Chapter 6 Chebyshev Differentiation Matrices

Consider $N+1$ Chebyshev node on $[-1,1]$

(Eq. 1) $x_j = \cos\left(j\pi / N\right)$ for $j = 0,1,2,\cdots,N$

Remark 1: under such configuration, $1 = x_0 > x_1 > x_2 > \cdots > x_N = -1$, this order is different than usual notation, we must be careful since our Chebyshev Differentiation matrix is based on this order.

Remark 2: Given $N+1$ points x_j and corresponding function value v_j, then we can construct a unique polynomial of degree N (if we consider v_j, then it may less than N), called $p(x)$, $p(x) = \sum_{j=0}^{N} v_j S_j(x)$ where $S_j(x)$ is a basis (polynomial of degree N) under $\{x_j\}$, say $S_j(x_k) = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}$. In fact under Lagrange polynomial, we can write down $S_j(x)$ explicitly

(Eq. 2) $$S_j(x) = \frac{\prod_{k=0,k\neq j}^{N} (x - x_k)}{\prod_{k=0,k\neq j}^{N} (x_j - x_k)}$$

Example 1: $N = 1$, $x_0 = 1$, $x_1 = -1$

$S_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{1}{2}(x+1)$ and $S_1(x) = \frac{x - x_0}{x_1 - x_0} = -\frac{1}{2}(x-1)$, then $p(x) = \frac{1}{2}(x+1)v_0 - \frac{1}{2}(x-1)v_1$, that means $p'(x) = \frac{1}{2}(v_0 - v_1)$. Note that if we sample $w_j = p'(x_j)$ for $j = 0,1$, then we have $w_0 = w_1$, or write in matrix form

(Eq. 3) $$\begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \triangleq D_1 \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}$$

Moreover we have $p'(x) = w_0 S_0(x) + w_1 S_1(x)$. This means that we can do twice differentiation, if we set $z_j = p^{(2)}(x_j)$, then

(Eq. 4) $$\begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = D_1 \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = D_1^2 \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$, that is $D_1^2 = 0_{2\times 2}$

Example 2: $N = 2$, $x_0 = 1$, $x_1 = 0$ and $x_2 = -1$

$S_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{1}{2}x(x+1)$, $S_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = -(x-1)(x+1)$ and $S_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{1}{2}(x-1)x$, then $p'(x) = v_0 S_0^{(1)}(x) + v_1 S_1^{(1)}(x) + v_2 S_2^{(1)}(x)$,

$$p'(x) = \left(x+\frac{1}{2}\right)v_0 - 2xv_1 + \left(x-\frac{1}{2}\right)v_2$$

(Eq. 5) $$\begin{pmatrix} w_0 \\ w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 3/2 & -2 & 1/2 \\ 1/2 & 0 & -1/2 \\ -1/2 & 2 & -3/2 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} = D_2 \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix}$$

$D_2^2 = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{pmatrix}$ matches $p^{(2)}(x) = v_0 - 2v_1 + v_2$ and $D_2^3 = 0_{3\times 3}$.

In general we have $S_j(x) = \frac{1}{a_j}\prod_{k=0,k\neq j}^{N}(x-x_k)$ where $a_j = \frac{1}{\prod_{k=0,k\neq j}^{N}(x_j - x_k)}$, then

(Eq. 6) $$\log S_j(x) = -\log a_j + \sum_{k=0,k\neq j}^{N} \log(x-x_k)$$

we differentiate (Eq. 6), then $\frac{1}{S_j(x)} S_j^{(1)}(x) = \sum_{k=0,k\neq j}^{N} \frac{1}{(x-x_k)}$, hence

(Eq. 7) $$S_j^{(1)}(x) = S_j(x) \sum_{k=0,k\neq j}^{N} \frac{1}{(x-x_k)} = S_j(x)\left\{ \frac{1}{(x-x_i)} + \sum_{k=0,k\neq i,j}^{N} \frac{1}{(x-x_k)} \right\}$$

Case 1: $x = x_j$, then

(Eq. 8) $$S_j^{(1)}(x_j) = \sum_{k=0,k\neq j}^{N} \frac{1}{(x_j - x_k)}$$

Case 2: $x = x_i$, then

(Eq. 9) $$\begin{aligned} S_j^{(1)}(x_i) &= \frac{S_j(x)}{(x-x_i)}\Big|_{x=x_i} + S_j(x_i) \sum_{k=0,k\neq i,j}^{N} \frac{1}{(x_i - x_k)} = \frac{S_j(x)}{(x-x_i)}\Big|_{x=x_i} \\ &= \frac{1}{a_j}\prod_{k=0,k\neq i,j}^{N}(x-x_k)\Big|_{x=x_i} = \frac{1}{a_j}\prod_{k=0,k\neq i,j}^{N}(x_i - x_k) = \frac{a_i}{a_j}\frac{1}{(x_i - x_j)} \end{aligned}$$

$p'(x_i) = \sum_{j=0}^{N} v_j S_j^{(1)}(x_i) = \sum_{j=0}^{N} D_{ij}^N v_j$, we have $D_{ij}^N = S_j^{(1)}(x_i)$

We implement such differentiation matrix as function **diff_matrix_for_polyInterpolate.m** in directory F:\course\2008spring\spectral_method\matlab

In particular, for Chebyshev nodes, we have simplified version

Theorem 1: for Chebyshev nodes $x_j = \cos(j\pi / N)$ for $j = 0,1,2,\cdots,N$, we have

$$D_{00}^N = \frac{2N^2+1}{6},\quad D_{NN}^N = -\frac{2N^2+1}{6},\quad D_{jj}^N = \frac{-x_j}{2\left(1-x_j^2\right)} \quad \text{for} \quad j = 1,2,\cdots,N-1$$

$$D_{ij}^N = \frac{c_i}{c_j}\frac{(-1)^{i+j}}{(x_i - x_j)} \quad \text{for} \quad i \neq j,\ \ i,j = 0,2,\cdots,N \quad \text{where} \quad c_i = \begin{cases} 2 & i = 0,N \\ 1 & otherwise \end{cases}$$

with identity $D_{ii}^N = -\sum_{j=0, j\neq i}^{N} D_{ij}^N$

We implement such special matrix in **cheb.m** in directory F:\course\2008spring\spectral_method\matlab

Exercise 1: Let D^N be Chebyshev differentiation matrix. We know $\left(D^N\right)^{N+1} = 0_{(N+1)\times(N+1)}$.

However we need consider rounding error (in order to check if rounding contaminates the result, we can use high precision package to point out), this error can be amplified due to so many multiplication, we demonstrate this as follows.

Table 1: use MATLAB, first number is double precision, second number is double-double and third number is quad-double.

High precision code is located at /home/macrold/backup/2008spring/spectral_method/cxx_example/chap6, see ex8.cpp

	$N=5$	$N=10$	$N=15$	$N=20$
$\left\|\left(D^N\right)^{N+1}\right\|_2$	9.8017E-011 3.7644E-27 5.1923E-60	4.8672E-002 3.3708E-18 6.6438E-51	1.0941E+009 2.7839E-8 3.2691E-40	4.8124E+019 1.7886E+2 4.6363E-31
$cond\left(D^N,2\right)$	1.6810E+017	2.1488E+017	6.8901E+016	3.0900E+017
$\det\left(D^N\right)$	1.0232E-012	1.3201E-006	-2.5372E+001	4.0785E+009
$\max\left(\max\left(D^N\right)\right)$	10.4721	40.8635	91.5231	162.4476

It is clear that $\left\|\left(D^N\right)^{N+1}\right\|_2 \neq 0$ is due to rounding, not condition number.

Exercise 2: Compare two methods, method 1 is $D_{jj}^N = \frac{-x_j}{2\left(1-x_j^2\right)}$ and method 2 is

$D_{ii}^N = -\sum_{j=0, j\neq i}^{N} D_{ij}^N$. We measure difference,

	$N=5$	$N=10$	$N=15$	$N=20$
$\max_i \left\|\Delta D_{ii}^N\right\|$	5.3291e-015 4.9303E-31 1.2534e-63	2.8422e-014 8.2830E-30 1.3839e-62	2.2737e-013 2.7610E-29 1.1668e-61	8.5265e-013 9.1236E-29 4.6870e-62