Chapter 10 Time-Stepping and Stability Region

Example 1: consider variable coefficient wave equation (we have do this in p6.m)

(Eq. 1) $u_t + c(x)u_x = 0$, $c(x) = 0.2 + \sin^2(x-1)$ for $x \in [0, 2\pi], t > 0$

with initial condition $u(x,0) = \exp\left(-100(x-1)^2\right)$ and periodic boundary condition.

Here we adopt leap-frog scheme for temporal discretization, and Fourier in space

(Eq. 2) $$\frac{u_j^{(n+1)} - u_j^{(n-1)}}{2\Delta t} = -c(x_j)\left(Du^{(n)}\right)_j \quad \text{for } j = 1, 2, \cdots, N$$

and extrapolate another initial condition, $u(x,0) = \exp\left(-100(x - 0.2\Delta t - 1)^2\right)$, which is wave backward with constant speed of 0.2. The result of spectral method is shown in Figure 1, note that

(1) $u(x,0)$ has compact support, we may regard it as periodic function before wave touch the boundary.

(2) From Figure 1, wave move faster at $x \in (2,4)$ and slower at $x \in (4,5)$, this is reasonable since $c(1+\pi/2) = c(2.57) = 1.2$ is maxima and $c(1+\pi) = c(4.14) = 0.2$ is minima.

Also, we use $\Delta t = \frac{h}{4} = \frac{1}{4}\frac{2\pi}{N} = 1.57N^{-1}$, $t_{\max} = 8$.

(3) period $T = \int_0^{2\pi} \frac{dt}{dx} dx = \int_0^{2\pi} \frac{1}{c(x)} dx = \frac{5\pi}{3}\sqrt{6} \approx 12.8255$.

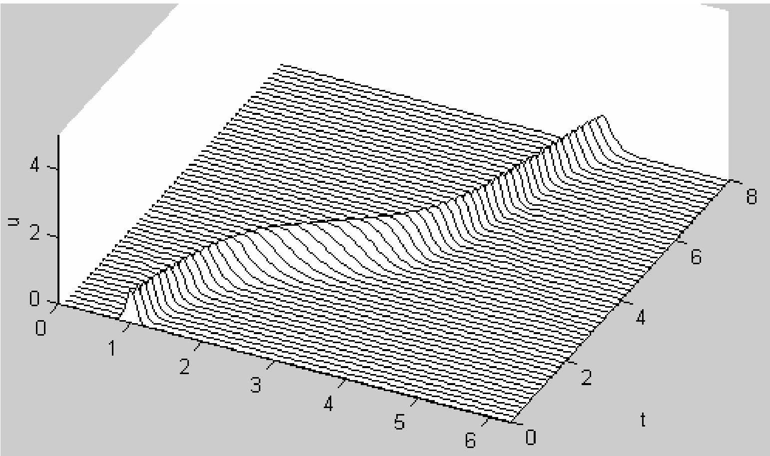


Figure 1: solution of wave equation under spectral method, $\Delta t = 1.57N^{-1}$

Question 1: Can we use larger Δt to save simulation time?

Let us consider a simple example, try $\Delta t = 1.9N^{-1}$ and $t_{\max} = 5$, then numerical simulation is catastrophic, see Figure 2. There are oscillation wave at $x = 0$, $x = 1 + \frac{\pi}{2} = 2.57$ and $x = 1 + \frac{3\pi}{2} = 5.7124$. This tells us that not any Δt will work, it depends on stability issue.

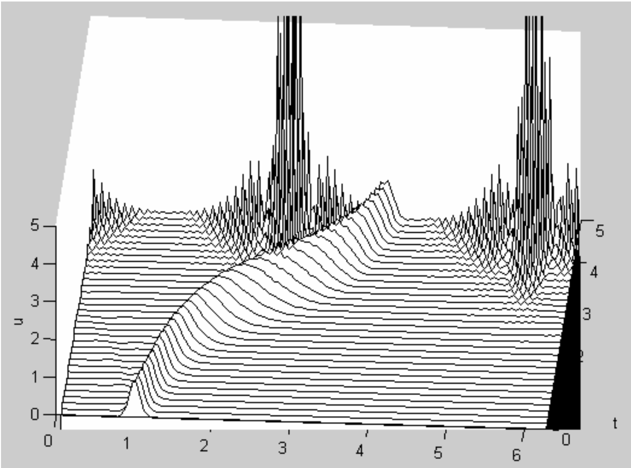


Figure 2: solution of wave equation under spectral method, $\Delta t = 1.9N^{-1}$ and $t_{\max} = 5$.

Time-dependent PDE is discretized in space, whether by spectral method or finite difference method, then the result is a coupled ODE, say $\frac{d}{dt}\vec{U} = A\vec{U}$ where $U_i = u(x_i)$. Under normal mode analysis, we have $A = V\Lambda V^{-1}$ and then $\frac{d}{dt}\vec{U} = A\vec{U}$ becomes $\frac{d}{dt}Z = \Lambda Z$, for $Z = V^{-1}\vec{U}$. The problem is then reduced to scalar problem $\frac{d}{dt}z_i = \lambda_i z_i$. Hence we only consider stability issue on model problem $\frac{d}{dt}u = \lambda u$, this is called **method of lines**. Hence stability issue means that we need to analyze stability region corresponding different time-discretized scheme in $\frac{d}{dt}u = \lambda u$.

Definition 1: stability region is a subset in complex plane consisting of $\lambda \in C$ for which the numerical approximation produces **bounded solution** when applied to the scalar linear model problem $\frac{d}{dt}u = \lambda u$.

Definition 2 (Rule of Thumb) : The method of line is stable if the eigenvalues of the linearized spatial discretization operator, scaled by Δt lie in the stability region of time-discretized operator.

In our problem we use Leap-frog scheme for $\frac{d}{dt}u = \lambda u$, say

(Eq. 3) $$\frac{u^{(n+1)} - u^{(n-1)}}{2\Delta t} = \lambda u^{(n)}$$

The characteristic equation for this recurrence relation is $g - g^{-1} = 2\lambda\Delta t$ by ansatz $u^{(n)} = g^n$ and for boundedness requirement of solution $\left\| u^{(n)} \right\| \le M$ we ask $|g| \le 1$ and if $|g| = 1$, then g must be simple or we will have Jordan form. Consider $g^2 - (2\lambda\Delta t)g - 1 = 0$, it has two solution

g_1, g_2 such that $g_1 g_2 = -1$, now if $g_1, g_2 \in R$, then $|g_1| < 1$ (stable) means $|g_2| = \dfrac{1}{|-g_1|} > 1$ (unstable). Hence we ask $g_1 = e^{i\theta}$ and $g_2 = -e^{-i\theta}$, under such configuration, $2\lambda\Delta t = g_1 - g_1^{-1} = e^{i\theta} - e^{-i\theta} = 2i\sin\theta$, or say $\lambda\Delta t = i\sin\theta$, possible candidate of λ in stable region is $\lambda\Delta t \in (-i, i)$, note that we eliminate $g = \pm i$ since g must be simple.

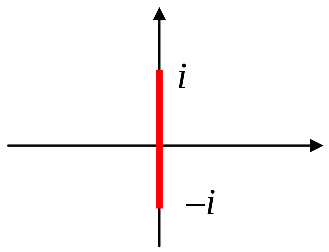


Figure 3: stability region of leap frog $\dfrac{u^{(n+1)} - u^{(n-1)}}{2\Delta t} = \lambda u^{(n)}$.

Remark 1: solution to recurrence $\dfrac{u^{(n+1)} - u^{(n-1)}}{2\Delta t} = \lambda u^{(n)}$ is $u^{(n)} = a g_1^n + b g_2^n$, a, b are determined by $u^{(0)}, u^{(1)}$, say $\begin{bmatrix} 1 & 1 \\ g_1 & g_2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} u^{(0)} \\ u^{(1)} \end{bmatrix}$.

Exercise 1: consider transport equation $u_t + u_x = 0$, if we use Fourier basis, that is,

(Eq. 4) $$S_N'(x_j) = \begin{cases} 0, & j = 0 \pmod N \\ \frac{1}{2}(-1)^j \cot\left(\frac{jh}{2}\right), & j \neq 0 \pmod N \end{cases}$$

(Eq. 5) $$D_N = \left[S_N'(x_i - x_j)\right] = \begin{pmatrix} 0 & & & -\frac{1}{2}\cot\frac{1h}{2} \\ -\frac{1}{2}\cot\frac{1h}{2} & \ddots & \ddots & \frac{1}{2}\cot\frac{2h}{2} \\ \frac{1}{2}\cot\frac{2h}{2} & \ddots & \ddots & -\frac{1}{2}\cot\frac{3h}{2} \\ -\frac{1}{2}\cot\frac{3h}{2} & & & \vdots \\ \vdots & \ddots & \ddots & \frac{1}{2}\cot\frac{1h}{2} \\ \frac{1}{2}\cot\frac{1h}{2} & & & 0 \end{pmatrix}$$

Eigenvalue pf Fourier differentiation matrix D_N is $\lambda_k = ik$ corresponding to eigenvector $\varphi_k = \exp(ikx)$ for $k = -N/2+1, \cdots, N/2-1$ with $\lambda = 0$ has multiplicity 2, why ?

(Eq. 6) $w_j = p'(x_j) = \sum_{k=1}^{N} v_k S_N'(x_j - x_k)$ where $S_N(x) = \dfrac{1}{2\pi} \mathrm{P} \sum_{k=-m}^{m} e^{ikx}$.

Clearly, $S_N'\left(x-x_j\right)=\frac{h}{2\pi}\sum_{\substack{k=-m-1\\k\neq 0}}^{m-1} ike^{ik\left(x-x_j\right)}+\frac{h}{4\pi}\left[ime^{im\left(x-x_j\right)}-imike^{-im\left(x-x_j\right)}\right]$ and then

(Eq. 7) $\quad S_N'\left(x-x_j\right)\cdot\vec{v}=\frac{1}{2\pi}\sum_{\substack{k=-m-1\\k\neq 0}}^{m-1} ike^{ikx}\hat{v}_k+\frac{h}{4\pi}\left[ime^{imx}\hat{v}_m-ime^{-imx}\hat{v}_{-m}\right]$

where $\hat{v}_k=h\sum_{j=1}^{N}e^{-ikx_j}v_j=\frac{2\pi}{N}\sum_{j=1}^{N}e^{-ikx_j}v_j$ for $k=-m+1,\cdots,m$

However we know $e^{imx_i}=e^{i(m+N)x_i}=e^{-imx_i}$ and $\hat{v}_m=\hat{v}_{-m}$, hence (Eq. 7) can be simplified as

(Eq. 8) $\quad D_N v=S_N'\left(x_i-x_j\right)\cdot\vec{v}=\frac{1}{2\pi}\sum_{\substack{k=-m-1\\k\neq 0}}^{m-1} ike^{ikx_i}\hat{v}_k$

Therefore when $v=1\left(e^{i0x}\right)$ and $v=e^{imx}$ we have $D_N v=0$

The stability condition for Fourier Spectral discretization in space coupled with leap frog scheme in time for $u_t+u_x=0$ is $\Delta t\left(\frac{N}{2}-1\right)<1$ ($\lambda\Delta t\in(-i,i)$), or $\Delta t<\frac{2}{N-2}$, we can restrict Δt a little, say $\Delta t<\frac{2}{N}<\frac{2}{N-2}$.

Remark 2: If we increase Δt across this threshold, then first mode to go to unstable is $\varphi_{m-1}=e^{i(m-1)x}=\exp\left(i\left(\frac{N}{2}-1\right)x\right)$.

Exercise 2: consider $u_t+c(x)u_x=0$, $c(x)=0.2+\sin^2(x-1)$, then $\max c=1.2$ and $\arg\max c(x)=1+\frac{\pi}{2},1+\frac{3\pi}{2}$. Consider Fourier discretization + leapfrog scheme $\frac{u_j^{(n+1)}-u_j^{(n-1)}}{2\Delta t}=-c\left(x_j\right)\left(Du^{(n)}\right)_j$, then maximum eigemvalue is about $(\max c)i\left(\frac{N}{2}-1\right)$, so stability condition is $\Delta t<\frac{2}{N-2}\frac{1}{1.2}=\frac{5}{3}\frac{1}{N-2}\sim\frac{5}{3}N^{-1}$.

Question 2: $\frac{5}{3}N^{-1}<1.9N^{-1}$ ($1.9N^{-1}$ is numerical threshold for unstable), why?

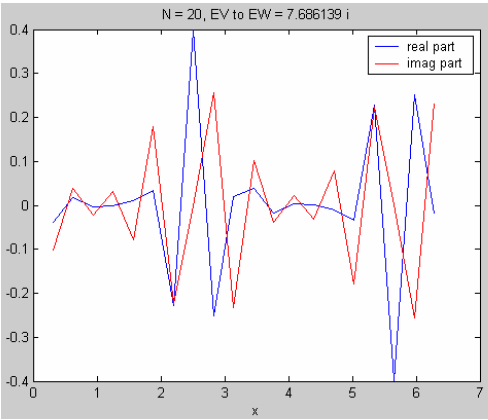
<Ans> This is because we estimate large eigenvalue by $(\max c)i\left(\frac{N}{2}-1\right)$, in fact we should find eigenvalue of $diag\left(c\left(x_j\right)\right)D_N$

Table 1: $\lambda_{\max}=\max|\Lambda|$, $1.2\left(\frac{N}{2}-1\right)$ is estimated largest eigenvalue, $\Delta t=1.9N^{-1}$ is numerical threshold of stability.

N	20	60	128	200
$\lambda_{\max}$	7.686139	29.051133	67.427430	108.899804
$1.2\left(\frac{N}{2}-1\right)$	10.8	34.8	75.6	118.8
$1.9N^{-1}$	0.095	0.031667	0.014844	0.0095
$1/\lambda_{\max}$	0.130104	0.034422	0.014831	0.009183

From above table we see $\lambda_{\max} < 1.2\left(\frac{N}{2}-1\right)$.

In Figure 6, largest eigenmode for $N = 128$ is the same as unstable mode visible in Figure 2, since $\Delta t = 1.9N^{-1}$ is slightly bigger than $\frac{1}{\lambda_{\max}}$, so largest eigenmode would be unstable at first.



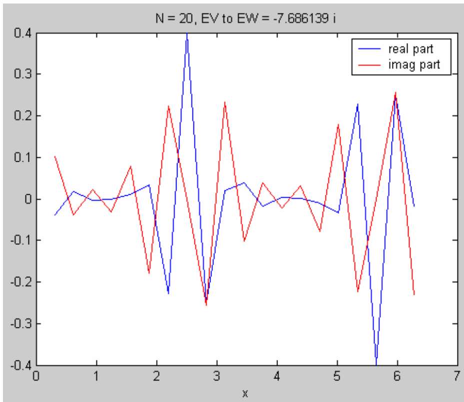
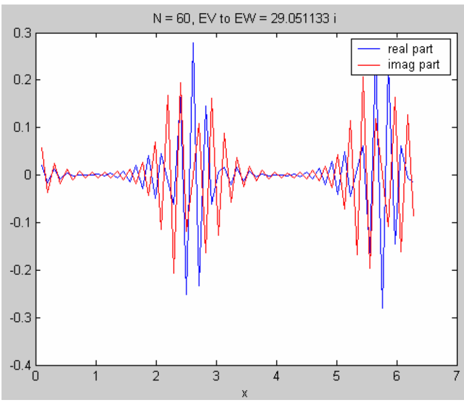


Figure 4: eigenmode corresponding to maximum eigenvalue $\lambda_{\max} = \pm 7.686139i$, $N = 20$



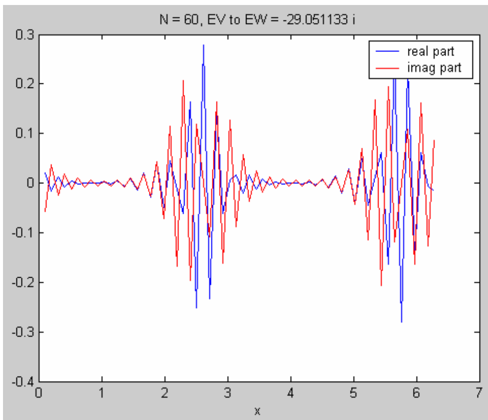


Figure 5: eigenmode corresponding to maximum eigenvalue $\lambda_{\max} = \pm 29.051133i$, $N = 60$

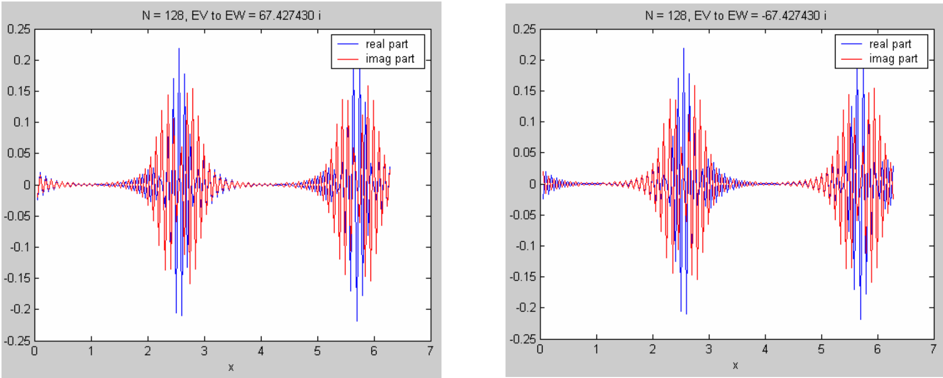


Figure 6: eigenmode corresponding to maximum eigenvalue $\lambda_{\max} = \pm 67.427430i$, $N = 128$

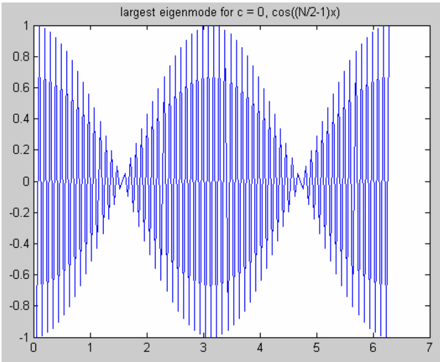


Figure 7: largest eigenmode $\varphi = \cos\left(\left(N/2-1\right)x\right)$, is different from Figure 6.

Question 3: Can you explain why largest eigenmode of $u_t + c(x)u_x = 0$ would localize at $x = \arg\max c$?

Example 2: consider second order wave equation (we have do this in p19.m)

(Eq. 9) $u_{tt} = u_{xx}$ for $x \in [-1,1], t > 0$ with chebyshev node

The exact solution $u(t,x) = \exp\left(-200(x+t)^2\right)$, it is a wave packet propagating backward.

In this problem we use chebyshev in space and Leap-frog in time,

(Eq. 10) $$\frac{v^{(n+1)} - 2v^{(n)} + v^{(n-1)}}{(\Delta t)^2} = \tilde{D}_N^2 v^{(n)}$$

Here $\tilde{D}_N^2$ is D_N^2 without first and last row and column, means that we use Dirichlet B.C.

Numerical simulation is plot in Figure 8 and Figure 9 with $N = 80$, $t_{\max} = 2.2$. One can see that when $\Delta t = 9.2N^{-2}$, then we have unstable mode near boundary.

Next we want to find stable region of such second order derivative of Leap-frog scheme and interpret unstable phenomenon in Figure 9.

As usual, let us consider model problem $u_{tt} = \lambda u$ under Leap-frog scheme

(Eq. 11) $$\frac{v^{(n+1)} - 2v^{(n)} + v^{(n-1)}}{(\Delta t)^2} = \lambda v^{(n)}$$

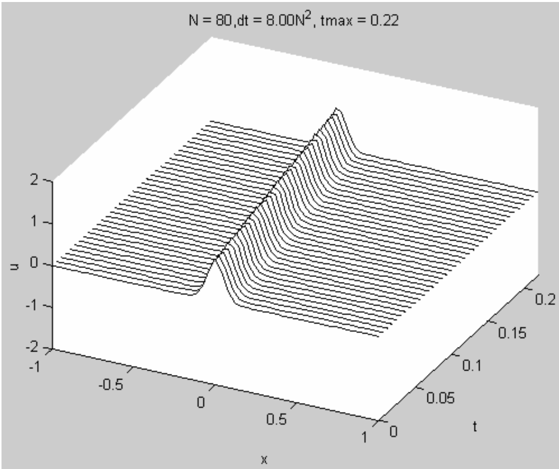


Figure 8: wave is propagating backward for $N = 80$, $dt = 8N^{-2}$

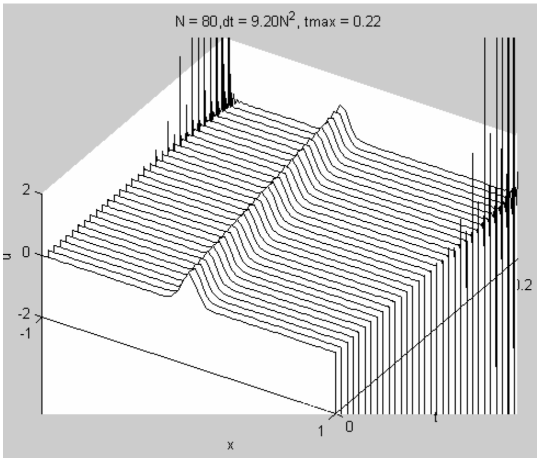


Figure 9: wave is propagating backward for $N = 80$, $dt = 9.2N^{-2}$, unstable mode occurs at boundary.

The characteristic equation for this recurrence relation is $g + g^{-1} = \lambda(\Delta t)^2 + 2$ by ansatz $u^{(n)} = g^n$. We solve $g^2 - \alpha g + 1 = 0$ for $\alpha = \lambda(\Delta t)^2 + 2$. As before if g is one root, then g^{-1} is the other root, we require $|g| \le 1$, say $g = \exp(i\theta)$ and then $2\cos(\theta) = \lambda(\Delta t)^2 + 2$ implies $-4\sin^2(\theta/2) = \lambda(\Delta t)^2$, or say $\lambda(\Delta t)^2 \in (-4, 0)$.

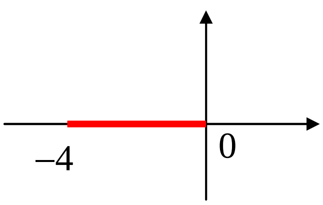


Figure 10: stability region of leap-frog formula for second derivative.

It suffices to find asymptotic behavior of eigenvalues of $\tilde{D}_N^2$, note that we have show that spectrum of $\tilde{D}_N^2$ approximates following continuous counterpart

(Eq. 12) $u_{xx} = \lambda u \quad -1 < x < 1$ with Dirichlet B.C $u(\pm 1) = 0$

The eigenmode is $\lambda_k = -\frac{\pi^2}{4}k^2$ and $u_k = \sin\left(\frac{k\pi}{2}(x+1)\right)$.

Prop 1: we show some known facts about $\tilde{D}_N^2$ without proof

(1) $\tilde{D}_N^2$ approximates Hermitian operator $\frac{d^2}{dx^2}$ but it is asymmetric.

(2) eigenvalue of $\tilde{D}_N^2$ is negative, real and $\lambda_{\max} \sim -0.048N^4$.

(3) large eigenmode of $\tilde{D}_N^2$ does not approximate to $\lambda_k = -\frac{\pi^2}{4}k^2$ and $u_k = \sin\left(\frac{k\pi}{2}(x+1)\right)$ since ppw is too small such that resolution is not enough, we called these modes are not physical, see Figure 11.

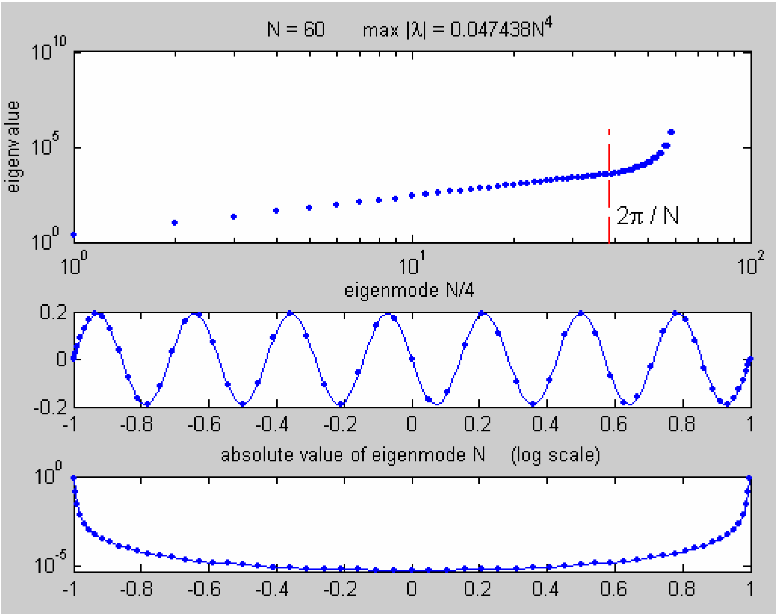


Figure 11: mode N is spurious and localized near the boundaries.

Under $\lambda_{\max} \sim -0.048N^4$, condition of stability region is $-0.048N^4(\Delta t)^2 \geq -4$, say $\Delta t \leq 9.1287N^{-2}$. When Δt exceeds the threshold, then largest mode is unstable, and oscillates near the boundary, see Figure 9.

Problem 1 (exercise 10.5): Chebyshev grids have an $O(N^{-2})$ spacing near the boundaries.

Therefore it is sometimes said, it is obvious that an explicit Chebyshev spectral method for a hyperbolic PDE such as $u_t = u_x$ must require time steps of size $O(N^{-2})$, "because of the CFL (Courant-Friedrichs-Lewy) condition". Explain why this argument is invalid.

<**ans**> we must find maximum eigenvalue of $\tilde{D}_N$ (for Dirichlet problem). We find eigenvalue of $\tilde{D}_N$ over $N = 8:8:300$, see Figure 12, if we use linear approximation, then $\log \lambda_{\max} = -2.3508 + 1.9858 \log N$, this means that $\lambda_{\max}(n) \sim 0.0953n^{1.988}$. In this example, we can say $\lambda_{\max}(n) \sim n^2$. If we consider D_N, then $\log \lambda_{\max} = -4.9263 + 2.1478 \log N$.

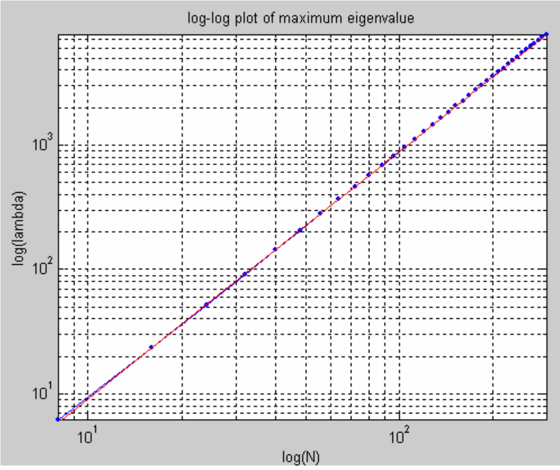


Figure 12: log-log plot of maximum eigenvalue of $\tilde{D}_N$, it seems that $\lambda_{\max}(n) \sim 0.0953 N^{1.9858}$

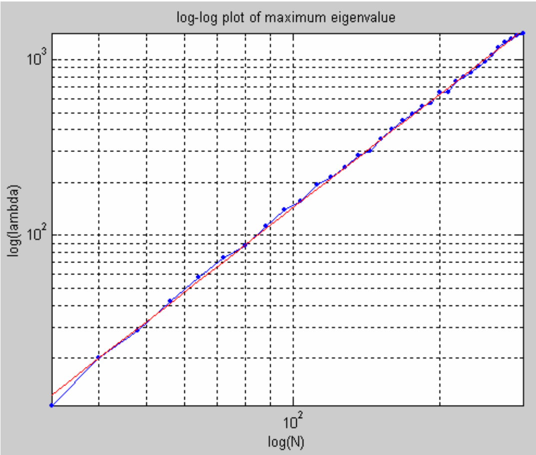


Figure 13: log-log plot of maximum eigenvalue of D_N, it seems that $\lambda_{\max}(n) \sim 0.0073 N^{2.1478}$

Example 3: consider second order wave equation (we have do this in p20.m)
(Eq. 13) $u_{tt} = u_{xx} + u_{yy}$, $-1 < x, y < 1, t > 0$

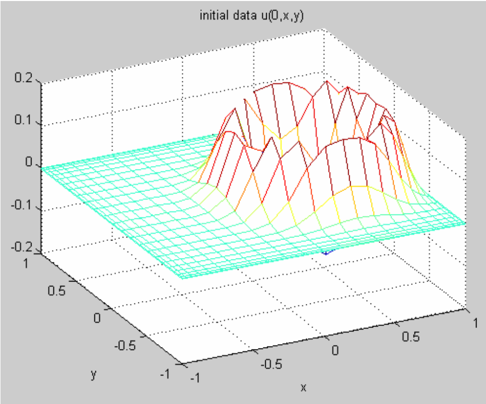
with initial data $u(0,x,y) = \exp\left[-40\left((x-0.4)^2 + y^2\right)\right]$ has compact support near $(x,y) = (0.4,0)$ and $u_t(0,x,y) = 0$.

Source code: **F:\course\2008spring\spectral_method\matlab\chap10_example3.m**

First we use $\Delta t = 6N^{-2}$, the result is good, see Figure 14. However if we add Δt to $\Delta t = 6.6N^{-2}$, then we have unstable mode near boundary, see Figure 15.

Since we use tensor product to construct Laplacian operator $\frac{d^2}{dx^2} + \frac{d^2}{dy^2}$, hence maximum eigenvalue of $\tilde{D}_N^2 \otimes I + I \otimes \tilde{D}_N^2$ is twice of $\tilde{D}_N^2$. $\lambda_{\max} \sim -2 \times 0.048 N^4$, then $\Delta t \le \frac{1}{\sqrt{2}} 9.1287 N^{-2} \sim 6.455 N^{-2}$.

Hence unstable mode appears in $\Delta t = 6.6N^{-2} > \mathit{threshold}$ and we know largest eigenmode is localized on the boundary.



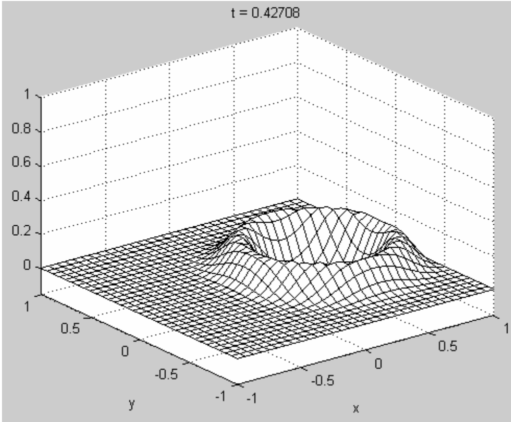


Figure 14: left panel is initial data $u(0,x,y)=\exp\left[-40\left((x-0.4)^2+y^2\right)\right]$, right panel is evolution of $u(t,x,y)$ at $t=0.42708$

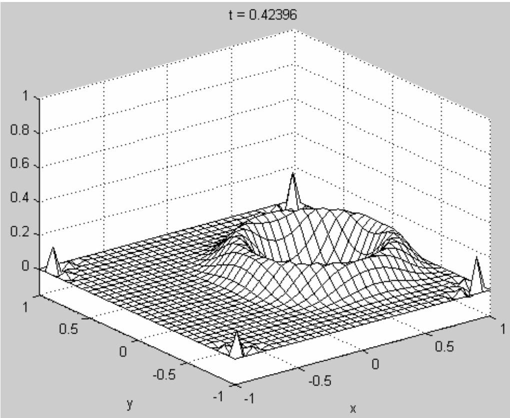


Figure 15: when $\Delta t = 6.6N^{-2}$, $t=0.42396$, we have unstable mode near boundary.

Example 4: KdV equation

(Eq. 14) $u_t + uu_x + u_{xxx} = 0$

There are two terms in the equation

(1) nonlinear term $uu_x = \frac{d}{dx}\frac{1}{2}u^2$ corresponding to Burger's equation $u_t + uu_x = 0$

(2) dispersive term $u_{xxx} = 0$, if we neglect nonlinear term, then $u_t + u_{xxx} = 0$ leads to $-iw + (ik)^3 = 0$ for $u = e^{ikx-iwt}$, then $w = -k^3$, group velocity $v_g = \frac{\partial w}{\partial k} = -k^2$.

Prop 2: solution of (Eq. 14) are called solitary (單獨的) waves, traveling wave of the form

(Eq. 15) $u(t,x) = 3a^2 \operatorname{sec}h^2\left(\frac{a}{2}\left(x - x_0 - 2a^2 t\right)\right)$ for any real a, x_0 and

(1) speed $v = 2a^2$ proportional to amplitude

(2) u decays exponentially by a factor $\exp\left(-\left(x-\left(x_0+2a^2t\right)\right)\right)$ since $\operatorname{sec}h(x) = \frac{2}{e^x + e^{-x}}$, so solitary wave is a localized wave in space.

(3) If solution of (Eq. 14) is composed of several solitary waves with different speed, then those solitary waves will interact cleanly, passing through one another with the only lasting effect of interaction being a phase shift.

Next we introduce **method of integration factor** to add stability, we use Fourier transform in space, then $u_t + \frac{d}{dx}\frac{1}{2}u^2 + u_{xxx} = 0$ becomes

(Eq. 16) $\hat{u}_t + \frac{ik}{2}F\left(u^2\right) + \left(ik\right)^3 \hat{u} = 0$

If we neglect nonlinear term, then $\hat{u}_t - ik^3\hat{u} = 0$ give us an idea to introduce integration factor $\hat{U} = e^{-k^3 t}\hat{u}$, such that $\hat{U}_t = 0$. Hence

(Eq. 17) $\hat{U}_t + \frac{i}{2}e^{-ik^3 t}kF\left(u^2\right) = 0$

Then we replace $u^2 = \left(F^{-1}\left(e^{ik^3 t}\hat{U}\right)\right)^2$ to simplify (Eq. 17) as

(Eq. 18) $\hat{U}_t + \frac{i}{2}e^{-ik^3 t}kF\left(\left(F^{-1}\left(e^{ik^3 t}\hat{U}\right)\right)^2\right) = 0$

with initial condition $u\left(0,x\right) = 3A^2 \sec h^2\left(\frac{A}{2}\left(x+2\right)\right) + 3B^2 \sec h^2\left(\frac{B}{2}\left(x+1\right)\right)$, composed of two solitons.

Prop 3: Recall forth-order Runge-Kutta method for $x' = f\left(t,x\right)$ $x\left(t+h\right) = x\left(t\right) + \frac{1}{6}\left(F_1 + 2F_2 + 2F_3 + F_4\right)$ $F_1 = hf\left(t,x\right)$, $F_2 = hf\left(t+h/2, x+F_1/2\right)$ $F_3 = hf\left(t+h/2, x+F_2/2\right)$, $F_4 = hf\left(t+h, x+F_3\right)$

If we apply Runge-Kutta algorithm into (Eq. 18) by setting

$f\left(t,\hat{U}\right) = -\frac{i}{2}e^{-ik^3 t}kF\left(\left(F^{-1}\left(e^{ik^3 t}\hat{U}\right)\right)^2\right)$, then set $\alpha_n = \exp\left(ik^3 t_n\right)$, then we have

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Algorithm 1 (directly use Runge-Kutta)

$$F_1 = \left(-\frac{i}{2}\alpha_{-n}k\Delta t\right)F\left(\left(F^{-1}\left(\alpha_n \hat{U}^{(n)}\right)\right)^2\right)$$

$$F_2 = \left(-\frac{i}{2}\alpha_{-n-1/2}k\Delta t\right)F\left(\left(F^{-1}\left(\alpha_{n+1/2}\left(\hat{U}^{(n)} + F_1/2\right)\right)\right)^2\right)$$

$$F_3 = \left(-\frac{i}{2}\alpha_{-n-1/2}k\Delta t\right)F\left(\left(F^{-1}\left(\alpha_{n+1/2}\left(\hat{U}^{(n)} + F_2/2\right)\right)\right)^2\right)$$

$$F_3 = \left(-\frac{i}{2}\alpha_{-n-1}k\Delta t\right)F\left(\left(F^{-1}\left(\alpha_{n+1}\left(\hat{U}^{(n)} + F_3\right)\right)\right)^2\right)$$

$$\hat{U}^{(n+1)} = \hat{U}^{(n)} + \frac{1}{6}\left(F_1 + 2F_2 + 2F_3 + F_4\right)$$

Under such configuration, stability condition is $\Delta t < 0.02N^{-2}$ since we need to compute $\alpha_n = \exp\left(ik^3 t_n\right)$ explicitly, however $\alpha_n = \exp\left(ik^3 t_n\right)$ is highly oscillatory when $k >> 1$, see Figure 16, source code: F:\course\2008spring\spectral_method\matlab\p27_2.m

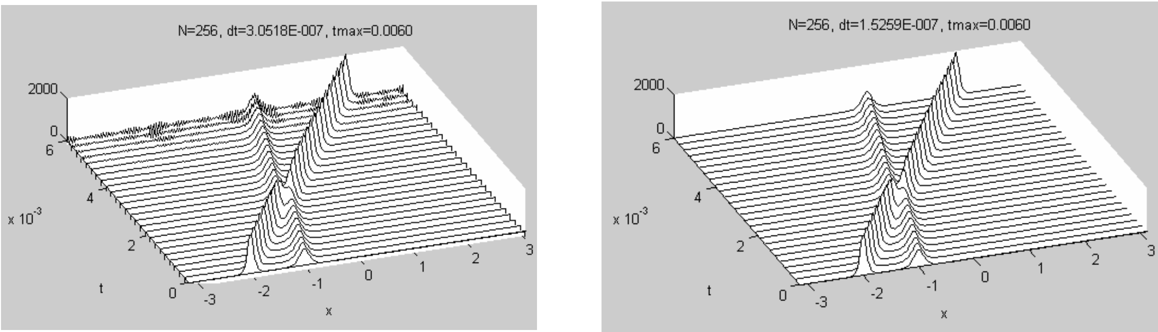


Figure 16: apply **Algorithm 1**, left panel: $\Delta t = 0.02N^{-2}$, unstable. Right panel: $\Delta t = 0.01N^{-2}$, stable.

If we rewrite **Algorithm 1** in $\hat{u}^{(n)} = \alpha_n \hat{U}^{(n)}$, then

Algorithm 2 (Runge-Kutta + implicit integration factor)

$$\tilde{F}_1 = \alpha_n F_1 = \left(-\frac{i}{2}k\Delta t\right) F\left(\left(F^{-1}\left(\hat{u}^{(n)}\right)\right)^2\right)$$

$$\tilde{F}_2 = \alpha_{n+1/2} F_2 = \left(-\frac{i}{2}k\Delta t\right) F\left(\left(F^{-1}\left(\alpha_{1/2}\left(\hat{u}^{(n)} + \tilde{F}_1/2\right)\right)\right)^2\right)$$

$$\tilde{F}_3 = \alpha_{n+1/2} F_3 = \left(-\frac{i}{2}k\Delta t\right) F\left(\left(F^{-1}\left(\alpha_{1/2}\hat{u}^{(n)} + \tilde{F}_2\right)\right)^2\right)$$

$$\tilde{F}_4 = \alpha_{n+1} F_4 = \left(-\frac{i}{2}k\Delta t\right) F\left(\left(F^{-1}\left(\alpha_1\hat{u}^{(n)} + \alpha_{1/2}\tilde{F}_3\right)\right)^2\right)$$

$$\hat{u}^{(n+1)} = \hat{u}^{(n)} + \frac{1}{6}\left(\alpha_1\tilde{F}_1 + 2\alpha_{1/2}\left(\tilde{F}_2 + \tilde{F}_3\right) + \tilde{F}_4\right)$$

source code: F:\course\2008spring\spectral_method\matlab\p27.m

Since we have hide $\alpha_n = \exp\left(ik^3 t_n\right)$ into $\hat{u}^{(n)}$ (we don't evaluate α_n explicitly), hence we expect that Δt can be larger than $0.02N^{-2}$. Numerical simulation is shown in Figure 17, when $\Delta t = 0.4N^{-2}$, result is stable, this means **Algorithm 2** allows time steps ten times than **Algorithm 1**. However result is unstable when $\Delta t = 0.45N^{-2}$.

Question 4 (Exercise 10.2): Can you explain unstable of $\Delta t = 0.45N^{-2}$ with reference to stablility region of Runge-Kutta?

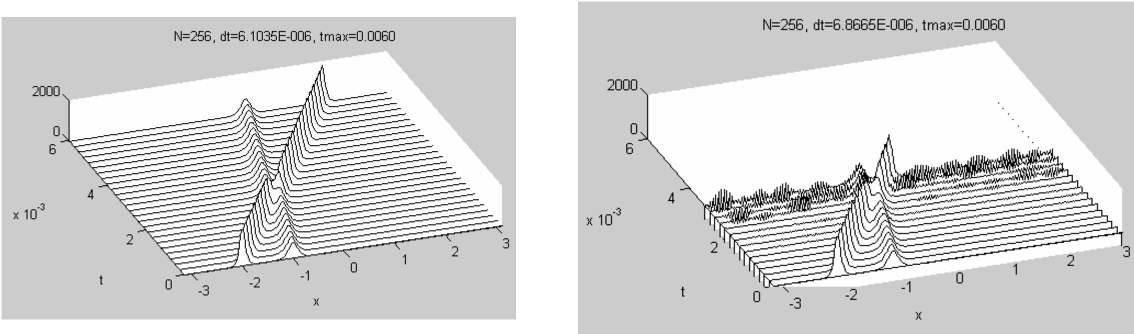


Figure 17: apply **Algorithm 2**, left panel: $\Delta t = 0.4N^{-2}$, stable. Right panel: $\Delta t = 0.45N^{-2}$, unstable.

Prop 4: If we apply 4-th Runge Kutta to model problem $\frac{d}{dt}u = \lambda u$ and define $\alpha = h\lambda$, and $u^{(n)} = g^n$ then $g = 1 + \alpha + \frac{1}{2}\alpha^2 + \frac{1}{6}\alpha^3 + \frac{1}{24}\alpha^4$. Stability region is $S = \left\{ \alpha \in Z : \left| g(\alpha) \right| \le 1 \right\}$

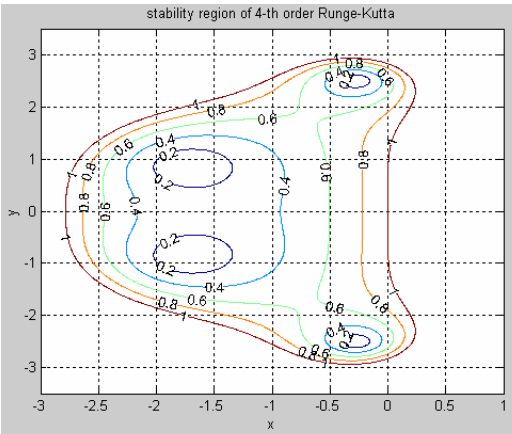


Figure 18: stability region of 4-th order Runge Kutta method.

Exercise 3 (10.3): Consider the first-order inear initial boundary value problem

(Eq. 19) $u_t = u_x$, $x \in [-1,1]$, $0 < t < 1$, $u(1,t) = 0$

with initial data $u_0(x) = u(0,x) = \exp\left[-60(x - 1/2)^2 \right]$. We use Chebyshev spectral discretization in space coupled with third-order Adams-Bashforth formula in time, say

(Eq. 20) $y_{n+1} = y_n + \frac{h}{12}\left[23f(t_n, y_n) - 16f(t_{n-1}, y_{n-1}) + 5f(t_{n-2}, y_{n-2}) \right]$

Take $N = 50$ and $\Delta t = \nu N^{-2}$ for $\nu = 7,8$. Plot $\varepsilon -$ *pseudospectrum* of Chebyshev spectral matrix for $\varepsilon = 10^{-2}, 10^{-3}, \cdots, 10^{-6}$.

In fact, analytic solution of $u_t = u_x$ is $u(t,x) = u_0(x + t)$, hence we can use analytic solution to give initial condition on $t_{-1} = -dt$ and $t_{-2} = -2dt$.

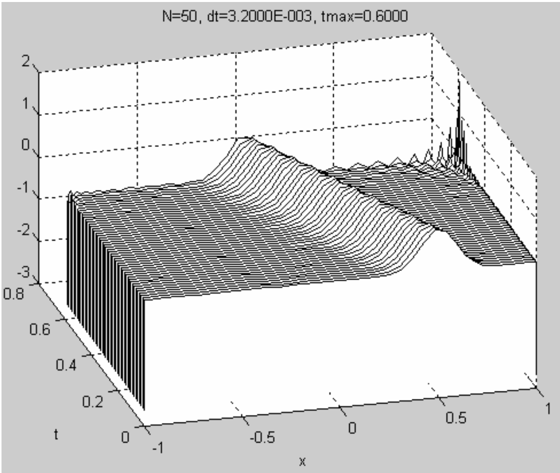
Experiment: Source code: F:\course\2008spring\spectral_method\matlab\chap10_ex3_v2.m

Let us set initial condition $u^{(0)} = \exp\left[-60(x - 1/2)^2 \right]$, $u^{(1)} = \exp\left[-60(x + dt - 1/2)^2 \right]$ and

$$u^{(2)} = \exp\left[-60\left(x+2dt-1/2\right)^2\right] \quad \text{and} \quad u^{(n+1)} = u^{(n)} + A\frac{h}{12}\left[23u^{(n)} - 16u^{(n-1)} + 5u^{(n-2)}\right]$$

where A is Chebyshev differential matrix deleting first row and column.

In right panel of Figure 19, $u(t,x)$ is propagated backward as we expected. But unstable mode occurs at boundary $x=1$ in left panel of Figure 19 since $\Delta t = 8N^{-2}$ is large.



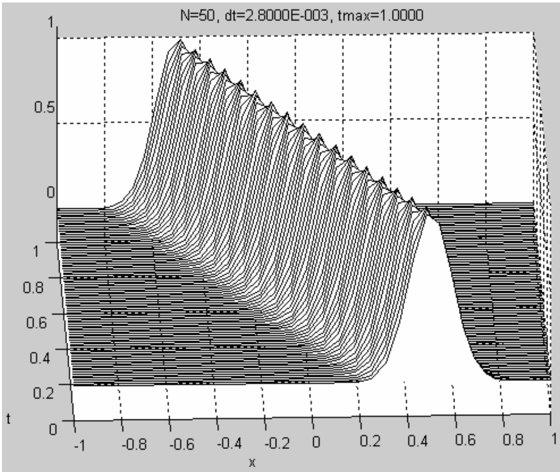


Figure 19: left panel: $\Delta t = 8N^{-2}$, unstable mode occurs at boundary. Right panel: $\Delta t = 7N^{-2}$, stable traveling wave.

From (Eq. 20), we have $g^3 = g^2 + \frac{h\lambda}{12}\left[23g^2 - 16g + 5\right]$, so we sweep all $|g| \le 1$ and plot region of $h\lambda$ (intersection of all $|g| \le 1$), see Figure 20.

Source code: **F:\course\2008spring\spectral_method\matlab\chap10_ex3.m**

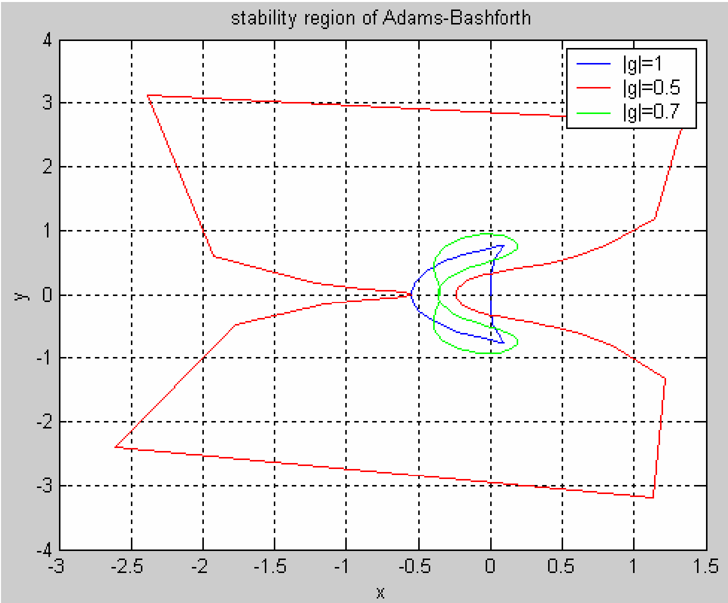


Figure 20: stability region of Adams-Bashforth

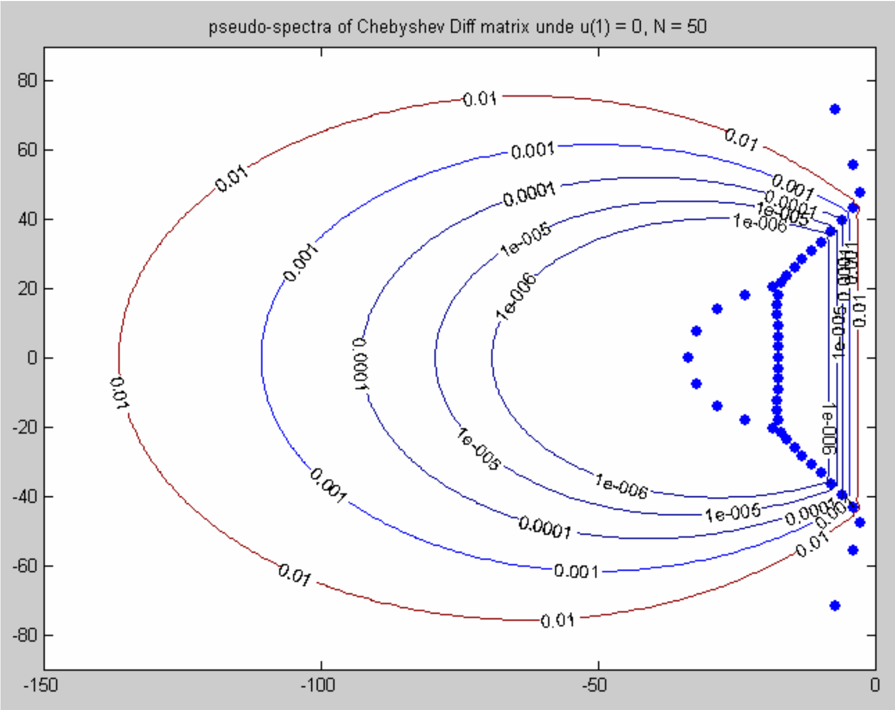


Figure 21: $\varepsilon - pseudospectrum$ of Chebyshev spectral matrix for $\varepsilon = 10^{-2}, 10^{-3}, \cdots, 10^{-6}$, $N = 50$