574\text{e-}005$

$\left|\hat{V}_2^{(10)} - \hat{V}_2^{(8)}\right| = 1.419280048400307\text{e-}004$

$\left|\hat{V}_3^{(10)} - \hat{V}_3^{(8)}\right| = 0.00171570157617$

$\left|\frac{1}{2}\hat{V}_4^{(8)} - \hat{V}_4^{(10)}\right| = 1.413004306399998\text{e-}004$

Hence $\left|p_{10}(x) - p_8(x)\right| \leq 6.396994825534096\text{e-}004$ for any $x \in [0, 2\pi]$.

Further $$p'_{10}(x) - p'_8(x) = \frac{1}{2\pi}\sum_{k=-3}^{3} ike^{ikx}\left(\hat{V}_k^{(10)} - \hat{V}_k^{(8)}\right) + \frac{1}{2\pi}\sum_{k=\pm 4} ike^{ikx}\left(\hat{V}_k^{(10)} - \frac{1}{2}\hat{V}_k^{(8)}\right)$$

$\left|p'_{10}(x) - p'_8(x)\right| \leq 0.00191185832541$ for any $x \in [0, 2\pi]$.

This number is about $\frac{1}{2}\|p_8' - V'\|_{I,\infty}$, hence pair phenomenon occurs at $n = 8,10$.

Table 5: we copy data from **Table 2** and add two new fields, one is $|err_n - err_{n+2}|$ and the other is $|p_n'(x) - p_{n+2}'(x)|$, $|err_n - err_{n+2}|$ is difference of two value in the field $L^\infty - err$, it measure derivation between pair, for example

$|err_8 - err_{10}| = |0.00431791109859 - 0.00316947217836| = 0.00114843892023$

$|p_n'(x) - p_{n+2}'(x)|$ measure derivation due to $\left|\hat{V}_k^{(n)} - \hat{V}_k^{(n+2)}\right|$.

	$n = 8,10$	$n = 12,14$
$L^\infty - err$	0.00431791109859 0.00316947217836	0.38249095565268E-4 0.25253568612049E-4
$\|err_n - err_{n+2}\|$	0.00114843892023	1.299552695321900e-5
$\|p_n'(x) - p_{n+2}'(x)\|$	0.00191185832541	1.807600691270616e-5

	$n = 16,18$	$n = 20,22$	$n = 24,26$
$L^\infty - err$	0.17618931913432E-6 0.10953028789507E-6	0.49879544938847E-9 0.29827984526776E-9	0.97200025805932E-12 0.59258153939368E-12
$\|err_n - err_{n+2}\|$	6.665903123925001e-8	2.00515604120710e-10	3.794187186656399e-13
$\|p_n'(x) - p_{n+2}'(x)\|$	8.527607765287214e-8	2.441827585710109e-10	5.089522571307543e-13

From above data, we would find $|err_n - err_{n+2}| \approx |p_n'(x) - p_{n+2}'(x)|$, this proves our idea that pair phenomenon occurs due to 3 reasons

(1) $\mathrm{Re}\hat{V}_{2k+1}^{(n)} = 0$ and $\mathrm{Im}\hat{V}_{2k}^{(n)} = 0$

(2) $\hat{V}_5^{(10)} = \hat{V}_7^{(14)} = \hat{V}_9^{(18)} = \hat{V}_{11}^{(22)} = 0$

(3) $\hat{V}_4^{(8)} \approx 2\hat{V}_4^{(10)}$, $\hat{V}_6^{(12)} \approx 2\hat{V}_6^{(14)}$, $\hat{V}_8^{(16)} \approx 2\hat{V}_8^{(18)}$ and $\hat{V}_{10}^{(20)} \approx 2\hat{V}_{10}^{(22)}$

(4) From chap3.doc, we have shown

Lemma 1: $\left|\hat{v}_k - \hat{V}_k\right| \le 2\sum_{p=1}^{\infty}\left|\hat{V}_{k+Np}\right|$ for $k = 0,1,2,\cdots,N-1$

Moreover from experimental result for $V = \exp(\sin(x))$ (see below), we found that $\left|\hat{V}_k\right| \sim \frac{1}{k^k}$, hence the error between discrete Fourier component $\left|\hat{v}_k^{(n)} - \hat{v}_k^{(n+2)}\right|$ for $k = 0,1,2,\cdots,n/2$ can be neglected.

So far, we cannot interpret reason 3.

We try to interpret reason 3.
First note that pair occurs at $n = 4k$ and $n = 4k + 2$

(Eq. 5) $$\hat{v}_{2k}^{(4k)} = h\sum_{j=1}^{N}(-1)^j v_j = 2h\left\{1 + (-1)^k \frac{1}{2}(e + 1/e) + \sum_{j=1}^{k-1}(-1)^j \left(e^{\sin(hj)} + e^{-\sin(hj)}\right)\right\}$$

(Eq. 6) $$\begin{aligned}\hat{v}_{2k}^{(4k+2)} &= \tilde{h}\sum_{j=1}^{N} e^{-i2kx_j} v_j = \tilde{h}\sum_{j=1}^{N}(-1)^j e^{-ix_j} v_j = \tilde{h}\sum_{j=1}^{N}(-1)^j \cos\left(x_j\right) v_j \\ &= 2\tilde{h}\left\{1 + \sum_{j=1}^{k}(-1)^j \cos\left(\tilde{h}j\right)\left(e^{\sin\left(\tilde{h}j\right)} + e^{-\sin\left(\tilde{h}j\right)}\right)\right\}\end{aligned}$$

where $\tilde{h} = \frac{2\pi}{4k+2} = \frac{\pi}{2k+1}$, $h = \frac{2\pi}{4k} = \frac{\pi}{2k}$, then $h - \tilde{h} = \frac{\pi}{2k(2k+1)}$

From our experimental result, $\hat{v}_{2k}^{(4k)} \approx 2\hat{v}_{2k}^{(4k+2)}$, this means that we only need to show

(Eq. 7) $$\begin{aligned}& h\left\{1 + (-1)^k \frac{1}{2}(e + 1/e) + \sum_{j=1}^{k-1}(-1)^j \left(e^{\sin(hj)} + e^{-\sin(hj)}\right)\right\} \\ &\approx 2\tilde{h}\left\{1 + \sum_{j=1}^{k}(-1)^j \cos\left(\tilde{h}j\right)\left(e^{\sin\left(\tilde{h}j\right)} + e^{-\sin\left(\tilde{h}j\right)}\right)\right\}\end{aligned}$$

Table 6: List difference between $\hat{v}_{2k}^{(4k)}$ and $\hat{v}_{2k}^{(4k+2)}$, define $\Delta = \left|\hat{v}_{2k}^{(4k)} - 2\hat{v}_{2k}^{(4k+2)}\right|$, $\Delta h = h - \tilde{h}$, we compare $\Delta h \cdot \hat{v}_{2k}^{(4k)}$ and Δ to determine if we can neglect difference of Δh and regard $h \approx h$.

However result is negative, we CANNOT regard $h \approx h$.

	$n = 4k = 8$	$n = 4k = 12$	$n = 4k = 16$	$n = 4k = 20$
$\hat{v}_{2k}^{(4k)}$	0.03439566712026	-0.00028260085474	0.00000125168893	-0.00000000345946
$\hat{v}_{2k}^{(4k+2)}$	0.01705653312949	-0.00014067458291	0.00000062411474	-0.00000000172646
Δ	2.826008612799996e-4	1.251688920000005e-6	3.459450000000128e-9	6.539999999999855e-12
Δh	0.15707963267949	0.07479982508547	0.04363323129986	0.02855993321445
$\Delta h \cdot \hat{v}_{2k}^{(4k)}$	0.00540285875702	2.113849450355667e-5	5.461523259816207e-8	9.880194655807041e-11

Define $t_1 = (-1)^k \frac{e}{2} + \sum_{j=1}^{k-1}(-1)^j \left(e^{\sin(hj)}\right)$, $t_2 = (-1)^k \frac{1}{2e} + \sum_{j=1}^{k-1}(-1)^j \left(e^{-\sin(hj)}\right)$,

$s_1 = \sum_{j=1}^{k}(-1)^j \cos\left(\tilde{h}j\right)\left(e^{\sin\left(\tilde{h}j\right)}\right)$ and $s_2 = \sum_{j=1}^{k}(-1)^j \cos\left(\tilde{h}j\right)\left(e^{-\sin\left(\tilde{h}j\right)}\right)$

$\Delta_1 = 2h\left(t_1 - s_1 + t_2 - s_2\right)$ and $\Delta_2 = 2\left(h - \tilde{h}\right)\left(1 + s_1 + s_2\right) = \frac{1}{2k}\hat{v}_{2k}^{(4k+2)}$

Then $\hat{v}_{2k}^{(4k)} = 2h\left(1 + t_1 + t_2\right)$, $\hat{v}_{2k}^{(4k+2)} = 2\tilde{h}\left(1 + s_1 + s_2\right)$ and then

$\hat{v}_{2k}^{(4k)} - \hat{v}_{2k}^{(4k+2)} = \Delta_1 + \Delta_2$

	$n = 4k = 8$	$n = 4k = 12$	$n = 4k = 16$	$n = 4k = 20$
$\hat{v}_{2k}^{(4k+2)}$	0.01705653312949	-1.4067458291e-4	6.2411474e-7	-1.72646e-9
Δ	2.826008612799996e-4	1.251688920000005e-6	3.459450000000128e-9	6.539999999999855e-12
t_1	-0.66897406741795	-0.63041950969349	-0.59800207628710	-0.57848539389295
t_2	-0.30912897080952	-0.36985035424424	-0.40199633001306	-0.42151461161294
s_1	-0.65635567309515	-0.61768932891181	-0.58985808333448	-0.57283605860995
s_2	-0.33007116910701	-0.38246739447485	-0.41014102268702	-0.42716394441257
ht_1	-0.52541100391058	-0.33008688338879	-0.23483486621187	-0.18173654636631
ht_2	-0.24278932592674	-0.19365319263688	-0.15786358964238	-0.13242272072240
$\tilde{h}s_1$	-0.41240043214754	-0.27721832255859	-0.20589931347379	-0.16360159576731
$\tilde{h}s_2$	-0.20738983200567	-0.17165096524569	-0.14316622486770	-0.12199773724045
Δ_1	0.01307500070840	-1.184805080196554e-4	5.495598519973221e-7	-1.560346593471024e-9
Δ_2	0.00426413328237	-2.344576381762684e-5	7.801434217854332e-8	-1.726463642529372e-10

Because $\tilde{h}k = \frac{\pi}{2} - \frac{\tilde{h}}{2}$, we can rewrite s_1 and s_2 as

$$s_1 = \sum_{j=1}^{k-1}(-1)^j \cos\left(\tilde{h}j\right)\left(e^{\sin\left(\tilde{h}j\right)}\right) + (-1)^k \sin\left(\tilde{h}/2\right)e^{\cos\left(\tilde{h}/2\right)}$$ and

$$s_2 = \sum_{j=1}^{k-1}(-1)^j \cos\left(\tilde{h}j\right)\left(e^{-\sin\left(\tilde{h}j\right)}\right) + (-1)^k \sin\left(\tilde{h}/2\right)e^{-\cos\left(\tilde{h}/2\right)}$$

$$t_1 - s_1 = \sum_{j=1}^{k-1}(-1)^j\left[\left(e^{\sin(hj)}\right) - \cos\left(\tilde{h}j\right)\left(e^{\sin\left(\tilde{h}j\right)}\right)\right] + (-1)^k\left[\frac{e}{2} - \sin\left(\tilde{h}/2\right)e^{\cos\left(\tilde{h}/2\right)}\right]$$

$$t_2 - s_2 = \sum_{j=1}^{k-1}(-1)^j\left[\left(e^{-\sin(hj)}\right) - \cos\left(\tilde{h}j\right)\left(e^{-\sin\left(\tilde{h}j\right)}\right)\right] + (-1)^k\left[\frac{1}{2e} - \sin\left(\tilde{h}/2\right)e^{-\cos\left(\tilde{h}/2\right)}\right]$$

Then if we want to show $\hat{v}_{2k}^{(4k)} \approx 2\hat{v}_{2k}^{(4k+2)}$, it suffices to show $\Delta_1 \approx \left(1 - \frac{1}{2k}\right)\hat{v}_{2k}^{(4k+2)}$.

Another view: if we divide $1 = \frac{1}{2} + \frac{1}{2}$, then we may regard summation in $\hat{v}_{2k}^{(4k)}$ as sum of Trapezoid rule, so is $\hat{v}_{2k}^{(4k+2)}$, then we rewrite them as

$$\hat{v}_{2k}^{(4k)} = h\sum_{j=0}^{k-1}(-1)^j\left(e^{\sin(hj)} - e^{\sin(hj+h)}\right) + h\sum_{j=0}^{k-1}(-1)^j\left(e^{-\sin(hj)} - e^{-\sin(hj+h)}\right)$$

Possible solution

$e^{\sin(x)} = e^{\sin(y)} + \int_y^x \cos(t) e^{\sin(t)} dt$

Then $e^{\sin(hj)} - e^{\sin(hj+h)} = -h\int_{hj}^{hj+h} \cos(t) e^{\sin(t)} dt$.

$\hat{v}_{2k}^{(4k)} = -h\sum_{j=0}^{k-1}(-1)^j \int_{hj}^{hj+h} \cos(t) e^{\sin(t)} dt + h\sum_{j=0}^{k-1}(-1)^j \int_{hj}^{hj+h} \cos(t) e^{-\sin(t)} dt$

In this section, we try to find Fourier component of $V = \exp(\sin(x))$, and decide the decay rate of $\hat{V}_k$ numerically since this decay rate determine the accuracy of discrete Fourier component.

From chapter 3 (see chap3.doc), we know

(Eq. 8) $V(x) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{ikx}\hat{V}_k$ with $\|V\|_{L^2}^2 = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\left|\hat{V}_k\right|^2$

(Eq. 9) $\hat{V}_k = \int_0^{2\pi} V(x) e^{-ikx} dx$

Because V is real, we have $\hat{V}_{-k} = \hat{V}_k^*$ and

(Eq. 10) $V(x) = \frac{1}{2\pi}\hat{V}_0 + \frac{1}{\pi}\sum_{k=1}^{\infty}\text{Re}\left(e^{ikx}\hat{V}_k\right) = \frac{1}{2\pi}\hat{V}_0 + \frac{1}{\pi}\sum_{k=1}^{\infty}\left(\cos(kx)\text{Re}\hat{V}_k - \sin(kx)\text{Im}\hat{V}_k\right)$

Now let $V(x) = \exp(\sin(x))$, then $V > 0$, $\|V\|_1 = \hat{V}_0$ and $\left|\hat{V}_k\right| \le \|V\|_1$ for all k.

$\text{Re}\hat{V}_k = \int_0^{2\pi} V(x)\cos(kx) dx = \int_0^{2\pi}\cos(kx) e^{\sin x} dx$

$\text{Im}\hat{V}_k = -\int_0^{2\pi} V(x)\sin(kx) dx = -\int_0^{2\pi}\sin(kx) e^{\sin x} dx$

$\|V\|_{L^2}^2 = \int_0^{2\pi}\exp(2\sin(x)) dx = 14.32305687810046$

$\hat{V}_0 = \int_0^{2\pi}\exp(\sin(x)) dx = 7.95492652101279$

$\text{Re}\hat{V}_1 = \int_0^{2\pi}\cos(x) e^{\sin x} dx = e^{\sin x}\Big|_0^{2\pi} = 0$ and $\text{Im}\hat{V}_1 = -\int_0^{2\pi}\sin(x) e^{\sin x} dx = -3.55099937842440$

$\text{Re}\hat{V}_2 = -0.85292776416409$ and $\text{Im}\hat{V}_2 = 0$

Prop 1: for $k > 2$, then

(Eq. 11) $\text{Re}\hat{V}_k = \frac{1}{2k}\left(\text{Im}\hat{V}_{k-1} + \text{Im}\hat{V}_{k+1}\right)$ and $\text{Im}\hat{V}_k = -\frac{1}{2k}\left(\text{Re}\hat{V}_{k-1} + \text{Re}\hat{V}_{k+1}\right)$

<proof> We use integration by parts

$$\text{Re}\hat{V}_k = \int_0^{2\pi}\cos(kx) e^{\sin x} dx = \frac{1}{k}\sin(kx) e^{\sin x}\Big|_0^{2\pi} - \frac{1}{k}\int_0^{2\pi}\sin(kx)\cos x e^{\sin x} dx$$

$$= -\frac{1}{k}\int_0^{2\pi}\sin(kx)\cos x e^{\sin x} dx = -\frac{1}{2k}\int_0^{2\pi}\left[\sin((k+1)x) + \sin((k-1)x)\right] e^{\sin x} dx$$

$$= \frac{1}{2k}\left(\text{Im}\hat{V}_{k-1} + \text{Im}\hat{V}_{k+1}\right)$$

$$\operatorname{Im}\hat{V}_k = -\int_0^{2\pi} \sin(kx) e^{\sin x} dx = \frac{1}{k}\cos(kx) e^{\sin x} \Big|_0^{2\pi} - \frac{1}{k}\int_0^{2\pi} \cos(kx)\cos x e^{\sin x} dx$$
$$= -\frac{1}{k}\int_0^{2\pi} \cos(kx)\cos x e^{\sin x} dx = -\frac{1}{2k}\int_0^{2\pi} \left[\cos((k+1)x) + \cos((k-1)x)\right] e^{\sin x} dx$$
$$= -\frac{1}{2k}\left(\operatorname{Re}\hat{V}_{k-1} + \operatorname{Re}\hat{V}_{k+1}\right)$$

We can rearrange (Eq. 11) to be

(Eq. 12) $\begin{bmatrix} \operatorname{Re}\hat{V}_{k+1} \\ \operatorname{Im}\hat{V}_{k+1} \end{bmatrix} = -\begin{bmatrix} \operatorname{Re}\hat{V}_{k-1} \\ \operatorname{Im}\hat{V}_{k-1} \end{bmatrix} + 2k \begin{bmatrix} -\operatorname{Im}\hat{V}_k \\ \operatorname{Re}\hat{V}_k \end{bmatrix}$ or $\hat{V}_{k+1} = -\hat{V}_{k-1} + 2k\sqrt{-1}\hat{V}_k$

Remark 2: one can derive (Eq. 12) by

$$\hat{V}_k = \int_0^{2\pi} V(x) e^{-ikx} dx = \frac{1}{ik}\int_0^{2\pi} e^{-ikx}\cos x e^{\sin x} dx = \frac{1}{2ik}\int_0^{2\pi} e^{-ikx}\left(e^{ix} + e^{-ix}\right) e^{\sin x} dx = \frac{1}{2ik}\left(\hat{V}_{k-1} + \hat{V}_{k+1}\right)$$

for example:

$\hat{V}_3 = -\hat{V}_1 + 4\sqrt{-1}\hat{V}_2 = (0, 0.13928832176804)$, $\hat{V}_4 = -\hat{V}_2 + 6\sqrt{-1}\hat{V}_3 = (0.01719783355585, 0)$

Lemma 2: for $V(x) = \exp(\sin(x))$, we have $\operatorname{Im}\hat{V}_{2k} = 0$ and $\operatorname{Re}\hat{V}_{2k+1} = 0$

(Eq. 13) $V(x) = \frac{1}{2\pi}\hat{V}_0 + \frac{1}{\pi}\sum_{k=1}^{\infty}\left(\cos(2kx)\operatorname{Re}\hat{V}_{2k}\right) - \frac{1}{\pi}\sum_{k=0}^{\infty}\sin((2k+1)x)\operatorname{Im}\hat{V}_{2k+1}$

<proof> we use induction

induction basis: $k = 0$, we have $\hat{V}_0 \in R$ and $\operatorname{Re}\hat{V}_1 = 0$, OK

$k = 1$, we have $\operatorname{Im}\hat{V}_2 = 0$ and $\operatorname{Re}\hat{V}_3 = 0$, OK

inductive hypothesis: Assume assertions holds for $k = 1, 2, \cdots, m$

Inductive step, for $k = m+1$

From (Eq. 12) we have

$$\operatorname{Im}\hat{V}_{2m} = -\operatorname{Im}\hat{V}_{2m-2} + 2(2m-1)\operatorname{Re}\hat{V}_{2m-1} = -\operatorname{Im}\hat{V}_{2(m-1)} + 2(2m-1)\operatorname{Re}\hat{V}_{2(m-1)+1} = 0$$
$$\operatorname{Re}\hat{V}_{2m+1} = -\operatorname{Re}\hat{V}_{2m-1} - 2\cdot 2m\operatorname{Im}\hat{V}_{2m} = -\operatorname{Re}\hat{V}_{2(m-1)+1} - 4m\operatorname{Im}\hat{V}_{2m} = 0$$

last equality comes from inductive hypothesis.

Moreover we use Matlab to verify these two condition, see Figure 10. Even when $k \le 40$, we still have $O(1.E-15)$ accuracy.

Another proof: we use $\sin(\pi - x) = \sin(x)$ to show the assertion

$$\int_{-\pi}^{\pi} e^{\sin x}\sin(2kx) dx = -\int_0^{\pi} e^{-\sin x}\sin(2kx) dx + \int_0^{\pi} e^{\sin x}\sin(2kx) dx$$

First we claim $\int_0^{\pi} e^{\sin x}\sin(2kx) dx = 0$

$$\int_0^{\pi} e^{\sin x}\sin(2kx) dx = \int_0^{\pi/2} e^{\sin x}\sin(2kx) dx + \int_{\pi/2}^{\pi} e^{\sin x}\sin(2kx) dx$$
$$\int_{\pi/2}^{\pi} e^{\sin x}\sin(2kx) dx = -\int_{\pi/2}^{0} e^{\sin(\pi-y)}\sin(2k(\pi-y)) dy = -\int_0^{\pi/2} e^{\sin y}\sin(2ky) dy$$

Hence $\int_0^{\pi} e^{\sin x}\sin(2kx) dx = 0$, the same cancellation holds for $\int_0^{\pi} e^{-\sin x}\sin(2kx) dx = 0$

Similarly

$$\int_{-\pi}^{\pi} e^{\sin x}\cos\left((2k+1)x\right)dx = -\int_0^{\pi} e^{-\sin x}\cos\left((2k+1)x\right)dx + \int_0^{\pi} e^{\sin x}\cos\left((2k+1)x\right)dx$$

we claim $\int_0^{\pi} e^{\sin x}\cos\left((2k+1)x\right)dx = 0$

$$\int_{\pi/2}^{\pi} e^{\sin x}\cos\left((2k+1)x\right)dx = -\int_{\pi/2}^{0} e^{\sin y}\cos\left(\pi-(2k+1)y\right)dy = -\int_0^{\pi/2} e^{\sin y}\cos\left((2k+1)y\right)dy$$

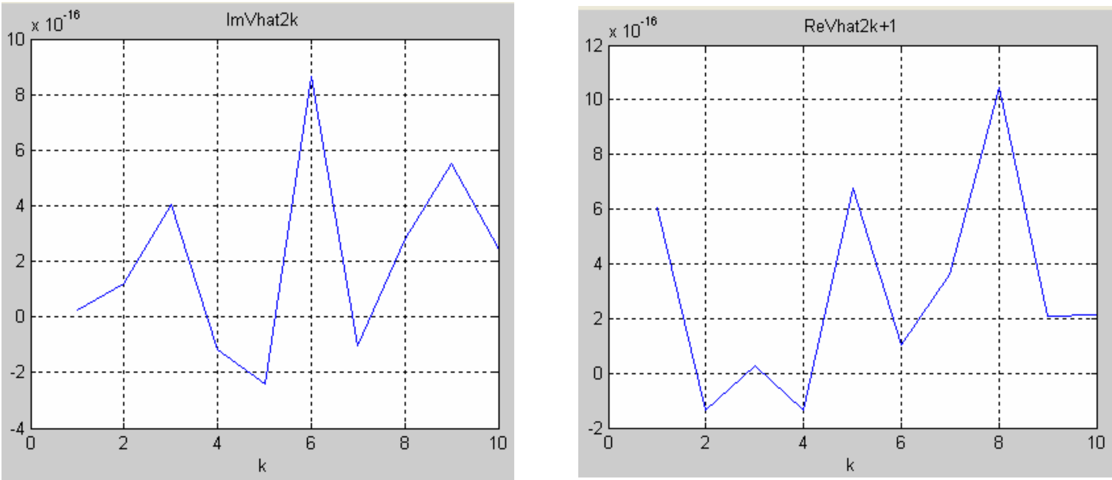

Figure 10: we verify condition $\mathrm{Im}\hat{V}_{2k}=0$ and $\mathrm{Re}\hat{V}_{2k+1}=0$ for $k=1:10$ by using trapezoid rule to do integration. It is clear that all data reach machine accuracy.

From **Lemma 2**, we can simplify (Eq. 10) as

(Eq. 14) $$V(x)=\frac{1}{2\pi}\hat{V}_0+\frac{1}{\pi}\sum_{k=1}^{\infty}\left(\cos(2kx)\mathrm{Re}\hat{V}_{2k}\right)-\frac{1}{\pi}\sum_{k=0}^{\infty}\left(\sin\left((2k+1)x\right)\mathrm{Im}\hat{V}_{2k+1}\right)$$

where

(Eq. 15) $$\begin{bmatrix}\mathrm{Re}\hat{V}_{2k}\\ \mathrm{Im}\hat{V}_{2k+1}\end{bmatrix}=-\begin{bmatrix}\mathrm{Re}\hat{V}_{2(k-1)}\\ \mathrm{Im}\hat{V}_{2(k-1)+1}\end{bmatrix}+2\begin{bmatrix}-(2k-1)\mathrm{Im}\hat{V}_{2k-1}\\ 2k\,\mathrm{Re}\hat{V}_{2k}\end{bmatrix}$$

Moreover we can simplify (Eq. 15) further

(Eq. 16) $$\begin{bmatrix}\mathrm{Re}\hat{V}_{2k}\\ \mathrm{Im}\hat{V}_{2k+1}\end{bmatrix}=\begin{bmatrix}-1 & -2(2k-1)\\ -4k & -\left(1+8k(2k-1)\right)\end{bmatrix}\begin{bmatrix}\mathrm{Re}\hat{V}_{2(k-1)}\\ \mathrm{Im}\hat{V}_{2(k-1)+1}\end{bmatrix}$$

$$\begin{bmatrix}\mathrm{Re}\hat{V}_{2}\\ \mathrm{Im}\hat{V}_{3}\end{bmatrix}=\begin{bmatrix}-1 & -2\\ -4 & -9\end{bmatrix}\begin{bmatrix}\mathrm{Re}\hat{V}_{0}\\ \mathrm{Im}\hat{V}_{1}\end{bmatrix},$$

$$\begin{bmatrix}\mathrm{Re}\hat{V}_{4}\\ \mathrm{Im}\hat{V}_{5}\end{bmatrix}=\begin{bmatrix}-1 & -6\\ -8 & -49\end{bmatrix}\begin{bmatrix}\mathrm{Re}\hat{V}_{2}\\ \mathrm{Im}\hat{V}_{3}\end{bmatrix}=\begin{bmatrix}25 & 56\\ 204 & 457\end{bmatrix}\begin{bmatrix}\mathrm{Re}\hat{V}_{0}\\ \mathrm{Im}\hat{V}_{1}\end{bmatrix},\ \ldots$$

$$\begin{bmatrix} \mathrm{Re}\hat{V}_8 \\ \mathrm{Im}\hat{V}_9 \end{bmatrix} = \begin{bmatrix} 351841 & 788192 \\ 5654440 & 12667041 \end{bmatrix} \begin{bmatrix} \mathrm{Re}\hat{V}_0 \\ \mathrm{Im}\hat{V}_1 \end{bmatrix}$$

This recursive formulation is not numerically stable, means that if we want to use this formula to find all $\hat{V}_k$, then rounding error would be disastrous. So we need high precision package to smooth accumulation error.

Next we use special technique to find $\hat{V}_0$

Lemma 3: $\int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx = \begin{cases} \dfrac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n}\dfrac{\pi}{2} & \text{if } n \geq 2,\ n \text{ is even} \\ \dfrac{2\cdot 4\cdot 6\cdots(n-1)}{3\cdot 5\cdots n} & \text{if } n \geq 3,\ n \text{ is odd} \end{cases}$

<proof> see integral table inside of cover of [1]

$\hat{V}_0 = \int_0^{2\pi} e^{\sin x} dx = \int_{-\pi}^{\pi} e^{\sin x} dx = \int_0^{\pi} \left(e^{\sin x} + e^{-\sin x}\right) dx = 2\int_0^{\pi/2} \left(e^{\sin x} + e^{-\sin x}\right) dx$, the last equality comes from $\sin(\pi - x) = \sin(x)$.

Using power series $\left(e^x + e^{-x}\right) = 2\left(1 + \sum_{k=1}^{\infty} \frac{1}{(2k)!} x^{2k}\right)$, then we have

$$\hat{V}_0 = 4\int_0^{\pi/2} \left(1 + \sum_{k=1}^{\infty} \frac{1}{(2k)!} \sin^{2k}(x)\right) dx = 4\left(\frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{1}{(2k)!} \int_0^{\pi/2} \sin^{2k}(x) dx\right)$$

where $\int_0^{\pi/2} \sin^{2k}(x) dx = \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots(2k)}\frac{\pi}{2} = \frac{\pi}{2}\frac{(2k)!}{\left(2\cdot 4\cdot 6\cdots(2k)\right)^2} = \frac{\pi}{2}\frac{(2k)!}{\left(2^k k!\right)^2} \equiv \frac{\pi}{2}\varphi_k$

Hence $\hat{V}_0 = 2\pi\left(1 + \sum_{k=1}^{\infty} \frac{\varphi_k}{(2k)!}\right) = 2\pi\left(1 + \sum_{k=1}^{\infty} \frac{1}{\left(2^k k!\right)^2}\right) \equiv 2\pi\left(1 + \sum_{k=1}^{\infty} a_k\right)$

Note that $a_k = \frac{1}{\left(2^k k!\right)^2} = \frac{1}{\left(2k 2^{k-1}(k-1)!\right)^2} = \frac{1}{(2k)^2} a_{k-1}$

Then we use following code to compute partial sum $\sum_{k=1}^{N} a_k$

```
a_k := 1,   sum := 0
for  k = 1 : N
        a_k ← a_k * 1/(2k)^2   ;   sum += a_k
end
```

Prop 2: If $V = e^{\sin x}$, then $\hat{V}_0 = 2\pi\left(1 + \sum_{k=1}^{\infty} a_k\right)$ where $a_k = \frac{1}{\left(2^k k!\right)^2}$ and $\sum_{k=N}^{\infty} a_k \leq \frac{e}{4^N (N!)^2}$

<**proof**> $\sum_{k=N}^{\infty} a_k = \sum_{m=0}^{\infty} \frac{1}{\left(2^{N+m}(N+m)!\right)^2} = \frac{1}{4^N (N!)^2} \sum_{m=0}^{\infty} \frac{1}{\left(2^m (N+1)(N+2)\cdots(N+m)\right)^2}$

We use $\frac{1}{\left(2^m (N+1)(N+2)\cdots(N+m)\right)^2} < \frac{1}{m!}$ and $e = \sum_{m=0}^{\infty} \frac{1}{m!}$.

	$N=10$	$N=15$	$N=20$	$N=25$	$N=30$	$N=40$	$N=50$
$\frac{e}{4^N (N!)^2}$	10^{-16}	10^{-42}	10^{-60}	10^{-80}	10^{-100}	10^{-142}	10^{-184}

	$N=100$	$N=200$	$N=300$	$N=400$	$N=500$	
$\frac{e}{4^N (N!)^2}$	10^{-375}	10^{-869}	10^{-1409}	10^{-1978}	10^{-2568}	

Prop 3: If $V = e^{\sin x}$, then $\operatorname{Im}\hat{V}_1 = -2\pi \sum_{k=1}^{\infty} a_k$ where $a_k = \frac{2k}{\left(2^k k!\right)^2}$ and $\sum_{k=N}^{\infty} a_k \le \frac{2e}{4^N (N!)^2}$

<**proof**> $\operatorname{Im}\hat{V}_1 = -\int_0^{2\pi} \sin(x) e^{\sin x} dx$, it suffices to show $\int_0^{2\pi} \sin(x) e^{\sin x} dx = 2\pi \sum_{k=1}^{\infty} a_k$.

(1) check $\int_0^{2\pi} \sin(x) e^{\sin x} dx = 2\int_0^{\pi/2} \sin(x)\left(e^{\sin x} - e^{-\sin x}\right) dx$

(2) $\left(e^x - e^{-x}\right) = x + \sum_{k=1}^{\infty} \frac{1}{(2k+1)!} x^{2k+1}$

(3) $\int_0^{2\pi} \sin(x) e^{\sin x} dx = 4\sum_{k=1}^{\infty} \frac{1}{(2k-1)!} \int_0^{2\pi} \sin^{2k}(x) dx = 4\sum_{k=1}^{\infty} \frac{1}{(2k-1)!} \frac{\pi}{2} \frac{(2k)!}{\left(2^k k!\right)^2}$

(4) $a_k = \frac{2k}{\left(2^k k!\right)^2} = \frac{1}{4k(k-1)} a_{k-1}$

$$\sum_{k=N}^{\infty} a_k = \sum_{m=0}^{\infty} \frac{2(N+m)}{\left(2^{N+m}(N+m)!\right)^2} = \frac{2}{4^N (N!)^2} \sum_{m=0}^{\infty} \frac{1}{\left(2^m (N+1)(N+2)\cdots(N+m-1)\right)^2 (N+m)}$$

Then we use following code to compute partial sum $\sum_{k=1}^{N} a_k$

```
a_k := 0.5,   sum := 0.5
for  k = 2 : N
        a_k <- a_k * 1/(4k(k-1))   ;   sum += a_k
end
```

Next we use high precision package to compute $\hat{V}_0$ and $\operatorname{Im}\hat{V}_1$ up to 128 digits, note that we use $N \geq 42$ to stabilize partial sum $\sum_{k=1}^{N} a_k$.

$\hat{V}_0$ = 7.9549265210128452 7451321966532939 4328161342771816 6385734005959553 8336060816469466 6995137357228568 7741332170437587 4113888148503023e0

Im$\hat{V}_1$ = -3.5509993784243618 9375715307444414 5068885827761984 4655200625893475 7625209545877072 0368124285904632 7616425367512080 1404294198552668e0

Table 7: We use high precision package with 1000 digits to compute $\hat{V}_k$ and estimate convergence order of $\hat{V}_k$, defined by $\left|\hat{V}_k\right| = \frac{1}{k^{m(k)}}$.

Source code: F:\course\2008spring\spectral_method\cxx_example\chap1

	$m(k)$		$m(k)$
ReV(0) = 7.9549265210128453e0		ImV(1) = -3.5509993784243619e0	
ReV(2) = -8.5292776416412149e-1		ImV(3) = 1.3928832176787595e-1	
ReV(4) = 1.7197833556865812e-2		ImV(5) = -1.7056533129494463e-3	
ReV(6) = -1.4130042737134921e-4	4.94744	ImV(7) = 1.0048184493255820e-5	5.914
ReV(8) = 6.2584446576772422e-7	6.86923	ImV(9) = -3.4673040972232835e-8	7.81773
ReV(10) = -1.7297282675331887e-9	8.76202	ImV(11) = 7.8475621569060340e-11	9.70361
ReV(12) = 3.2645930138612418e-12	10.6434	ImV(13) = -1.2538923639053624e-13	11.582
ReV(14) = -4.4728677072995018e-15	12.5199	ImV(15) = 1.4894058615019287e-16	13.4573
ReV(16) = 4.6501227937157073e-18	14.3944	ImV(17) = -1.3665675129023628e-19	15.3313
ReV(18) = -3.7932498476738764e-21	16.2682	ImV(19) = 9.9756773976728090e-23	17.2051
ReV(20) = 2.4924365582089317e-24	18.1421	ImV(21) = -5.9311648370821361e-26	19.0792
ReV(22) = -1.3473266344345731e-27	20.0164	ImV(23) = 2.9276455700143845e-29	20.9539
ReV(24) = 6.0967222795625615e-31	21.8915	ImV(25) = -1.2188758243549561e-32	22.8293
ReV(26) = -2.3431577877807459e-34	23.7674	ImV(27) = 4.3377470896829986e-36	24.7056
ReV(28) = 7.7435935192659899e-38	25.6441	ImV(29) = -1.3347188940443096e-39	26.5828
ReV(30) = -2.2239338089943938e-41	27.5217	ImV(31) = 3.5860864767329276e-43	28.4608
ReV(32) = 5.6019341997869982e-45	29.4002	ImV(33) = -8.4858886924868411e-47	30.3397
ReV(34) = -1.2476627456831309e-48	31.2795	ImV(35) = 1.7820218415512617e-50	32.2195
ReV(36) = 2.4745659724766378e-52	33.1597	ImV(37) = -3.3434136808250048e-54	34.1001
ReV(38) = -4.3984866613426193e-56	35.0407	ImV(39) = 5.6381820461407180e-58	35.9815
ReV(40) = 7.0466535285927712e-60	36.9225	ImV(41) = -8.5922326650106772e-62	37.8637
ReV(42) = -1.0227432840158852e-63	38.805	ImV(43) = 1.1890792772412452e-65	39.7466
ReV(44) = 1.3510558841438762e-67	40.6883	ImV(45) = -1.5009919463409708e-69	41.6302
ReV(46) = -1.6313243700245210e-71	42.5723	ImV(47) = 1.7352591841154628e-73	43.5146

ReV(48) = 1.8073695598597204e-75	44.457	ImV(49) = -1.8440665013114252e-77	45.3996
ReV(50) = -1.8438857452372014e-79	46.3424	ImV(51) = 1.8075607422380681e-81	47.2853
ReV(52) = 1.7378815437191885e-83	48.2283	ImV(53) = -1.6393677011213502e-85	49.1716
ReV(54) = -1.5178053055730738e-87	50.1149	ImV(55) = 1.3797110243049970e-89	51.0584
ReV(56) = 1.2317883757703079e-91	52.0021	ImV(57) = -1.0804344225222833e-93	52.9459
ReV(58) = -9.3134094904861371e-96	53.8899	ImV(59) = 7.8921625891422459e-98	54.8339
ReV(60) = 6.5763529828695323e-100	55.7782	ImV(61) = -5.3900969880710571e-102	56.7225
ReV(62) = -4.3465742284267052e-104	57.667	ImV(63) = 3.4494482194268623e-106	58.6116
ReV(64) = 2.6947194885863902e-108	59.5563	ImV(65) = -2.0727403628291311e-110	60.5012
ReV(66) = -1.5701690851976058e-112	61.4462	ImV(67) = 1.1717036829142283e-114	62.3913
ReV(68) = 8.6150092539842003e-117	63.3365	ImV(69) = -6.2424372377078596e-119	64.2819
ReV(70) = -4.4586594735415350e-121	65.2273	ImV(71) = 3.1397474971062253e-123	66.1729
ReV(72) = 2.1802765069509171e-125	67.1186	ImV(73) = -1.4932709690467884e-127	68.0644
ReV(74) = -1.0089214260595890e-129	69.0102	ImV(75) = 6.7258478596712626e-132	69.9562
ReV(76) = 4.4247108899611723e-134	70.9024	ImV(77) = -2.8730693028074481e-136	71.8486
ReV(78) = -1.8416363770214422e-138	72.7949	ImV(79) = 1.1655465399828615e-140	73.7413
ReV(80) = 7.2843848521037795e-143	74.6878	ImV(81) = -4.4963646256784895e-145	75.6344
ReV(82) = -2.7415850462649380e-147	76.5811	ImV(83) = 1.6514980399108078e-149	77.5279
ReV(84) = 9.8300012997093223e-152	78.4748	ImV(85) = -5.7821559641664032e-154	79.4218
ReV(86) = -3.3616062643676250e-156	80.3689	ImV(87) = 1.9318945408827106e-158	81.316
ReV(88) = 1.0976323170853944e-160	82.2633	ImV(89) = -6.1662812416461892e-163	83.2106
ReV(90) = -3.4256072372713951e-165	84.1581	ImV(91) = 1.8821455767799992e-167	85.1056
ReV(92) = 1.0228753179651741e-169	86.0532	ImV(93) = -5.4991724078868826e-172	87.0009
ReV(94) = -2.9250098213943509e-174	87.9486	ImV(95) = 1.5394366550286099e-176	88.8965
ReV(96) = 8.0176839992052904e-179	89.8444	ImV(97) = -4.1327181194190573e-181	90.7924
ReV(98) = -2.1084753231915019e-183	91.7405	ImV(99) = 1.0648596371350890e-185	92.6886
ReV(100) = 5.3241664025762733e-188	93.6369	ImV(101) = -2.6356619834325278e-190	94.5852
ReV(102) = -1.2919604256718823e-192	95.5336	ImV(103) = 6.2715061887867499e-195	96.482
ReV(104) = 3.0150781811827308e-197	97.4306	ImV(105) = -1.4357192666992787e-199	98.3792
ReV(106) = -6.7721114245441988e-202	99.3278	ImV(107) = 3.1644695908567913e-204	100.277
ReV(108) = 1.4650011066538491e-206	101.225	ImV(109) = -6.7200484477269938e-209	102.174
ReV(110) = -3.0545049364460404e-211	103.123	ImV(111) = 1.3758754570485746e-213	104.072
ReV(112) = 6.1421798204873893e-216	105.021	ImV(113) = -2.7177259399354918e-218	105.971
ReV(114) = -1.1919623317795924e-220	106.92	ImV(115) = 5.1823478021086980e-223	107.869
ReV(116) = 2.2337294591867709e-225	108.818	ImV(117) = -9.5456795389410712e-228	109.768
ReV(118) = -4.0447074560274482e-230	110.717	ImV(119) = 1.6994271629333847e-232	111.667
ReV(120) = 7.0808245992629515e-235	112.616	ImV(121) = -2.9259110276340004e-237	113.566
ReV(122) = -1.1991238867047224e-239	114.516	ImV(123) = 4.8744074477763628e-242	115.466

ReV(124) = 1.9654551737166463e-244	116.415	ImV(125) = -7.8616959079921701e-247	117.365
ReV(126) = -3.1196718603786482e-249	118.315	ImV(127) = 1.2281983797660650e-251	119.265
ReV(128) = 4.7975772843065435e-254	120.215	ImV(129) = -1.8594983589878912e-256	121.165
ReV(130) = -7.1518117784269494e-259	122.116	ImV(131) = 2.7296596884347195e-261	123.066
ReV(132) = 1.0339472798414659e-263	124.016	ImV(133) = -3.8869653249621741e-266	124.967
ReV(134) = -1.4503401527580714e-268	125.917	ImV(135) = 5.3715570542648909e-271	126.867
ReV(136) = 1.9748106550868906e-273	127.818	ImV(137) = -7.2072428548484198e-276	128.768
ReV(138) = -2.6112858423585874e-278	129.719	ImV(139) = 9.3929938718587938e-281	130.67
ReV(140) = 3.3545981842720248e-283	131.62	ImV(141) = -1.1895589712441229e-285	132.571
ReV(142) = -4.1885363598235693e-288	133.522	ImV(143) = 1.4645054229194620e-290	134.473
ReV(144) = 5.0850273907962369e-293	135.424	ImV(145) = -1.7534370145803019e-295	136.375
ReV(146) = -6.0048513361321557e-298	137.326	ImV(147) = 2.0424429712458045e-300	138.277
ReV(148) = 6.9000669490463556e-303	139.228	ImV(149) = -2.3154328083236962e-305	140.179
ReV(150) = -7.7180241741020677e-308	141.13	ImV(151) = 2.5556093075861283e-310	142.082

From the experimental result, we found $m(k) \sim k$, see Figure 11. Hence we expect that $\left|\hat{V}_k\right| = \frac{1}{k^{m(k)}} \sim \frac{1}{k^k} \sim \frac{1}{k!}$.

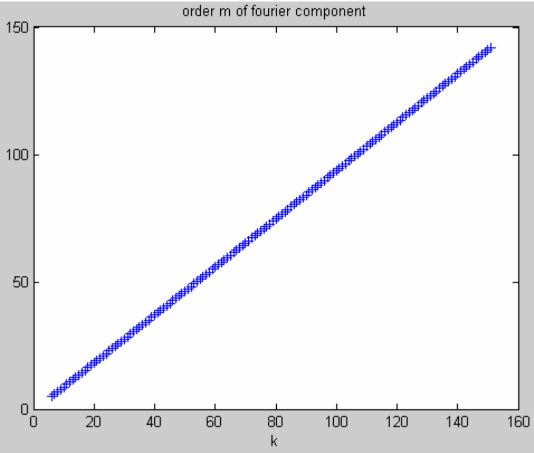

Figure 11: decay rate of $\left|\hat{V}_k\right| = \frac{1}{k^{m(k)}}$

This is reasonable since we know from chap3.doc

If $V^{(k)}$ is periodic with period 2π for $k = 0,1,2,\cdots,m$, then

(Eq. 17) $\hat{V}_k = \int_0^{2\pi} V(x) e^{-ikx} dx = \frac{1}{(ik)^m} \int_0^{2\pi} V^{(m)} e^{-ikx} dx$

Reference

[1] Finney, Weirm, Giordano, Thomas's Calculus