Chapter 5 polynomial interpolation and Runge phenomenon

Consider the function $f(x)=\frac{1}{1+25x^2}$, Runge found that if this function is interpolated at equidistant points $x_i=-1+(i-1)\frac{2}{n}$ between -1 and 1 with a polynomial $P_n(x)$, the resulting interpolation oscillates toward the end of the interval, i.e. close to -1 and 1. It can even be proven that the interpolation error tends to infinity when degree of polynomial increases, that is, $\lim_{n\to\infty}\left(\max_{-1\le x\le 1}\left|f(x)-P_n(x)\right|\right)=\infty$.

Preliminary:

Theorem 1(Newton Approximation, see p224 [1]): Assume that $P_n(x)$ is the Newton polynomial to interpolate $f(x)$ such that $f(x)=P_n(x)+E_n(x)$. If $f\in C^{n+1}[a,b]$, then for each $x\in[a,b]$, there corresponds a number $c=c(x)\in(a,b)$ such that error term

(Eq. 1) $E_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}\prod_{j=0}^{n}(x-x_j)$.

Moreover if $x_j=a+hj$ is equi-distributed and let $x=a+(b-a)t$, then

(Eq. 2) $E_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}\prod_{j=0}^{n}\left(t-\frac{j}{n}\right)$.

If we $\left\|f^{(n+1)}\right\|_\infty$ cannot be controlled, then Error would be large, that is Runge phenomenon.

Question 1: why oscillates near boundary points -1 and 1?

<ans> $f'(x)=\frac{50x}{(1+25x^2)^2}$, $f^{(2)}(x)=\frac{5000(1+25x^2)-50(1+25x^2)^2}{(1+25x^2)^4}$

	f	$f^{(1)}$	$f^{(2)}$	$f^{(3)}$	$f^{(4)}$	$f^{(5)}$
$\|\cdot\|_{(-1,1)}$	1	3.2439	50	580	1.5E4	3.1105E5

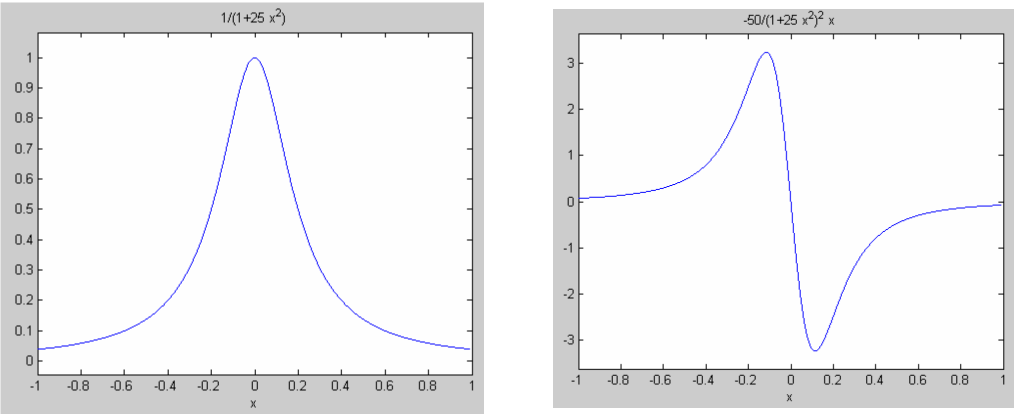

Figure 1: left panel is f (even), right panel is $f^{(1)}$ (odd)

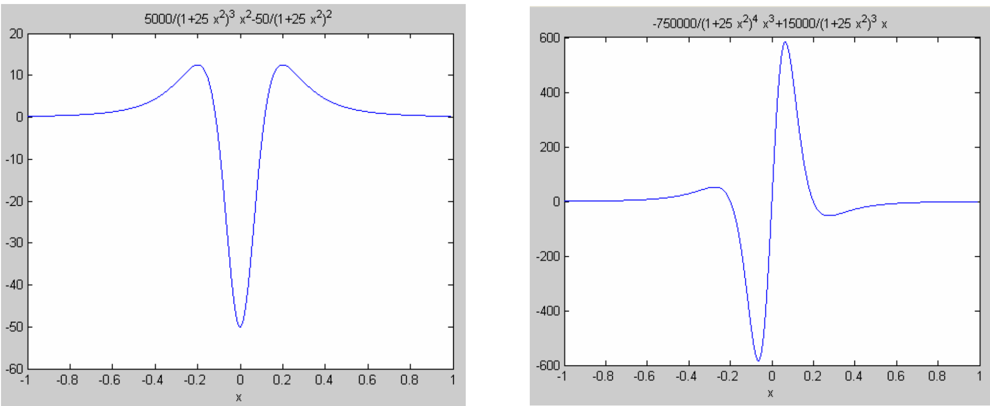

Figure 2: left panel is $f^{(2)}$ and right panel is $f^{(3)}$

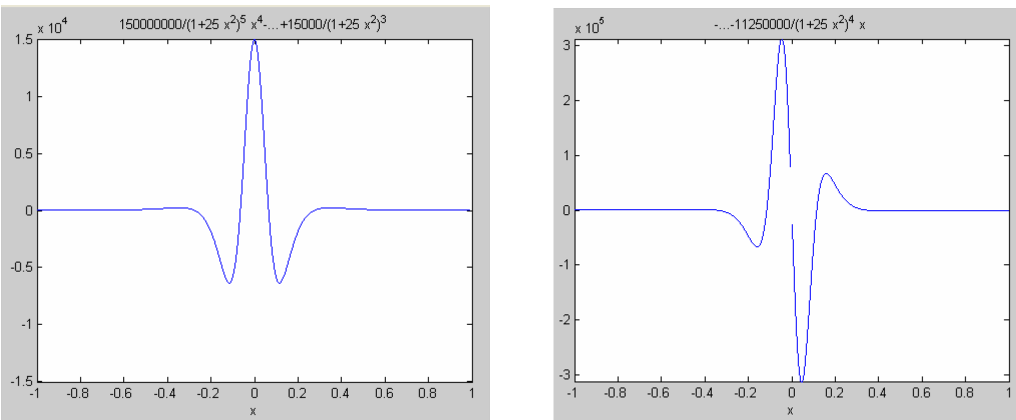

Figure 3: left panel is $f^{(4)}$ and right panel is $f^{(5)}$

In page 275 of [2], Consider the function $f(x)=\dfrac{1}{1+x^2}$ on $[-5,5]$ with equidistant points

$x_i = -5 + (i-1)\frac{10}{n}$. The author shows

(Eq. 3) $R_n(x) = f(x) - P_n(x) = f[x_0, x_1, \cdots, x_n, x] \prod_{j=0}^{n} (x - x_j)$

where $f[x_0, x_1, \cdots, x_n, x] = f(x) \frac{(-1)^{r+1}}{\prod_{j=0}^{r}(1+x_j^2)} \begin{cases} 1 & \text{if } n = 2r+1 \\ x & \text{if } n = 2r \end{cases}$

for case $n = 2r+1$, we can re-write

(Eq. 4) $R_n(x) = (-1)^{r+1} f(x) g_n(x)$, $g_n(x) = \prod_{j=0}^{r} \frac{x^2 - x_j^2}{1+x_j^2}$

Since $\frac{1}{26} \le f(x) \le 1$, hence $R_n(x) \to 0$ if and only if $g_n(x) \to 0$.

In order to estimate $g_n(x)$, we write $|g_n(x)| = \left[\exp\left(h \ln |g_n(x)|\right)\right]^{1/h}$. It suffices to estimate $h \ln |g_n(x)| = h \prod_{j=0}^{r} \ln \left| \frac{x^2 - x_j^2}{1+x_j^2} \right|$. Note that we can use integral test

(Eq. 5) $q(x) \equiv \lim_{h \to 0} h \ln |g_n(x)| = \int_{-5}^{0} \ln \left| \frac{x^2 - \xi^2}{1+\xi^2} \right| d\xi$ for $\min_{0 \le j \le r} |x + x_j| \ge \theta \frac{h}{2}$, $0 < \theta < 1$.

We have $q(x) = \begin{cases} = 0 & x = 3.63 \\ < 0 & |x| < 3.63 \\ > 0 & 5 \ge |x| > 3.63 \end{cases}$, and from $\lim_{n} |g_n(x)| = \lim\left[\exp(q(x))\right]^{1/h}$, we conclude that if $5 \ge |x| > 3.63$, then $\lim_{n} |g_n(x)| = \infty$.

Consider monic polynomial p of degree n, $p(z) = \prod_{k=1}^{N} (z - z_k)$ where $\{z_k\}$ are roots, counting multiplicity, then $|p(z)| = \prod_{k=1}^{N} |z - z_k| \in R$, then $\log |p(z)| = \sum_{k=1}^{N} \log |z - z_k|$, we define $\varphi_N(z) = \frac{1}{N} \sum_{k=1}^{N} \log |z - z_k|$, then $|p(z)| = \exp(N \varphi_N(z))$. In order to estimate growth rate (or asymptotic behavior) of $|p(z)|$, it suffices to consider limit behavior of $\varphi_N(z)$.

Remark 1: 2D Laplace equation $\Delta f = \frac{1}{s}\frac{\partial}{\partial s}\left(s \frac{\partial f}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 f}{\partial \phi^2}$, then $\Delta \log |z - z_k| = 0$ for $z \ne z_k$.
In fact $\Delta \log r = 2\pi\delta$, one can check this by Gauss theorem.
$\varphi_N(z) = \frac{1}{N} \sum_{k=1}^{N} \log |z - z_k|$ is solution of $\Delta f = 2\pi \frac{1}{N} \sum_{k=1}^{N} \delta(z - z_k)$. Let charge distribution be $\rho(x, y)$ satisfying $\int_a^b \rho dx = 1$ (in fact, charge is negative, since Gauss law is $\nabla \cdot E = 4\pi\rho$ and $E = -\nabla V$). Then we expect that $\lim \varphi_N(z) = \varphi(z) = \int_a^b \rho(x) \log |z - x| dx$.

Example 1: uniformly distribution $\rho(x)=\frac{1}{2}$ over $[-1,1]$

(Eq. 6) $\varphi(z)=\int_{-1}^{1}\rho(x)\log|z-x|\,dx=-1+\frac{1}{2}\mathrm{Re}\left[(z+1)\log(z+1)-(z-1)\log(z-1)\right]$

First we focus on $z\in[-1,1]$, then $\min\limits_{z\in[-1,1]}\varphi(z)=\varphi(0)=-1$ and $\max\limits_{z\in[-1,1]}\varphi(z)=\varphi(\pm 1)=-1+\log 2=$ -0.3069, see Figure 4.

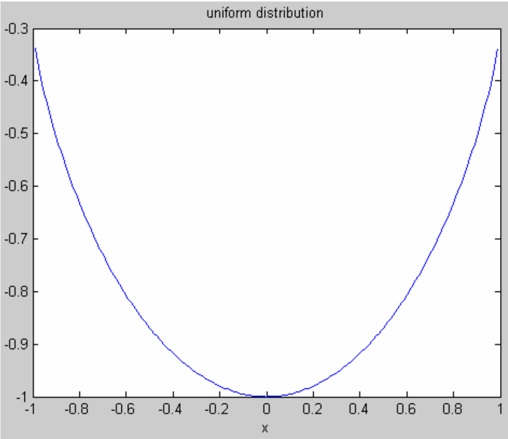

Figure 4: plot $\varphi(z)$ for $z\in[-1,1]$

Hence $|p(z)|=\exp(N\varphi_N(z))\sim\begin{cases}(2/e)^N & \text{near } x=\pm 1\\(1/e)^N & \text{near } x=0\end{cases}$. This means that $|p(z)|$ is larger near endpoints.

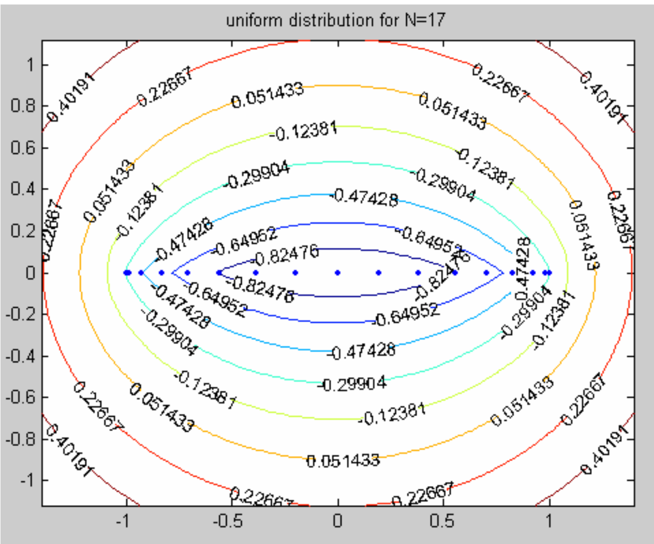

Figure 5: plot $\varphi(z)$ over square region, uniform distribution.

$\varphi(z)=\left\{\max\limits_{z\in[-1,1]}\varphi(z)=\text{-0.3069}\right\}$ is between level curve -0.47428 and -0.29904 in Figure 5.

If we consider standard benchmark $f(x)=\frac{1}{1+16x^2}$ which has analytic extension $f(z)=\frac{1}{1+16z^2}$ with two poles at $z=\pm 0.25i$, from Figure 5, we know $z=\pm 0.25i$ is inside

$\varphi(z)=\left\{\max_{z\in[-1,1]}\varphi(z)\le -0.3069\right\}$, hence we know Runge phenomenon occurs, see Figure 6.

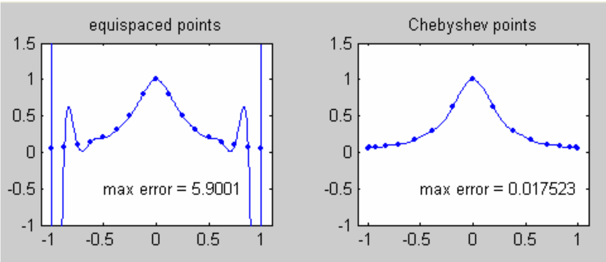

Figure 6: Runge phenomenon occurs at uniform grid (left panel)

However if we use $f(x)=\dfrac{1}{1+x^2}$, with poles $z=\pm i$ is outside $\varphi(z)=\left\{\max_{z\in[-1,1]}\varphi(z)\le -0.3069\right\}$, then we expect that interpolation is well for uniform distribution grid.

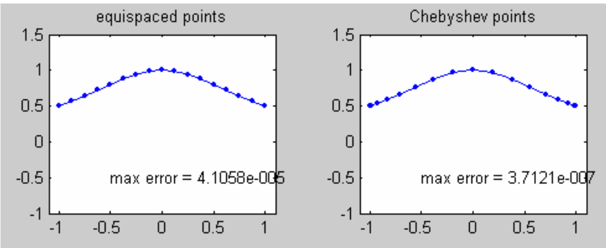

Figure 7: Runge phenomenon disappear for $f(x)=\dfrac{1}{1+x^2}$

Example 2: Chebyshev density $\rho(x)=\dfrac{1}{\sqrt{1-x^2}}$, the potential

(Eq. 7) $\varphi(z)=\int_{-1}^{1}\rho(x)\log|z-x|\,dx=\log\dfrac{\left|z-\sqrt{z^2-1}\right|}{2}$

$$\varphi(x)=\log\frac{\left|x-i\sqrt{1-x^2}\right|}{2}=\log(1/2)\quad\text{over}\quad[-1,1]$$

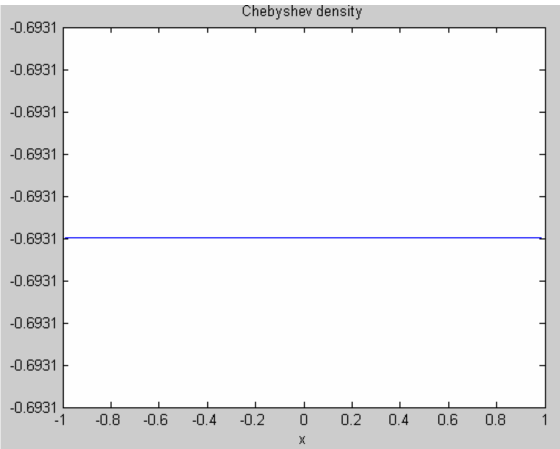

Figure 8: plot $\varphi(z)$ for $z\in[-1,1]$. Under chebyshev density, potential is constant, means no tangential electric force on this interval

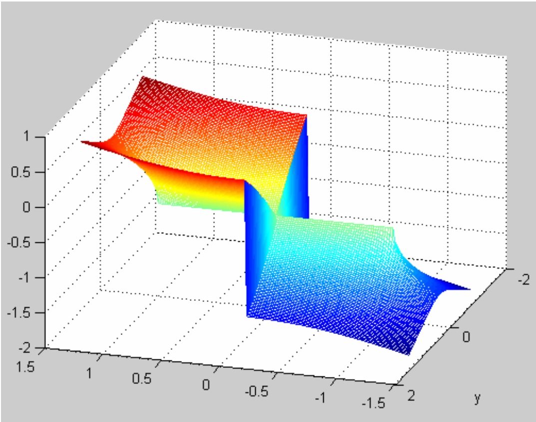

Figure 9: plot $\varphi(z)$ over square region, one can see that discontinuity occurs near x = 0.

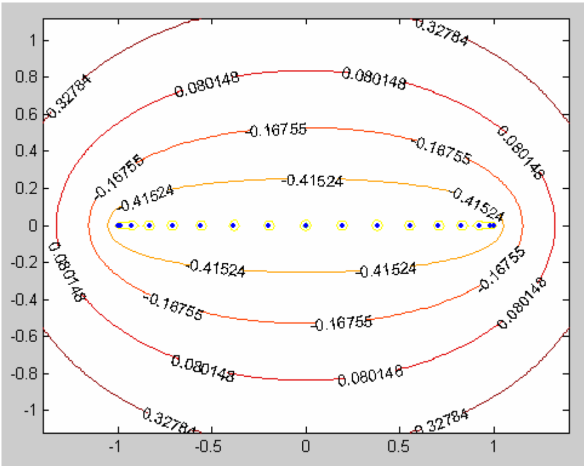

Figure 10: plot $\varphi_N(z)$ for N=17 over square region, the level curve does not reveal discontinuity

Reference

[1] John H. Mathews, Numerical Methods Using Matlab, 3rd edition.

[2] Eugene Isaacson, Analysis of Numerical Methods.