An interesting application, for the $\star$-Sylvester equation

$$AX \pm X^\star B^\star = C \ , \quad A, B, X \in \mathbb{C}^{n\times n}$$

arises from the eigensolution of the palindromic linearization

$$(\lambda Z + Z^\star)x = 0 \ , \quad Z = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{C}^{2n\times 2n}.$$

Appying congruence, we have

$$\begin{bmatrix} I_n & 0 \\ X & I_n \end{bmatrix} (\lambda Z + Z^\star) \begin{bmatrix} I_n & X^\star \\ 0 & I_n \end{bmatrix} =$$

$$\begin{bmatrix} \lambda A + A^\star & \lambda(AX^\star + B) + (XA + C)^\star \\ \lambda(XA + C) + (AX^\star + B)^\star & \lambda\mathcal{R}(X) + \mathcal{R}(X)^* \end{bmatrix}$$

with

$$\mathcal{R}(X) \equiv XAX^\star + XB + CX^\star + D. \tag{0.1}$$

If we can solve the $\star$-Riccati equation

$$\mathcal{R}(X) = 0$$

the palindromic linearization can then be "square-rooted". We then have to solve the generalized eigenvalue problem for the pencil $\lambda(AX^\star + B) + (XA + C)^\star$, with the reciprocal eigenvalues in $\lambda(XA + C) + (AX^\star + B)^\star$ obtained for free.

It is easy to show from the $\star$-Riccati equation that its solution corresponds to the (stabilizing) deflating subspaces of $\lambda Z + Z^\star$ spanned by

$$(S_1, S_2) \equiv \left(\begin{bmatrix} X^\star \\ I \end{bmatrix}, \begin{bmatrix} I \\ -X \end{bmatrix} \right).$$

It turns out that the palindromic symmetry in the problem leads to the orthogonality property $S_1^\star S_2 = 0$, allowing the above congruence to annihilate the lower-right corner of the transformed pencil, thus square-rooting the problem.

Solving the $\star$-Riccati equation is of course as difficult as the original eigenvalue problem of $\lambda Z + Z^\star$. The usual invariance/deflating subspace approach for Riccati equations leads back to the original difficult eigenvalue problem.

Let

$$R = AX^* + B,$$
$$S = A^*X^* + C^*,$$

then we have

$$Z \begin{bmatrix} X^* \\ I \end{bmatrix} = \begin{bmatrix} X^* \\ I \end{bmatrix} R$$
$$Z^* \begin{bmatrix} I \\ -X \end{bmatrix} = \begin{bmatrix} I \\ -X \end{bmatrix} S,$$

thus,

$$Z^{-*} Z \begin{bmatrix} X^* \\ I \end{bmatrix} = \begin{bmatrix} X^* \\ I \end{bmatrix} S^{-1}R$$
$$Z^* Z^{-1} \begin{bmatrix} I \\ -X \end{bmatrix} = \begin{bmatrix} I \\ -X \end{bmatrix} SR^{-1}.$$

Some Observation:

- Let $W \equiv Z^*Z^{-1}$, if $\lambda \in \sigma(W)$ then $\frac{1}{\lambda} \in \sigma(W)$ since

$$\det(W - \lambda I) = (-\lambda)^n \det(Z^* Z^{-1}) \overline{\det(W - \frac{1}{\overline{\lambda}} I)}.$$

- In CARE, the Hamiltonian matrix $\mathcal{H} \in \mathbb{C}^{2n \times 2n}$ is satisfying

$$(\mathcal{H}\mathcal{J})^* = \mathcal{H}\mathcal{J},$$

  where $\mathcal{J} = \begin{bmatrix} & -I \\ I & \end{bmatrix}$. We can get the partition of $\mathcal{H}$ as the form

$$\mathcal{H} = \begin{bmatrix} H_1 & H_2 \\ H_3 & -H_1^* \end{bmatrix},$$

  where $H_2 = H_2^*$ and $H_3 = H_3^*$. It is corresponding to the CARE

$$XH_2X + XH_1 + H_1^*X - H_3 = 0.$$

  What is the similar results for the T-Rittati equation?, What is the structure of $W = Z^*Z^{-1}$?

Another obvious application of Newton's method lead to the iterative process

$$\delta X_{k+1}(AX_k^\star + B) + (X_k A + C)\delta X_{k+1}^\star = -\mathcal{R}(X_k)$$

which is a $\star$-Sylvester equation for δX_{k+1}.

Algorithm of Newton's iteration for T-Riccati Eq.

`Set`

$$X_0 \text{ is given}$$

For $k = 0, 1, \ldots,$ compute X_{k+1} until convergence

`Solve the T-Sylvester Eq.`

$$(C + X_k A)X_{k+1}^\top + X_{k+1}(B + AX_k^\top) = X_k AX_k^\top - D. \tag{0.2a}$$

End of algorithm

Algorithm of Fixed-point iteration for T-Riccati Eq.

`Set`

$$X_0 \text{ is given}$$

for $k = 0, 1, \ldots,$ compute X_{k+1} until convergence

`Solve`

$$X_{k+1}(AX_k^\top + B) = -(CX_k^\top + D), \tag{0.3a}$$

or

$$(X_k A + C)X_{k+1}^\top = -(X_k^\top B + D), \tag{0.3b}$$

End of algorithm

Example 0.1. *We generating the coefficient matrices A, B, C, D and the solution X by the Matlab command*

$$\begin{aligned} A &= randn(n), \\ B &= randn(n), \\ C &= randn(n), \\ X &= randn(n) \end{aligned}$$

and

$$D = -(XAX^\top + XB + CX^\top).$$

We compare the numerical behavior of the Fixed-point algorithm and the Newton's method with respect to the numbers of iterations (ITs), the CPU times in seconds, and the "normalized" residuals (NRes):

$$\text{NRes} = \frac{\|\widetilde{X}A\widetilde{X}^\top + \widetilde{X}B + C\widetilde{X}^\top + D\|_\infty}{\|\widetilde{X}\|_\infty(\|A\|_\infty\|\widetilde{X}^\top\|_\infty + \|B\|_\infty) + \|C\|_\infty\|\widetilde{X}\|_\infty + \|D\|_\infty},$$

where $\widetilde{X}$ is the approximate solution to the solution of (0.1)*.*

- *Fixed-point Iteration: iteration number:200-300. Newton Iteration: iteration number: 8-20.*

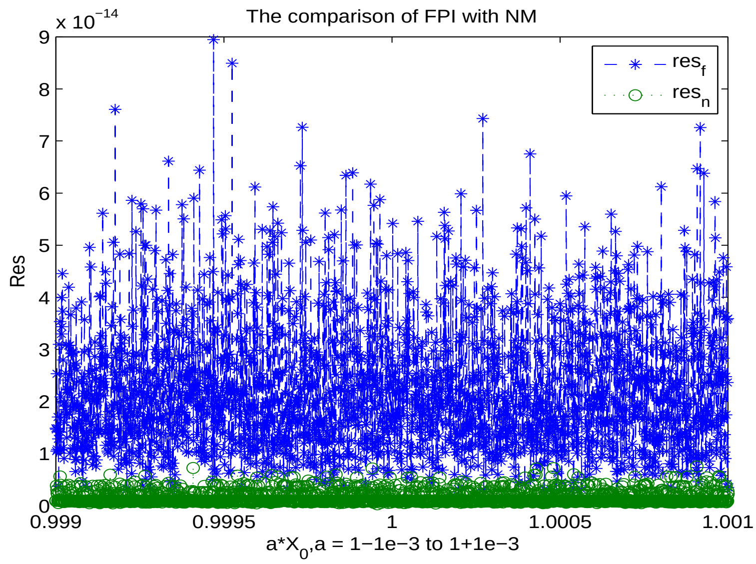

Figure 0.1: The graph of FPI and NM w.r.t $X_0 = aX$.

- *Figure 0.1: Let $X_0 = a * X$, where $a = 1 - 1e - 3 : 1e - 6 : 1 + 1e - 3$, res_f and res_n are the residuals of FPI and NM, respectively.*

- *In this test, we let $X_0 = aX$, then NTI and FPI are converge to two different solution of $X_n = X$ and X_f of* (0.1)*, respectively.*

We take $n = 5, 10, 15, 20, 25, 30$. *The IT counts, CPU times and NRes for FPI and NTI are listed in Table 1.*

Table 1: Numerical results for n increase.

Methods		FPI NTI	Methods		FPI NTI
	IT			IT	
$n = 5$	CPU		n=10	CPU	
	NRes			NRes	
	IT			IT	
$n = 15$	CPU		n=20	CPU	
	NRes			NRes	
	IT			IT	
$n = 25$	CPU		n=30	CPU	
	NRes			NRes	