Linear optimization

Simplex method

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THE LINEAR PROGRAMMING PROBLEM

Standard – Form Linear Problem

$$\begin{aligned} &\text{Minimize} \quad z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ &\text{Subject to } \ \mathbf{Ax} = \mathbf{b}\,,\ \mathbf{x} \geq 0 \end{aligned}$$

where $\mathbf{x} = (\,x_1, \cdots, x_n)^T, \mathbf{c} = (\,c_1, \cdots, c_n)^T, \mathbf{b} = (\,b_1, \cdots, b_m)^T,$

and $\mathbf{A} =$ matrix of (a_{ij})

z : *objective function*
$\mathbf{c}$: *cost vector*
c_i : *cost coefficient* $\quad i = 1, \cdots, m$
$\mathbf{b}$: *right-hand-side vector*
b_i : *right-hand-side coefficient* $\quad i = 1, \cdots, m$
$\mathbf{A}$: *constraint matrix*
$\sum_{j=1}^{n} a_{ij}\mathrm{x}_j = b_i$: ith *technological constraint*

Embedded Assumption

1. Proportionality assumption
2. Additivity assumption
3. Divisibility assumption
4. Certainty assumption

Converting to Standard Form

Linear Inequalities and Equations.
If ith *technological constrain* has the form

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i$$

then add a nonnegative *slack variable* $s_i \geq 0$

$$\sum_{j=1}^{n} a_{ij}x_j + s_i = b_i$$

If ith *technological constrain* has the form

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i$$

then add a nonnegative *surplus variable* $e_i \geq 0$

$$\sum_{j=1}^{n} a_{ij}x_j - e_i = b_i$$

Restricted and Unrestricted Variables.

If $x_i \geq l_i$, then let $x_i = \overline{x_i} + l_i$ and $\overline{x_i} \geq 0$
If $x_i \leq u_i$, then let $x_i = u_i - \overline{x_i}$ and $\overline{x_i} \geq 0$
If $x_i \in R$, then let $x_i = \overline{x_i} - \widehat{x_i}$ and $\overline{x_i} \geq 0,\ \widehat{x_i} \geq 0$

Maximization and Minimization

$$\text{maximum}\left(\sum_{j=1}^{n} c_j x_j\right) = -\,\text{minimum}\left(\sum_{j=1}^{n} -c_j x_j\right)$$

Geometry of Linear Programming

BASIC TERMINOLOGY OF LINEAR PROGRAMMING

$P = \{\mathbf{x} \in R^n | \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$: feasible domain or feasible region when P is not empty, the linear program is said to be *consistent* .

For a consistent linear program with feasible solution $\mathbf{x}^*$, if $\mathbf{c}^{\mathrm{T}}\mathbf{x}^*$ attains the minimum, then x^* is an *optimal solution*.

if $\exists\ C$ s.t. $\mathbf{c}^{\mathrm{T}}\mathbf{x} \geq C \ \forall\, \mathbf{x} \in P$, then the linaer program is *bounded*.

SOME DEFNITION

hyperplane : $H = \{\mathbf{x} \in R^n \big| \mathbf{a}^{\mathrm{T}}\mathbf{x} = \beta\}$

closed half space : $H_L = \{\mathbf{x} \in R^n \middle| \mathbf{a}^\mathrm{T}\mathbf{x} \leq \beta\}$,

$H_U = \{\mathbf{x} \in R^n \middle| \mathbf{a}^\mathrm{T}\mathbf{x} \geq \beta\}$

normal of H : $\boldsymbol{a}$

hyperplane $H = \{\mathbf{x} \in R^n \middle| \mathbf{c}^\mathrm{T}\mathbf{x} = \beta\}$ **depict the contours of objective function**.

polyhedral set or *polyhedron* :
a set formed by the intersection of finite closed halfspaces.
polytope : nonempty and bounded polyhedron.

feasible domain P **is a polyhedral set**!

Let $\mathbf{x}^1, \cdots, \mathbf{x}^p \in R^n, \lambda_1, \cdots, \lambda_p \in R$

Linear combination : $\lambda_1\mathbf{x}^1 + \cdots + \lambda_p\mathbf{x}^p$

affine combination : Linear combination & $\lambda_1 + \cdots + \lambda_p = 1$

convex combination : affine combination & $0 \leq \lambda_1, \cdots, \lambda_p \leq 1$

For a set $S \subset R^n$ and $S \neq \phi$
If for any two points $\mathbf{x}^1, \mathbf{x}^2 \in S$, their every affine(convex) combination is contained in S, then we say S is affine(convex)

by above defintions, we know *feasible domain* is convex!

Separation Theorem

Let S be a convex subset of R^n and $\mathbf{x}$ be a boundary point of S. Then there is a hyperplane H containing $\mathbf{x}$ with S contained in either H_L or H_U.

supporting hyperplane H :
a hyperplane H that satisfies (i)$H \cap S \neq \phi$ and (ii)$H_L \subset S$.

Graphical method : the intersection set of the polyhedral set P

and the supporting hyperplane with the negative cost vector $-\mathrm{c}$ as its normal provides optimal solutions to our linear programming problem.

$F = P \cap H$ is called a face of P.
(1) F is a zero-dimension set, we have a *vertex*.
(2) one-dimension, an *edge*.
(3) one dimension less than P, a *facet*.

EXTREME POINTS AND BASIC FEASIBLE SOLUTIONS

A point $\mathbf{x}$ in a convex set C is said to be an *extreme point* of C if $\mathbf{x}$ is not a convex combination of any other two distincts points in C.

Let $\mathbf{A} = (\mathbf{A}_1 | \cdots | \mathbf{A}_n)$, we call column $\mathbf{A}_j$ the corresponding column

of the jth component x_j of $\mathbf{x}$.

<u>Theorem</u>
A point $\mathbf{x}$ of the polyhedral set P is an extreme point of P iff the columns of $\mathbf{A}$ corresponding to the positive components of $\mathbf{x}$ are linearly independent.

For an $m \times n$ matix $\mathbf{A}$ (assuming $m \leq n$), if there exist m linearly independent columns of $\mathbf{A}$, then we say $\mathbf{A}$ has *full row rank* or *full rank* in short.

in the case, we can group those m linearly independent columns together to form a *basis* $\mathbf{B}$ and leave the remaining $n-m$ columns as *nonbasis* $\mathbf{N}$. i.e. $\mathbf{A}$ can be rearrange to be $[\mathbf{B} \mid \mathbf{N}]$
also $\mathbf{x} = \left(\frac{\mathbf{x}_B}{\mathbf{x}_N}\right)$, componet of $\mathbf{x}_B$: basic variables
componet of $\mathbf{x}_N$: nonbasic variables
sine $\mathbf{B}$ is invertible, we can set $\mathbf{x}_\mathrm{N} = \mathbf{0}$, then solve $\mathbf{B}\mathbf{x}_\mathrm{B} = \mathbf{b}$
The vector $\mathbf{x}$ becomes a *basic solution*.

if $\mathbf{x}_B = \mathrm{B}^{-1}\mathrm{b} \geq 0$, then we say $\mathbf{x}$ is a *basic feasible solution*.

Note : There are at most $C(n,m)$ basic solutions.

<u>Corollary.</u>
$\mathbf{x}$ is an extreme point of P iff $\mathbf{x}$ is a basic feasible solution corresponding to some basis $\mathbf{B}$ **.**

<u>Corollary.</u>
For a given linear program in its standard form, there are at most $C(n,m)$ extreme point in P.

NONDEGENERACY AND ADJACENCY

we define a basic feasible solution to be *nondegenerate*, if it has exactly m positive basic variables. Otherwise, we call *degenerate*. Moreover, we say a linear programming problem is *nondegenerate* if all basic feasible solution are nondegenerate. In this case, there is a one-to-one corespondence between the extreme point and basic feasible solution .

Two basic feasible solution of P are *adjacent*, if they use $m - 1$ basic variables in common to form basis.

RESOLUTION THEOREM FOR CONVEX POLYHEDRONS

An *extremal direction* of a polyhedral set P is a nonzero vector $\mathbf{d} \in R^n$ s.t. $\forall \mathbf{x}^0 \in P$ the ray $\{\mathbf{x} \in R^n | \mathbf{x} = \mathbf{x}^0 + \lambda\mathbf{d}, \lambda \geq 0\} \subseteq P$.

$\mathbf{d}$ is an extremal direction iff $\mathbf{Ad} = 0$, $\mathbf{d} \geq 0$.
P is unbounded iff P has an extremal direction.

<u>Resolution Theorem</u>

Let $V = \{\mathbf{v}^i \in R^n | i \in I\}$ be the set of all extreme points of P with a finite index set I. Then for each $\mathbf{x} \in P$, we have

$$\mathbf{x} = \sum_{i \in I} \lambda_i \mathbf{v}^i + \mathbf{d} \quad , \qquad \text{where} \sum_{i \in I} \lambda_i = 1 \, , \ \lambda_i \geq 0 \ , \qquad \forall i \in I$$

and $\mathbf{d}$ is either the zero vector or an extremal direction of P.

Corollary .

If P is bounded(a polytope), then each $\mathbf{x} \in P$ is a convex combination of the extreme points of P .

Corollary .

If P is nonempty, then it has at least one extreme point .

Fundamental Theorem of Linear Programming

For a consistent linear program in its standard form with a feasible domain P, the minmum objective value $z = \mathbf{c}^{\mathrm{T}}\mathbf{x}$ over P is either unbounded below or is achievable at least at one extreme point of P.

The Revised Simplex Method

(1)How do we start with an extreme point ?

(2)How do we move from one extreme point to a better neighboring extreme point in an efficient way ?

(3)When do we stop the process ?

For a given basic feasible solution $\mathbf{x}^* = \left(\frac{\mathbf{x}_B^*}{\mathbf{x}_N^*}\right)$,

$$\mathbf{A} = (\mathbf{B}|\mathbf{N}) \text{ and } \mathbf{c} = \left(\frac{\mathbf{c}_B}{\mathbf{c}_N}\right)$$

the linear programming problem becomes

$$\text{Minimize } z = \mathbf{c}_B^T \mathbf{x}_B + \mathbf{c}_N^T \mathbf{x}_N$$

$$\text{subject to } \mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathrm{b};\ \mathbf{x}_B \geq 0;\ \mathbf{x}_N \geq 0$$

$$\begin{aligned}
\text{s.t.}\quad \mathbf{x}_B &= \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N \\
z &= \mathbf{c}_B^T (\mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N) + \mathbf{c}_N^T \mathbf{x}_N \\
&= \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} + (\mathbf{c}_N^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N})\mathbf{x}_N \\
&= \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} + \mathbf{r}^T \left(\frac{\mathbf{x}_B}{\mathbf{x}_N}\right)
\end{aligned}$$

where $\mathbf{r} = \left(\dfrac{0}{\mathbf{c}_N - \mathbf{B}^{-1}\mathbf{N}^T\mathbf{c}_B}\right)$

$$z - z^* = \mathbf{r}^T \left(\frac{\mathbf{x}_B}{\mathbf{x}_N}\right) \qquad \forall\, \mathbf{x} \in P$$

Note : if every component of $\mathbf{c}_N - \mathbf{B}^{-1}\mathbf{N}^T\mathbf{c}_B$ is nonnegative, then $z - z^* \geq 0 \quad \forall\, \mathbf{x} \in P$, i.e. $\mathbf{x}^*$ is optimal !
we named $\mathbf{r}$ the *reduced cost vector*.

Conclusion :

If $\mathbf{x}^* = \left(\dfrac{\mathbf{x}_B^*}{\mathbf{x}_N^*}\right) = \left(\dfrac{\mathbf{B}^{-1}\mathbf{b}}{0}\right) \geq 0$ is a basic feasible solution with nonnegative reduced costs vector $\mathbf{r}$, then $\mathbf{x}^*$ is an optimal solutin to the linear programming problem.

Stopping rule of simplex method
− checking for optimality

assume $\widetilde{\mathbf{B}}$ is the index set of basic variables in $\mathbf{x}^*$,
and $\widetilde{\mathbf{N}}$ is the index set of nonbasic variables in $\mathbf{x}^*$,

if $r_q = c_q - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N}_q \geq 0$ for each $q \in \widetilde{\mathbf{N}}$

then we can terminate the simplex method with an optimal solutin $\mathbf{x}^*$.

Note : $\mathbf{N}_q = \mathbf{A}_q$ for each $q \in \widetilde{\mathbf{N}}$

Iterations of the Simplex Method
− Moveing for Improvement

Direction of Translation

$$\mathbf{x}_B = \mathbf{B}^{-1}(\mathbf{b} - x_q\mathbf{A}_q)$$

$$\mathbf{d}^q = \left(\frac{-\mathbf{B}^{-1}\mathbf{A}_q}{\mathbf{e}_q}\right) \quad \text{for each } q \in \widetilde{\mathbf{N}}$$

$$\mathbf{A}\mathbf{d}^q = (\mathbf{B}|\mathbf{N})\begin{pmatrix}-\mathbf{B}^{-1}\mathbf{A}_q\\ \hline \mathbf{e}_q\end{pmatrix} = \mathbf{A}_q - \mathbf{N}_q = \mathbf{0}$$

$$\mathbf{c}^{\mathrm{T}}(\mathbf{x}^* + \alpha\mathbf{d}^q) < \mathbf{c}^{\mathrm{T}}\mathbf{x}^* \quad \text{for } \alpha > 0.$$

$$\mathbf{c}^{\mathrm{T}}\mathbf{d}^q = (\mathbf{c}_B^T|\mathbf{c}_N^T)\begin{pmatrix}-\mathbf{B}^{-1}\mathbf{A}_q\\ \hline \mathbf{e}_q\end{pmatrix} = c_q - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{A}_q < 0$$

i.e. $r_q < 0$ assure the corresponding edge direction is a good direction of translation .

Conclusion :

Let $\mathbf{x}^* = \begin{pmatrix}\mathbf{B}^{-1}\mathbf{b}\\ \hline 0\end{pmatrix} \geq 0$ be a basic feasible solution . If $r_q < 0$ for some nonbasic variable x_q , then the edge direction $\mathbf{d}^q$ leads to an improvement in the objective value .

And if there is a feasible edge direction $\mathbf{d}^q \geq \mathbf{0}$ with $r_q < 0$, then the linear programming problem is unbounded below.

Step Length

$$\alpha = \underset{j \in \widetilde{\mathbf{B}}}{\text{Minimum}} \left(-\frac{x_{\mathrm{j}}^*}{d_{\mathrm{j}}^q} \middle| d_{\mathrm{j}}^q < 0 \right) \quad \textit{minimun ratio test}$$

STARTING THE SIMPLEX METHOD

Two – Phase Method

consider $\mathbf{x}^a = (x_{\mathbf{1}}^{\boldsymbol{a}}, x_{\mathbf{2}}^{\boldsymbol{a}}, \cdots, x_{\boldsymbol{m}}^{\boldsymbol{a}})^T \in R^m$ *artificial variables*

Phase I problem :

$$\text{Minimize} \quad z = \sum_{i=1}^{m} x_{\boldsymbol{i}}^{\boldsymbol{a}}$$

Subject to $\mathbf{Ax} + \mathbf{x}^a = \mathbf{b}$; $\mathbf{x} \geq 0$; $\mathbf{x}^a \geq 0$

$\left(\frac{\mathbf{x}}{\mathbf{x}^{a*}}\right) = \left(\frac{\mathbf{0}}{\mathbf{b}}\right)$ is a basic feasible solution to Phase I problem

Big − M Method

impose a large penalty $M > 0$ *for artificial variables, then solve*

$$\text{Minimize} \quad z = \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} M x_{\boldsymbol{i}}^{\boldsymbol{a}}$$

Subject to $\mathbf{Ax} + \mathbf{x}^a = \mathbf{b}$; $\mathbf{x} \geq 0$; $\mathbf{x}^a \geq 0$

$\left(\frac{\mathbf{x}}{\mathbf{x}^{a*}}\right) = \left(\frac{\mathbf{0}}{\mathbf{b}}\right)$ is a starting basic feasible solution.

In fact, how big the M we need choose is not easy to determine.

THE REVISED SIMPLEX METHOD

Step 1 : Compute the "simplex multipliers" $\mathbf{w}$, by solving

$$\mathbf{B}^T\mathbf{w} = \mathbf{c}_B$$

Step 2 : Compute the reduced costs

$$r_j = c_j - \mathbf{w}^T A_j \quad \forall\, j \notin \widetilde{\mathbf{B}}.$$

Step 3(check for optimality): If $r_j \geq 0 \quad \forall\, j \notin \widetilde{\mathbf{B}}$. then STOP.

The current solution is OPTIMAL.

Step 4(enter the basis): Choose $q \notin \widetilde{\mathbf{B}}$ s.t. $r_q < 0$.

Step 5(edge direction): Compute $\mathbf{d}^q$ by solving

$$\mathbf{Bd} = -\mathbf{A}_q \text{ and set } \mathbf{d}^q = \left(\frac{\mathbf{d}}{\mathbf{e}_q}\right).$$

Step 6(check for unboundedness): If $\mathbf{d} \geq \mathbf{0}$, then STOP.

The problem is unbounded below.

Step 7(leave the basis and step − length):

Find an index j_p and step − length α according to

$$\alpha = -\frac{x_{j_p}}{d_{j_p}} = \min_{1 \le i \le m} \left(-\frac{x_{j_i}}{d_{j_i}} \middle| d_{j_i} < 0 \right)$$

Step 8(**update**): Set

$$x_q \leftarrow \alpha$$

$$x_{j_i} \leftarrow x_{j_i} + \alpha d_{j_i} \quad \text{for} \quad 1 \le i \le m$$

$$\mathbf{B} \leftarrow \mathbf{B} + \left[\mathbf{A}_q - \mathbf{A}_{j_p}\right] \mathbf{e}_{\boldsymbol{p}}^{\boldsymbol{T}}$$

$$\widetilde{\mathbf{B}} \leftarrow \widetilde{\mathbf{B}} \cup \{q\} \backslash \{j_p\}$$

Go to **Step 1** .

<u>Reference</u>

Shu – Cherng Fang , Sarat Puthenpura , "*Linear Optimization and Extentions*."