On the $\star$-Sylvester Equation $AX \pm X^\star B^\star = C$

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Outline

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- In [Braden1998], the Lyapunov-like linear matrix equation

$$A^\star X + X^\star A = B \ , \quad A, X \in \mathbb{C}^{m\times n} \ \ (m \neq n)$$

with $(\cdot)^\star = (\cdot)^T$ was considered using generalized inverses. Applications occur in Hamiltonian mechanics.

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- In this talk, we consider the $\star$-Sylvester equation

$$AX \pm X^\star B^\star = C \ , \quad A, B, X \in \mathbb{C}^{n\times n}. \tag{1.1}$$

This includes the special cases of the T-Sylvester equation when $\star = T$ and the H-Sylvester equation when $\star = H$.

- Some related equations of (1.1), e.g., $AXB^\star \pm X^\star = C$, $AXB^\star \pm CX^\star D^\star = E$, $AX \pm X^\star A^\star = C$, $AX \pm YB = C$, $AXB \pm CYD = E$, $AXA^\star \pm BYB^\star = C$ and $AXB \pm (AXB)^\star = C$ will also be studied.

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- Our tools include the (generalized and periodic) Schur, (generalized) singular value and QR decompositions.
- An interesting application, for the $\star$-Sylvester equation (1.1)

$$AX \pm X^\star B^\star = C \ , \quad A, B, X \in \mathbb{C}^{n\times n}$$

arises from the eigensolution of the palindromic linearization [Chu2007]

$$(\lambda Z + Z^{\star})x = 0 \ , \quad Z = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{C}^{2n \times 2n}.$$

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Appying congruence, we have

$$\begin{bmatrix} I_n & 0 \\ X & I_n \end{bmatrix} (\lambda Z + Z^\star) \begin{bmatrix} I_n & X^\star \\ 0 & I_n \end{bmatrix} =$$

$$\begin{bmatrix} \lambda A + A^\star & \lambda(AX^\star + B) + (XA + C)^\star \\ \lambda(XA + C) + (AX^\star + B)^\star & \lambda\mathcal{R}(X) + \mathcal{R}(X)^* \end{bmatrix}$$

with

$$\mathcal{R}(X) \equiv XAX^\star + XB + CX^\star + D.$$

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How solve the $\star$-Riccati equation?

$$\mathcal{R}(X) = 0.$$

- If we can solve $\mathcal{R}(X) = 0$, then the palindromic linearization can then be "square-rooted". We then have to solve the generalized eigenvalue problem for the pencil $\lambda(AX^\star + B) + (XA + C)^\star$, with the reciprocal eigenvalues in $\lambda(XA + C) + (AX^\star + B)^\star$ obtained for free.

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- the application of Newton's method lead to the iterative process

$$\delta X_{k+1}(AX_k^\star + B) + (X_k A + C)\delta X_{k+1}^\star = -\mathcal{R}(X_k)$$

which is a $\star$-Sylvester equation for δX_{k+1}.

Outline

Recalled the $\star$-Sylvester equation (1.1)

$$AX \pm X^{\star}B^{\star} = C \ , \quad A, B, X \in \mathbb{C}^{n\times n}.$$

With the Kronecker product, (1.1) can be written as

$$\mathcal{P}\,\mathrm{vec}(X) = \mathrm{vec}(C) \ , \quad \mathcal{P} \equiv I \otimes A \pm (B \otimes I)E, \qquad (2.1)$$

where

$$A \otimes B \equiv [a_{ij}B] \in \mathbb{C}^{n^2\times n^2}, \ \mathrm{vec}(X) \equiv \begin{bmatrix} X(:,1) \\ X(:,2) \\ \vdots \\ X(:,n) \end{bmatrix} \in \mathbb{C}^{n^2\times 1}.$$

And,

$$E \equiv \sum_{1\le i,j\le n} e_i e_j^\top \otimes e_j e_i^\top, \quad e_i = \begin{bmatrix} \vdots \\ 1 \\ \vdots \end{bmatrix}.$$

Note that $E\,\mathrm{vec}(X) = \mathrm{vec}(X^T)$, $E(A \otimes B)E = B \otimes A$.

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- The application of Gaussian elimination and the like will be inefficient.
- The approach ignores the structure of the original problem.

Another approach will be to transform (1.1) by some unitary P and Q, so that (1.1) becomes:

- For $\star = T$:

$$PAQ \cdot \overline{Q}^T XP^T \pm PX^T\overline{Q} \cdot Q^T B^T P^T = PCP^T. \qquad (2.2)$$

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- For $\star = H$:

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Let $(Q^H A^H P^H, Q^H B^H P^H)$ be in (upper-triangular) generalized Schur form.

The transformed equations in (2.2) and (2.3) then have the form

$$\begin{bmatrix} a_{11} & 0^T \\ a_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12}^\star \\ x_{21} & X_{22} \end{bmatrix} \pm \begin{bmatrix} x_{11}^\star & x_{21}^\star \\ x_{12} & X_{22}^\star \end{bmatrix} \begin{bmatrix} b_{11}^\star & b_{21}^\star \\ 0 & B_{22}^\star \end{bmatrix} \quad (2.4)$$

$$= \begin{bmatrix} c_{11} & c_{12}^\star \\ c_{21} & C_{22} \end{bmatrix}.$$

Multiply the matrices out, we have

$$
\begin{aligned}
a_{11}x_{11} \pm b_{11}^{\star}x_{11}^{\star} &= c_{11}, && (2.5)\\
a_{11}x_{12}^{\star} \pm x_{21}^{\star}B_{22}^{\star} &= c_{12}^{\star} \mp x_{11}^{\star}b_{21}^{\star}, && (2.6)\\
A_{22}x_{21} \pm b_{11}^{\star}x_{12} &= c_{21} - x_{11}a_{21}, && (2.7)\\
A_{22}X_{22} \pm X_{22}^{\star}B_{22}^{\star} &= \widetilde{C}_{22} \equiv C_{22} - a_{21}x_{12}^{\star} \mp x_{12}b_{21}^{\star}. && (2.8)
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A_{22}X_{22} \pm X_{22}^{\star}B_{22}^{\star} &= \widetilde{C}_{22} \equiv C_{22} - a_{21}x_{12}^{\star} \mp x_{12}b_{21}^{\star}. &(2.8)
\end{aligned}$$

From (2.5) for $\star = T$, we have

$$(a_{11} \pm b_{11})x_{11} = c_{11}. \tag{2.9}$$

Let $\lambda_1 \equiv a_{11}/b_{11} \in \sigma(A, B)$. The solvability condition of the above equation is

$$a_{11} \pm b_{11} \neq 0 \Leftrightarrow \lambda_1 \neq \mp 1. \tag{2.10}$$

From (2.5) when $\star = H$, we have

$$a_{11}x_{11} \pm \overline{b}_{11}\overline{x}_{11} = c_{11}. \tag{2.11}$$

To solve (2.11) is to write it together with its complex conjugate in the composite form

$$\begin{bmatrix} a_{11} & \pm b_{11}^{\star} \\ \pm b_{11} & a_{11}^{\star} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{11}^{\star} \end{bmatrix} = \begin{bmatrix} c_{11} \\ c_{11}^{\star} \end{bmatrix}.$$

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The determinant of the matrix operator in above:

$$d = |a_{11}|^2 - |b_{11}|^2 \neq 0 \Leftrightarrow |\lambda_{11}| \neq 1. \tag{2.12}$$

requiring that no eigenvalue $\lambda \in \sigma(A, B)$ lies on the unit circle.

From (2.6) and (2.7), we obtain

$$\begin{bmatrix} a_{11}^\star I & \pm B_{22} \\ \pm b_{11}^\star I & A_{22} \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{21} \end{bmatrix} = \begin{bmatrix} \widetilde{c}_{12} \\ \widetilde{c}_{21} \end{bmatrix} \equiv \begin{bmatrix} c_{12} \\ c_{21} \end{bmatrix} + x_{11} \begin{bmatrix} \mp b_{21} \\ -a_{21} \end{bmatrix}. \tag{2.13}$$

With $a_{11} = b_{11} = 0$, x_{11} will be undetermined. However, $\sigma(A, B) = \mathbb{C}$ and this case will be excluded by (2.16).

- If $a_{11} \neq 0$, (2.13) is then equivalent to

$$\begin{bmatrix} a_{11}^\star I & \pm B_{22} \\ 0 & A_{22} - \frac{b_{11}^\star}{a_{11}^\star} B_{22} \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{21} \end{bmatrix} = \begin{bmatrix} \widetilde{c}_{12} \\ \widehat{c}_{21} \end{bmatrix} \equiv \begin{bmatrix} \widetilde{c}_{12} \\ \widetilde{c}_{21} \mp \frac{b_{11}^\star}{a_{11}^\star} \widetilde{c}_{12} \end{bmatrix}. \tag{2.14}$$

The solvability condition of (2.13) and (2.14) is

$$\det \widetilde{A}_{22} \neq 0 \ , \quad \widetilde{A}_{22} \equiv A_{22} - \frac{b_{11}^\star}{a_{11}^\star} B_{22}$$

or that λ and $\lambda^{-\star}$ cannot be in $\sigma(A, B)$ together. Note that $\widetilde{A}_{22}$ is still lower-triangular, just like A or B.

- If $b_{11} \neq 0$, (2.13) is equivalent to

$$\begin{bmatrix} 0 & B_{22} - \frac{a_{11}^{\star}}{b_{11}^{\star}} A_{22} \\ b_{11}^{\star} I & \pm A_{22} \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{21} \end{bmatrix} = \begin{bmatrix} \widehat{c}_{12} \\ \pm\widetilde{c}_{21} \end{bmatrix} \equiv \begin{bmatrix} \pm\widetilde{c}_{12} - \frac{a_{11}^{\star}}{b_{11}^{\star}}\widetilde{c}_{21} \\ \pm\widetilde{c}_{21} \end{bmatrix} \tag{2.15}$$

  with an identical solvability condition (2.16). Lastly, (2.8) is of the same form as (1.1) but of smaller size.

We summarize the solvability condition for (1.1) in the following theorem:

Theorem1

The ⋆-Sylvester equation (1.1):

$$AX \pm X^{\star}B^{\star} = C \ , \quad A, B \in \mathbb{C}^{n\times n}$$

is uniquely solvable if and only if the condition:

$$\lambda \in \sigma(A, B) \Rightarrow \lambda^{-\star} \notin \sigma(A, B) \tag{2.16}$$

is satisfied. Here, the convention that 0 and ∞ are mutually reciprocal is followed.

The process in this subsection is summarized below: (with BS denoting back-substitution)

Algorithm SSylvester

(For the unique solution of $AX \pm X^{\star}B^{\star} = C$; $A, B, C, X \in \mathbb{C}^{n\times n}$.)

- `Compute` the lower-triangular generalized Schur form (PAQ, PBQ) using QZ.
- `Store` (PAQ, PBQ, PCP^{*}) in (A, B, C).
- `Solve` (2.9) for $\star = T$, or (2.11) for $\star = H$; `if` fail, `exit`.

If $n = 1$ or $|a_{11}|^2 + |b_{11}|^2 \leq$ tolerance, exit.
If $|a_{11}| \geq |b_{11}|$, then
if $\widetilde{A}_{22} \equiv A_{22} - \frac{b_{11}^\star}{a_{11}^\star} B_{22}$ has any negligible diagonal elements, then exit.
Else if $\widetilde{B}_{22} \equiv B_{22} - \frac{a_{11}^\star}{b_{11}^\star} A_{22}$ has any negligible diagonal elements, then exit.
Else compute $x_{21} = \widetilde{B}_{22}^{-1} \widehat{c}_{12}$ by BS, $x_{12} = \left(\pm \widetilde{c}_{21} \mp A_{22} x_{21}\right) / b_{11}^\star$ (c.f. (2.15)).

- Apply **Algorithm TSylvester** to $A_{22} X_{22} \pm X_{22}^\star B_{22}^\star = \widetilde{C}_{22}$, $n \leftarrow n - 1$.
- Output $X \leftarrow QX\overline{P}$ for $\star = T$, or $X \leftarrow QXP$ for $\star = H$.

End of algorithm

Cost

Let the operation count of the Algorithm SSylvester be $f(n)$ complex flops, we assume that A, B are lower-triangular matrices. (After a QZ procedure $(66n^3)$)

- Inverting $\widetilde{A}_{22} \equiv A_{22} - \frac{b_{11}^\star}{a_{11}^\star} B_{22}$ or $\widetilde{B}_{22} \equiv B_{22} - \frac{a_{11}^\star}{b_{11}^\star} A_{22}$ $(\frac{1}{2}n^2$ flops).
- Computing $x_{21} = \widetilde{B}_{22}^{-1} \widehat{c}_{12}$ and $x_{12} = (\pm \widetilde{c}_{21} \mp A_{22} x_{21}) / b_{11}^\star$ $(n^2$ flops).
- Forming $\widetilde{C}_{22} = C_{22} - a_{21} x_{12}^\star \mp x_{12} a_{21}^\star$ $(2n^2)$.

Thus $f(n) \approx f(n-1) + \frac{7}{2}n^2$, ignoring $O(n)$ terms. This implies that $f(n) \approx \frac{7}{6}n^3$ and the total operation count for Algorithm SSylvester is $67\frac{1}{6}n^3$ complex flops, ignoring $O(n^2)$ terms.

Solvability Condition

- The solvability condition of

$$AX \pm X^{\star} B^{\star} = C$$

is obviously identical to that of

$$\mathcal{P}\, \mathrm{vec}(X) = \mathrm{vec}(C), \quad \mathcal{P} = I \otimes A \pm (B \otimes I) E.$$

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However, E reshuffles the columns of $B \otimes I$, making the analysis of the matrix operator $\mathcal{P}$ difficult.

- Consider the trivial example when $n = 2$, $A = [a_{ij}]$ and $B = [b_{ij}]$ are lower-triangle matrices, we have

$$\mathcal{P} = \begin{bmatrix} a_{11} \pm b_{11}^\star & & & \\ \pm b_{21}^\star & a_{11} & \pm b_{22}^\star & \\ a_{21} & \pm b_{11}^\star & a_{22} & \\ & a_{21} \pm b_{21}^\star & & a_{22} \end{bmatrix}.$$

The eigenvalues of the corresponding $\widetilde{\mathcal{P}}$ are $\lambda_{ii} = a_{ii} \pm b_{ii}^{\star}$ $(i = 1, 2)$ and those of the middle block W_{12} where

$$W_{ij} \equiv \begin{bmatrix} a_{ii} & \pm b_{jj}^{\star} \\ \pm b_{ii}^{\star} & a_{jj} \end{bmatrix}.$$

The characteristic polynomial of W_{ij}, identical to that for W_{ji}, is $\lambda^2 - (a_{ii} + a_{jj})\lambda + \det W_{ij}$ with $\det W_{ij} = a_{ii}a_{jj} - b_{ii}^{\star}b_{jj}^{\star}$, and the eigenvalues are

$$\lambda_{W_{ij}} = \frac{1}{2}\left[a_{ii} + a_{jj} \pm \sqrt{(a_{ii} - a_{jj})^2 + 4b_{ii}^{\star}b_{jj}^{\star}}\right].$$

Note that some λ_{ii} or $\lambda_{W_{ij}} = 0$ if and only if (2.16) in Theorem 1 is violated.

Outline

Example1

- In MATLAB commands: $\hat{A} = tril(randn(n), -1) + diag(a)$, $\hat{B} = tril(randn(n), -1) + diag(b)$ and $C = randn(n)$, where $a, b \in \mathbb{R}^n$.
- To guarantee condition (2.16), let $b = randn(n, 1)$, $a = 2b$. In Table1, we list the CPU time ratios, corresponding residuals and their ratios, with increasing dimensions $n = 16, 25, 30, 35, 40$.
- Note that the operation counts for the SSA and KRP methods are approximately $67n^3$ and $\frac{2}{3}n^6$ flops respectively (the latter for the LU decomposition of the $n^2 \times n^2$ matrix in (2.1).

Table1: Results for Example1

n	$\frac{t_{KRP}}{t_{ASS}}$	Res(ASS)	Res(KRP)	$\frac{\mathrm{Res(KRP)}}{\mathrm{Res(ASS)}}$
16	1.00e+00	1.8527e-17	2.1490e-17	1.16
25	1.31e+01	2.3065e-17	2.8686e-17	1.24
30	2.61e+01	3.1126e-18	5.7367e-18	2.20
35	6.48e+01	7.0992e-18	1.2392e-17	1.75
40	1.05e+02	1.7654e-18	6.4930e-18	3.68

- The results in Table1 show that the advantage of ASS over KRP in CPU time grows rapidly as n increases, as predicted by the operation counts.

- The results in Table1 show that the advantage of ASS over KRP in CPU time grows rapidly as n increases, as predicted by the operation counts.
- Even with better management of sparsity or parallellism, the $O(n^6)$ operation count makes the KRP approach uncompetitive even for moderate size n. The residuals from ASS is also better than that from KRP, as (2.1) is solved by Gaussian elimination in an unstructured way.

Example2

- We let $a = [\alpha + \epsilon, \beta]^\top$, $b = [\beta, \alpha]^\top$, where α, β are two randomly numbers greater than 1, with the spectral set $\sigma(A, B) = \{\frac{\alpha+\varepsilon}{\beta}, \frac{\beta}{\alpha}\}$, and $|\lambda_1\lambda_2 - 1| = \frac{\varepsilon}{\alpha}$. Judging from (2.16), (1.1) has worsening condition when ε decreases.
- We report a comparison of absolute residuals for the ASS and KRP approaches for $\varepsilon = 10^{-1}, 10^{-3}, 10^{-5}, 10^{-7}$ and 10^{-9} in Table2.
- The results show that if (2.1) is solved by Guassian elimination, its residual will be larger than that for ASS especially for smaller ε. Note that the size of X reflects partially the condition of (1.1). The KRP approach copes less well than the ASS approach for an ill-conditioned problem.

Table2: Results for Example2

ϵ	Res(ASS)	Res(KRP)	$\frac{\mathrm{Res(KRP)}}{\mathrm{Res(ASS)}}$	$O(\|X\|)$
1.0e-1	2.0673e-15	2.4547e-15	1.19	10^1
1.0e-3	8.6726e-13	4.3279e-13	0.50	10^3
1.0e-5	2.3447e-12	2.4063e-12	1.03	10^3
1.0e-7	5.9628e-10	1.1786e-09	1.98	10^6
1.0e-9	5.8632e-08	3.4069e-07	5.81	10^8

Outline

1. Introduction
2. ⋆-Sylvester Equation
3. Numerical Examples
4. Related Equations
5. Conclusions

Generalized $\star$-Sylvester equation I

The more general version of the $\star$-Sylvester equation I:

$$AXB^{\star} \pm X^{\star} = C$$

with $A, B^{\star}, X^{\star} \in \mathbb{C}^{m \times n}$ is uniquely solvable if and only if the condition:

$$\lambda \in \sigma(AB) \Rightarrow \lambda^{-\star} \notin \sigma(AB)$$

is satisfied.

Generalized ⋆-Sylvester equation II

The more general version of the ⋆-Sylvester equation II:

$$AXB^{\star} \pm CX^{\star}D^{\star} = E$$

is uniquely solvable if and only if the condition:

$$\lambda_1, \lambda_2 \in \sigma(A, C); \lambda_3, \lambda_4 \in \sigma(D, B) \;\Rightarrow \lambda_1\lambda_2 \neq \lambda_3\lambda_4$$

is satisfied.

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- We have considered the solution of the $\star$-Sylvester equation which has not been fully investigated before. The solvability conditions have been derived and algorithms have been proposed.

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- We have considered the solution of the $\star$-Sylvester equation which has not been fully investigated before. The solvability conditions have been derived and algorithms have been proposed.
- It is interesting and exciting that the $\star$ above the second X in (1.1) makes the equation behave very differently.
  1. For the ordinary continuous-time Sylvester equation $AX \pm XB = C$, the solvability condition is: $\sigma(A) \cap \sigma(\mp B) = \phi$.
  2. For the ordinary discrete-time Sylvester equation $X \pm AXB = C$, the the solvability condition is: if $\lambda \in \sigma(A)$, then $\mp\lambda^{-1} \notin \sigma(B)$.

  (1.1) looks like a Sylvester equation associated with continuous-time but (2.16) is satisfied when $\sigma(A, B)$ in totally inside the unit circle, hinting at a discrete-time type of stability behaviour.

Thank you for your attention!