Chapter 14 Gaussian Elimination (IV)
Bunch-Kaufman diagonal pivoting
(partial pivoting)

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Reference: [1] James R. *Bunch* and Linda *Kaufman*, Some Stable Methods for Calculating Inertia and Solving Symmetric Linear Systems, Mathematics of Computation, volume 31, number 137, January 1977, page 163-179

[2] Cleve Ashcraft, Roger G. Grimes, and John G. Lewis, Accurate Symmetric Indefinite Linear Equation Solvers, SIAM J. MATRIX ANAL. APPL. Vol. 20, No. 2, 1998, pp. 513-561

OutLine

- Review Bunch-Parlett diagonal pivoting
  - expensive pivot-selection strategy
  - pivot induces swapping row/column, not efficient
- Partial pivoting: basic idea
- Implementation of partial pivoting
- Example

Recall Criterion for pivot strategy (complete pivoting) [1]

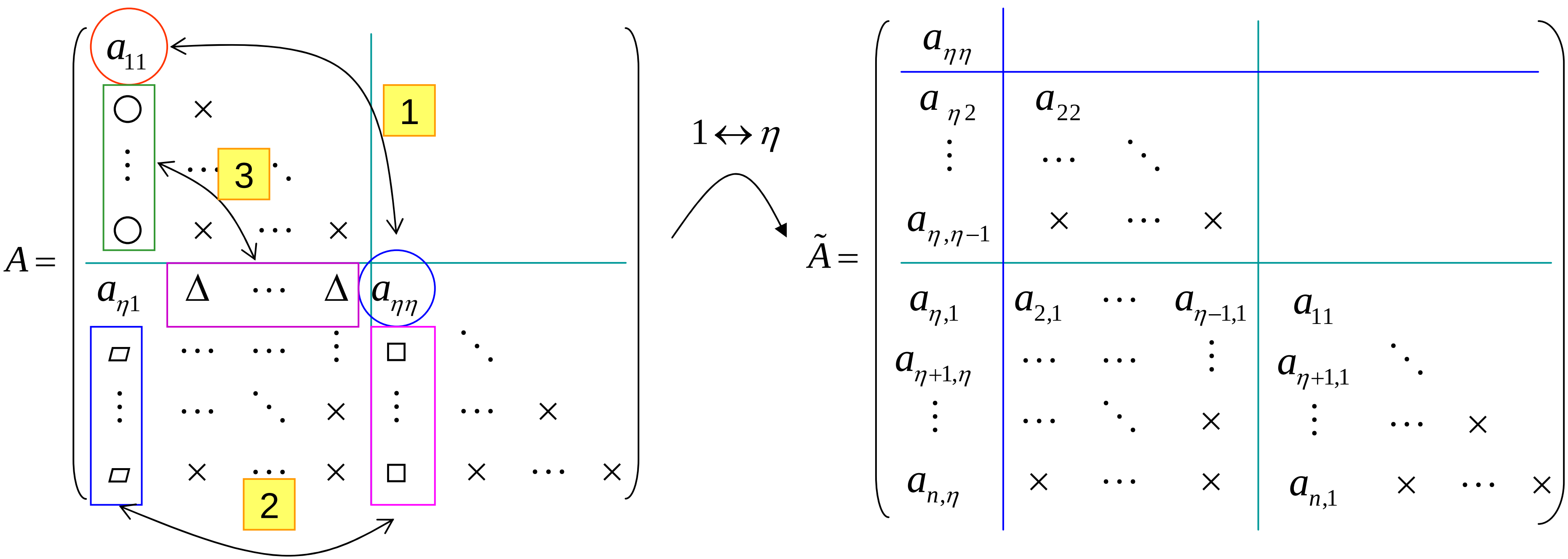
$$A=\begin{pmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \cdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}=A^T$$

$$\mu_0=\max_{1\le i,j\le n}\left|A_{ij}\right|=\left|A_{rq}\right|$$ expensive

$$\mu_1=\max_{1\le i\le n}\left|A_{ii}\right|=\left|A_{\eta\eta}\right|$$

Case 1: $\mu_1 \ge \alpha\mu_0$

Define permutation matrix $P=\left(\eta,2:\eta-1,1,\eta+1:n\right)$ and do symmetric permutation



Recall Criterion for pivot strategy (complete pivoting) [2]

Then do LDL^T on $\tilde{A}$ with 1×1 pivot, $\tilde{A} = \begin{pmatrix} E & c^T \\ c & B \end{pmatrix} = \begin{pmatrix} 1 & \\ cE^{-1} & I \end{pmatrix} \begin{pmatrix} E & \\ & A^{(2)} \end{pmatrix} L^T$, $E = a_{\eta\eta}$

$$c = \begin{pmatrix} A(\eta, 2:\eta-1) \\ a_{\eta,1} \\ A(\eta+1:n,\eta) \end{pmatrix} = \begin{pmatrix} a_{\eta,2} \\ \vdots \\ a_{\eta,\eta-1} \\ \hline a_{\eta,1} \\ \hline a_{\eta+1,\eta} \\ \vdots \\ a_{n,\eta} \end{pmatrix} = \begin{pmatrix} a_{2,\eta} \\ \vdots \\ a_{\eta-1,\eta} \\ \hline a_{1,\eta} \\ \hline a_{\eta+1,\eta} \\ \vdots \\ a_{n,\eta} \end{pmatrix} \qquad \begin{pmatrix} a_{1,\eta} \\ a_{2,\eta} \\ \vdots \\ a_{\eta-1,\eta} \\ a_{\eta,\eta} \\ a_{\eta+1,\eta} \\ \vdots \\ a_{n,\eta} \end{pmatrix}$$

$$B = \tilde{A}(2:n, 2:n), \; A^{(2)} = B - cE^{-1}c^T$$

$$cE^{-1} = \frac{c}{a_{\eta\eta}} \rightarrow \left| cE^{-1} \right|_\infty \le \max_{j\ne\eta} \left| \frac{a_{j\eta}}{a_{\eta\eta}} \right| \le \frac{1}{\nu} \max_{j\ne\eta} \left| a_{j\eta} \right| \le \frac{\mu_0}{\nu}$$

where $\nu = \left| \det E \right| = \left| a_{\eta\eta} \right|$

Therefore $\left| L \right|_\infty = \max\left(1, \left| cE^{-1} \right|_\infty\right) \le \max\left(1, \frac{\mu_0}{\nu}\right) \le \frac{\mu_0}{\nu}$

Observation: if we define $\lambda = \max_{j\ne\eta} \left| A_{j,\eta} \right| =$ maximum of off-diagonal elements in column η

, then $$\left| L \right|_\infty \le \max\left(1, \frac{\lambda}{\nu}\right) \le \frac{\mu_0}{\nu}$$

Recall Criterion for pivot strategy (complete pivoting) [3]

$$A^{(2)} = B - cE^{-1}c^T = \tilde{A}(2:n,2:n) - L(2:n,1)\left[\tilde{A}(2:n,1)\right]^T$$

$$A_{ij}^{(2)} = \tilde{A}_{i+1,j+1} - L_{i+1,1}\tilde{A}_{j+1,1} \longrightarrow \boxed{\left|A^{(2)}\right|_\infty \le \mu_0 + |L|_\infty \lambda}$$

Question: To compute $\mu_0 = \max_{1\le i,j\le n} |A_{ij}| = |A_{rq}|$ is expensive, if we don't want to compute it,

Can we have other choice such that controllability of $|L|_\infty$ and $\left|A^{(2)}\right|_\infty$ holds ?

Observation: It suffices to change pivoting condition $\mu_1 \ge \alpha\mu_0$ to $\mu_1 \ge \alpha\lambda$, then

$$\boxed{|L|_\infty \le \max\left(1, \frac{\lambda}{\nu}\right) \le \max\left(1, \frac{1}{\alpha}\right) = \frac{1}{\alpha}} \quad \text{and} \quad \boxed{\left|A^{(2)}\right|_\infty \le \mu_0 + \frac{1}{\alpha}\lambda \le \left(1 + \frac{1}{\alpha}\right)\mu_0}$$

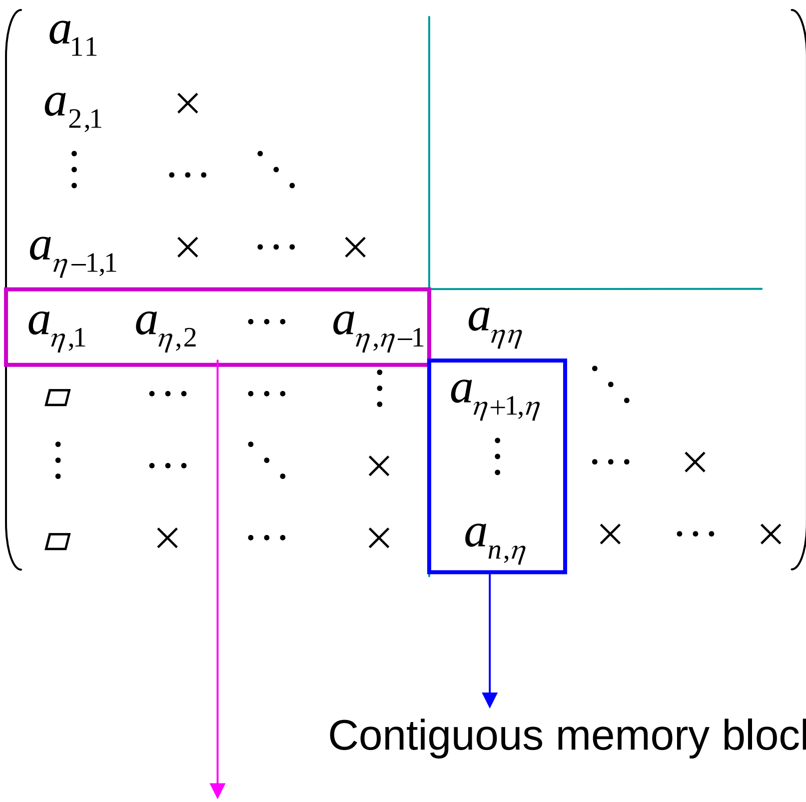
The same as that we do by using pivoting criterion (complete pivoting) $\mu_1 \ge \alpha\mu_0$

However computing $\lambda = \max_{j\ne\eta} |A_{j,\eta}|$ is cheaper than $\mu_0 = \max_{1\le i,j\le n} |A_{ij}|$

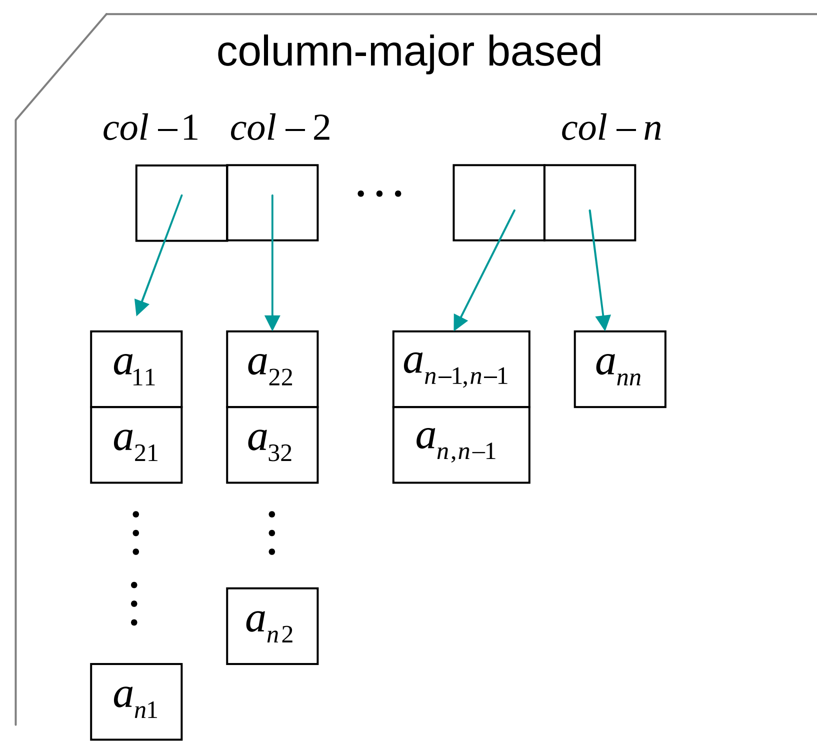
since $\lambda = \max_{j\ne\eta} |A_{j,\eta}| = O(n)$ but $\mu_0 = \max_{1\le i,j\le n} |A_{ij}| = O\left(n^2/2\right)$

Recall Criterion for pivot strategy (complete pivoting) [4]

Technical problem: computing $\lambda = \max_{j \neq \eta} |A_{j,\eta}|$ does not attain optimal.



$$\lambda = \max_{j \neq \eta} |A_{j,\eta}| = \max\left(\left(|a_{\eta,1}| \quad \cdots \quad |a_{\eta,\eta-1}| \right), \begin{pmatrix} |a_{\eta+1,\eta}| \\ \vdots \\ |a_{n,\eta}| \end{pmatrix} \right)$$



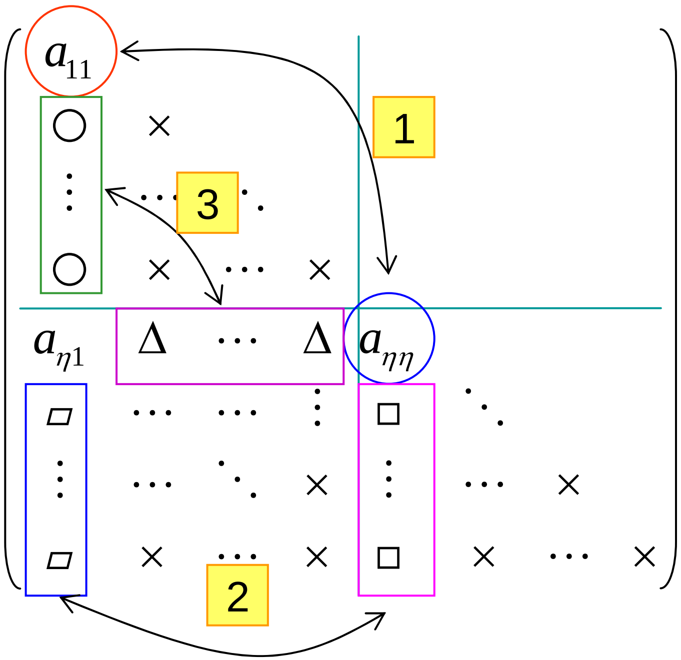
Contiguous memory block, fast

NOT contiguous memory block, ***slowly.***

How to improve this ?

Recall Criterion for pivot strategy (complete pivoting) [5]

Second, if $\mu_1 \geq \alpha\lambda$, we need to interchange row/column $\eta \leftrightarrow 1$



3 relates to read/write on row vector $(\Delta \quad \cdots \quad \Delta)$

which is NOT contiguous, slowly.

Question: why we persist on computing $\mu_1 = \max_{1 \leq i \leq n} |A_{ii}| = |A_{\eta\eta}|$, why not choose first column?

say $E = a_{11}$ and $\lambda = \max_{j \neq 1} |A_{j,1}| =$ maximum of off-diagonal elements in column 1

Again, if $\nu \geq \alpha\lambda$, then we also have

$$|L|_\infty \leq \max\left(1, \frac{\lambda}{\nu}\right) \leq \max\left(1, \frac{1}{\alpha}\right) = \frac{1}{\alpha}$$

and

$$|A^{(2)}|_\infty \leq \mu_0 + \frac{1}{\alpha}\lambda \leq \left(1 + \frac{1}{\alpha}\right)\mu_0$$

OutLine

- Review Bunch-Parlett diagonal pivoting
- Partial pivoting: basic idea
  - select pivot by at most computation of two columns
  - potential risk of growth rate of lower triangle matrix
- Implementation of partial pivoting
- Example

Partial pivoting: basic idea [1]

$A = \begin{pmatrix} E & c^T \\ c & B \end{pmatrix}$, choose $E = a_{11}$ and $\lambda_1 = \max|c| =$ maximum of off-diagonal elements in column 1

If $|a_{11}| \geq \alpha\lambda_1$ [α is determined later] , then

$$A = \begin{pmatrix} 1 & \\ cE^{-1} & I \end{pmatrix} \begin{pmatrix} E & \\ & A^{(2)} \end{pmatrix} L^T \longrightarrow \begin{cases} |L|_\infty \leq \max\left(1, \frac{\lambda_1}{\nu}\right) \leq \frac{1}{\alpha} \\ \left|A^{(2)}\right|_\infty \leq \mu_0 + \frac{\lambda_1}{\alpha} \leq \left(1 + \frac{1}{\alpha}\right)\mu_0 \end{cases}$$

where $\mu_0 = \max_{1 \leq i, j \leq n} |A_{ij}|$

Pros (優點): ~~compute $\mu_0 = \max_{1 \leq i, j \leq n} |A_{ij}|$ and $\mu_1 = \max_{1 \leq i \leq n} |A_{ii}|$~~

Cons (缺點): It is difficult to satisfy $|a_{11}| \geq \alpha\lambda_1$, we need to deal with the case $|a_{11}| < \alpha\lambda$ carefully such that controllability of $|L|_\infty$ and $\left|A^{(2)}\right|_\infty$ holds.

Comparison with $PA = LU$: suppose $|a_{11}| < \alpha\lambda_1$ for $|a_{r1}| = \lambda_1$

we CANNOT move A_{r1} to A_{11} due to symmetric permutation,

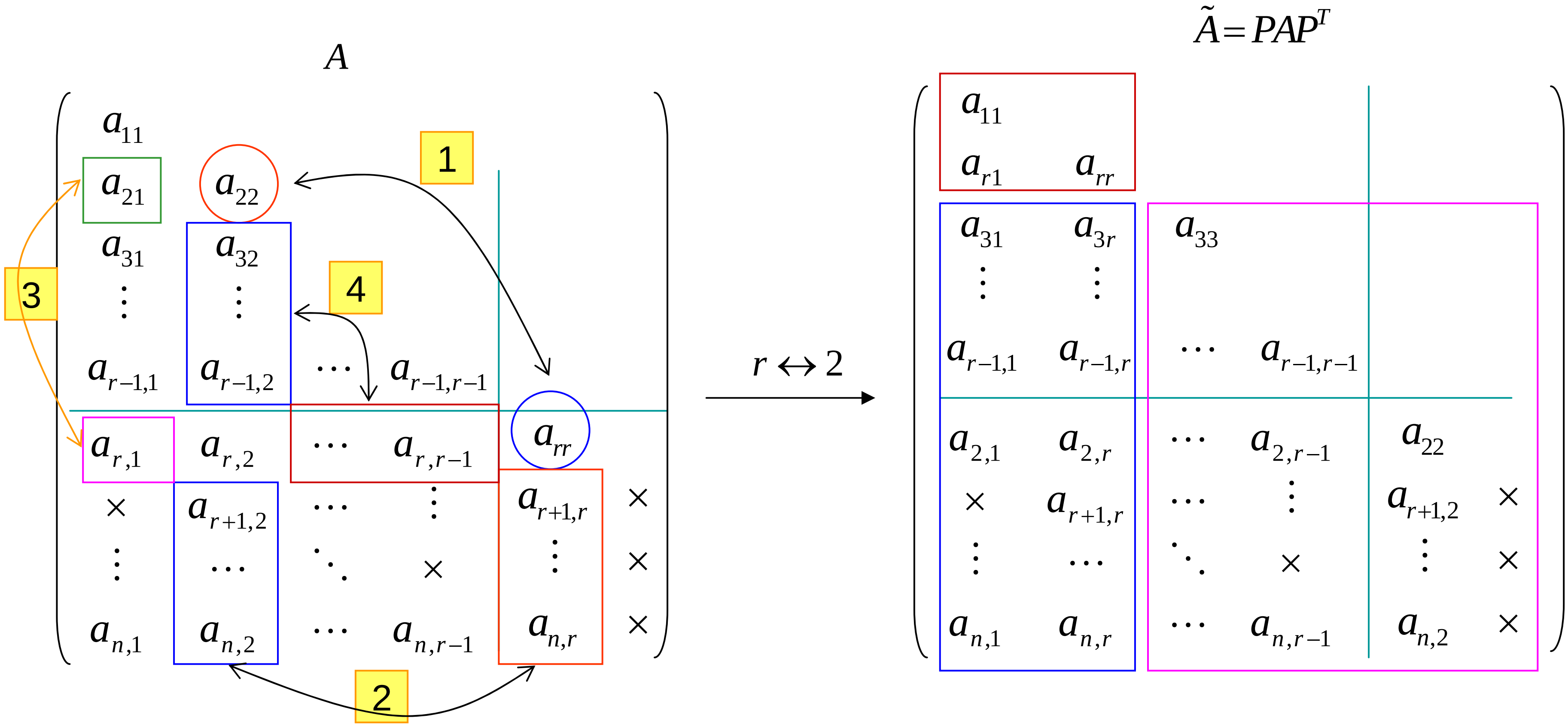
It is impossible to keep stability by just computing one column as we do in $PA = LU$

Partial pivoting: basic idea [2]

Question: how to deal with the case $|a_{11}| < \alpha\lambda_1$ for $|a_{r1}| = \lambda_1$

<ans> from procedure of complete pivoting, we need 2x2 pivot.

define permutation $P = (1, r, 3 : r-1, 2, r+1 : n)$, change a_{r1} to a_{21}



Partial pivoting: basic idea [3]

$$A = \begin{pmatrix} E & c^T \\ c & B \end{pmatrix} = L \begin{pmatrix} E & \\ & A^{(3)} \end{pmatrix} L^T$$

$$E = \begin{pmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} \end{pmatrix}, \quad c = \begin{pmatrix} a_{31} & a_{32} \\ \vdots & \vdots \\ a_{r-1,1} & a_{r-1,2} \\ \hline a_{r,1} & a_{r,2} \\ \hline a_{r+1,1} & a_{r+1,2} \\ \vdots & \vdots \\ a_{n,1} & a_{n,2} \end{pmatrix}$$

$$L = \begin{pmatrix} I & \\ cE^{-1} & I \end{pmatrix} = \begin{pmatrix} I & \\ (l_1 \mid l_2) & I \end{pmatrix}$$

$$\forall \;\; 3 \le i \le n$$

$$\begin{pmatrix} l_{i1} & l_{i2} \end{pmatrix} = \begin{pmatrix} a_{i,1} & a_{i,2} \end{pmatrix} \frac{1}{\det E} \begin{pmatrix} a_{22} & -a_{21} \\ -a_{21} & a_{11} \end{pmatrix}$$

$$r \leftrightarrow 2 \;\longrightarrow$$

$$\tilde{A} = \begin{pmatrix} \tilde{E} & \tilde{c}^T \\ \tilde{c} & \tilde{B} \end{pmatrix} = \tilde{L} \begin{pmatrix} \tilde{E} & \\ & \tilde{A}^{(3)} \end{pmatrix} \tilde{L}^T$$

$$\tilde{E} = \begin{pmatrix} a_{11} & a_{r1} \\ a_{r1} & a_{rr} \end{pmatrix}, \quad \tilde{c} = \begin{pmatrix} a_{31} & a_{3r} \\ \vdots & \vdots \\ a_{r-1,1} & a_{r-1,r} \\ \hline a_{2,1} & a_{2,r} \\ \hline a_{r+1,1} & a_{r+1,r} \\ \vdots & \vdots \\ a_{n,1} & a_{n,r} \end{pmatrix}$$

$$L = \begin{pmatrix} I & \\ \tilde{c}\tilde{E}^{-1} & I \end{pmatrix} = \begin{pmatrix} I & \\ (\tilde{l}_1 \mid \tilde{l}_2) & I \end{pmatrix}$$

$$\forall \;\; i = 3 : r-1, 2, r+1 : n$$

$$\begin{pmatrix} l_{i1} & l_{ir} \end{pmatrix} = \begin{pmatrix} a_{i,1} & a_{i,r} \end{pmatrix} \frac{1}{\det \tilde{E}} \begin{pmatrix} a_{rr} & -a_{r1} \\ -a_{r1} & a_{11} \end{pmatrix}$$

we may say $\forall \;\; i \ne 1, r$

Partial pivoting: basic idea [4]

$$B = \begin{pmatrix} a_{33} & & & & \\ \vdots & \ddots & & & \\ a_{r3} & a_{r4} & \cdots & a_{rr} & \\ a_{r+1,3} & & & a_{r+1,r} & \\ & & & \vdots & \ddots \\ a_{n,3} & \cdots & & a_{n,r} & \cdots & a_{nn} \end{pmatrix}$$

$$B_{i,j} = a_{i+2,j+2}$$

$$\tilde{B} = \begin{pmatrix} a_{33} & & & & \\ \vdots & \ddots & & & \\ a_{23} & a_{24} & \cdots & a_{22} & \\ a_{r+1,3} & & & a_{r+1,2} & \\ & & & \vdots & \ddots \\ a_{n,3} & \cdots & & a_{n,2} & \cdots & a_{nn} \end{pmatrix}$$

$$A^{(3)} = B - cE^{-1}c^T$$

$$A_{ij}^{(3)} = B_{ij} - \begin{pmatrix} c_{i,1} & c_{i,2} \end{pmatrix} E^{-1} \begin{pmatrix} c_{j,1} \\ c_{j,2} \end{pmatrix}$$

$$= a_{i+2,j+2} - \begin{pmatrix} a_{i+2,1} & a_{i+2,2} \end{pmatrix} \frac{1}{\det E} \begin{pmatrix} a_{22} & -a_{21} \\ -a_{21} & a_{11} \end{pmatrix} \begin{pmatrix} a_{j+2,1} \\ a_{j+2,2} \end{pmatrix}$$

$$\xrightarrow{r \leftrightarrow 2}$$

$$\tilde{A}^{(3)} = \tilde{B} - \tilde{c}\tilde{E}^{-1}\tilde{c}^T$$

$$\forall \; i+2 \neq 1, r \;\text{ and }\; j+2 \neq 1, r$$

$$\tilde{A}_{ij}^{(3)} = a_{i+2,j+2} - \begin{pmatrix} a_{i+2,1} & a_{i+2,r} \end{pmatrix} \frac{1}{\det \tilde{E}} \begin{pmatrix} a_{rr} & -a_{r1} \\ -a_{r1} & a_{11} \end{pmatrix} \begin{pmatrix} a_{j+2,1} \\ a_{j+2,r} \end{pmatrix}$$

Partial pivoting: basic idea [5]

Under the case $|a_{11}| < \alpha\lambda_1$, we want to find a condition for a_{rr} such that $\left\|\tilde{L}\right\|_\infty$ is bounded.

$\forall \; i \neq 1, r$

$$\begin{pmatrix} l_{i1} & l_{ir} \end{pmatrix} = \begin{pmatrix} a_{i,1} & a_{i,r} \end{pmatrix} \frac{1}{\det \tilde{E}} \begin{pmatrix} a_{rr} & -a_{r1} \\ -a_{r1} & a_{11} \end{pmatrix} \longrightarrow \begin{cases} l_{i1} = \dfrac{1}{\det \tilde{E}} \left(a_{i,1} a_{rr} - a_{i,r} a_{r1} \right) \\ l_{ir} = \dfrac{1}{\det \tilde{E}} \left(-a_{i,1} a_{r1} + a_{i,r} a_{11} \right) \end{cases}$$

$$\begin{cases} |l_{i1}| \leq \dfrac{|a_{i,1}||a_{rr}| + |a_{i,r}||a_{r1}|}{\left|\det \tilde{E}\right|} \leq \dfrac{\lambda_1 |a_{rr}| + |a_{i,r}| \lambda_1}{\left|\det \tilde{E}\right|} \\ |l_{ir}| \leq \dfrac{|a_{i,1}||a_{r1}| + |a_{i,r}||a_{11}|}{\left|\det \tilde{E}\right|} \leq \dfrac{\lambda_1 \lambda_1 + |a_{i,r}||a_{11}|}{\left|\det \tilde{E}\right|} \end{cases} \xrightarrow{\left\|\tilde{L}\right\|_\infty \leq \, ?} \begin{cases} \text{Lower bound of } \left|\det \tilde{E}\right| \\ \text{upper bound of } |a_{i,r}|, \; |a_{r,r}| \end{cases}$$

define $\lambda_r \equiv \max_{j \neq r} |a_{j,r}| =$ maximum of off-diagonal elements in column ***r***

1. $\lambda_r \equiv \max_{j \neq r} |a_{j,r}| \geq |a_{1,r}| = \lambda_1$

2. If $|a_{rr}| < \alpha \lambda_r$, then upper bound of $|a_{i,r}|$, $|a_{r,r}|$ holds.

Question: how to achieve lower bound of $\left|\det \tilde{E}\right|$

Partial pivoting: basic idea [6]

$$\left|\det \tilde{E}\right| = \left|a_{11}a_{rr} - a_{r1}^2\right| = \left|a_{11}a_{rr} - \lambda_1^2\right| = \begin{cases} a_{11}a_{rr} - \lambda_1^2 & \text{if } a_{11}a_{rr} > \lambda_1^2 \\ \lambda_1^2 - a_{11}a_{rr} & \text{if } a_{11}a_{rr} < \lambda_1^2 \end{cases}$$

From assumption $\left|a_{11}\right| < \alpha\lambda_1$, to keep inequality, the natural choice is $\lambda_1^2 > a_{11}a_{rr}$

Since if we add assumption $\left|a_{rr}\right| < \alpha\lambda_r$, then $\left|a_{11}a_{rr}\right| < \left|a_{11}\right|\left(\alpha\lambda_r\right)$ and moreover

If we add third assumption

3 $\left|a_{11}\right|\lambda_r < \alpha\lambda_1^2$, then $\left|a_{11}a_{rr}\right| < \left|a_{11}\right|\left(\alpha\lambda_r\right) < \alpha^2\lambda_1^2$ and then $\left|\det \tilde{E}\right| = \lambda_1^2 - a_{11}a_{rr} > \left(1-\alpha^2\right)\lambda_1^2$

Lower bound of $\left|\det \tilde{E}\right|$ holds.

To sum up, in order to control $\left|\tilde{L}\right|_\infty$ under 2x2 pivoting, we need three conditions

(1) $\left|a_{11}\right| < \alpha\lambda_1$ (2) $\left|a_{rr}\right| < \alpha\lambda_r$ (3) $\left|a_{11}\right|\lambda_r < \alpha\lambda_1^2$

$$\begin{cases} \left|l_{i1}\right| \leq \dfrac{\lambda_1\left|a_{rr}\right| + \left|a_{i,r}\right|\lambda_1}{\left|\det \tilde{E}\right|} < \dfrac{\lambda_1\left(\alpha\lambda_r\right) + \lambda_r\lambda_1}{\left(1-\alpha^2\right)\lambda_1^2} = \dfrac{\lambda_r}{\lambda_1}\dfrac{1}{1-\alpha} \\ \left|l_{ir}\right| \leq \dfrac{\lambda_1\lambda_1 + \left|a_{i,r}\right|\left|a_{11}\right|}{\left|\det \tilde{E}\right|} < \dfrac{\lambda_1^2 + \lambda_r\left|a_{11}\right|}{\left(1-\alpha^2\right)\lambda_1^2} < \dfrac{\lambda_1^2 + \alpha\lambda_1^2}{\left(1-\alpha^2\right)\lambda_1^2} = \dfrac{1}{1-\alpha} \end{cases} \longrightarrow \left|\tilde{L}\right|_\infty \leq \dfrac{\lambda_r}{\lambda_1}\dfrac{1}{1-\alpha} \quad \left(\because \lambda_r \geq \lambda_1\right)$$

Partial pivoting: basic idea [7]

Question: does the three condition keep controllability of $\left|\tilde{A}^{(3)}\right|_{\infty}$

$$\forall\ i+2\neq 1,r \ \text{ and } \ j+2\neq 1,r \quad \begin{cases} \tilde{A}_{ij}^{(3)} = a_{i+2,j+2} - \begin{pmatrix} a_{i+2,1} & a_{i+2,r} \end{pmatrix} \dfrac{1}{\det \tilde{E}} \begin{pmatrix} a_{rr} & -a_{r1} \\ -a_{r1} & a_{11} \end{pmatrix} \begin{pmatrix} a_{j+2,1} \\ a_{j+2,r} \end{pmatrix} \\ = a_{i+2,j+2} - \dfrac{1}{\det \tilde{E}} \begin{pmatrix} a_{i+2,1}a_{rr} - a_{i+2,r}a_{r1} & -a_{i+2,1}a_{r,1} + a_{i+2,r}a_{11} \end{pmatrix} \begin{pmatrix} a_{j+2,1} \\ a_{j+2,r} \end{pmatrix} \\ = a_{i+2,j+2} - \dfrac{\left(a_{i+2,1}a_{rr} - a_{i+2,r}a_{r1}\right)a_{j+2,1} + \left(-a_{i+2,1}a_{r,1} + a_{i+2,r}a_{11}\right)a_{j+2,r}}{\det \tilde{E}} \end{cases}$$

$$\longrightarrow \quad \left|\tilde{A}^{(3)}\right|_{\infty} \leq \mu_0 + \frac{\left(\lambda_1 \left|a_{rr}\right| + \lambda_r \lambda_1\right)\lambda_1 + \left(\lambda_1 \lambda_1 + \lambda_r \left|a_{11}\right|\right)\lambda_r}{\left|\det \tilde{E}\right|}$$

(2) $\left|a_{rr}\right| < \alpha\lambda_r$

$$\longrightarrow \quad \left|\tilde{A}^{(3)}\right|_{\infty} \leq \mu_0 + \frac{\left(\lambda_1 \alpha \lambda_r + \lambda_r \lambda_1\right)\lambda_1 + \left(\lambda_1^2 + \lambda_r \left|a_{11}\right|\right)\lambda_r}{\left|\det \tilde{E}\right|}$$

(3) $\left|a_{11}\right|\lambda_r < \alpha\lambda_1^2$

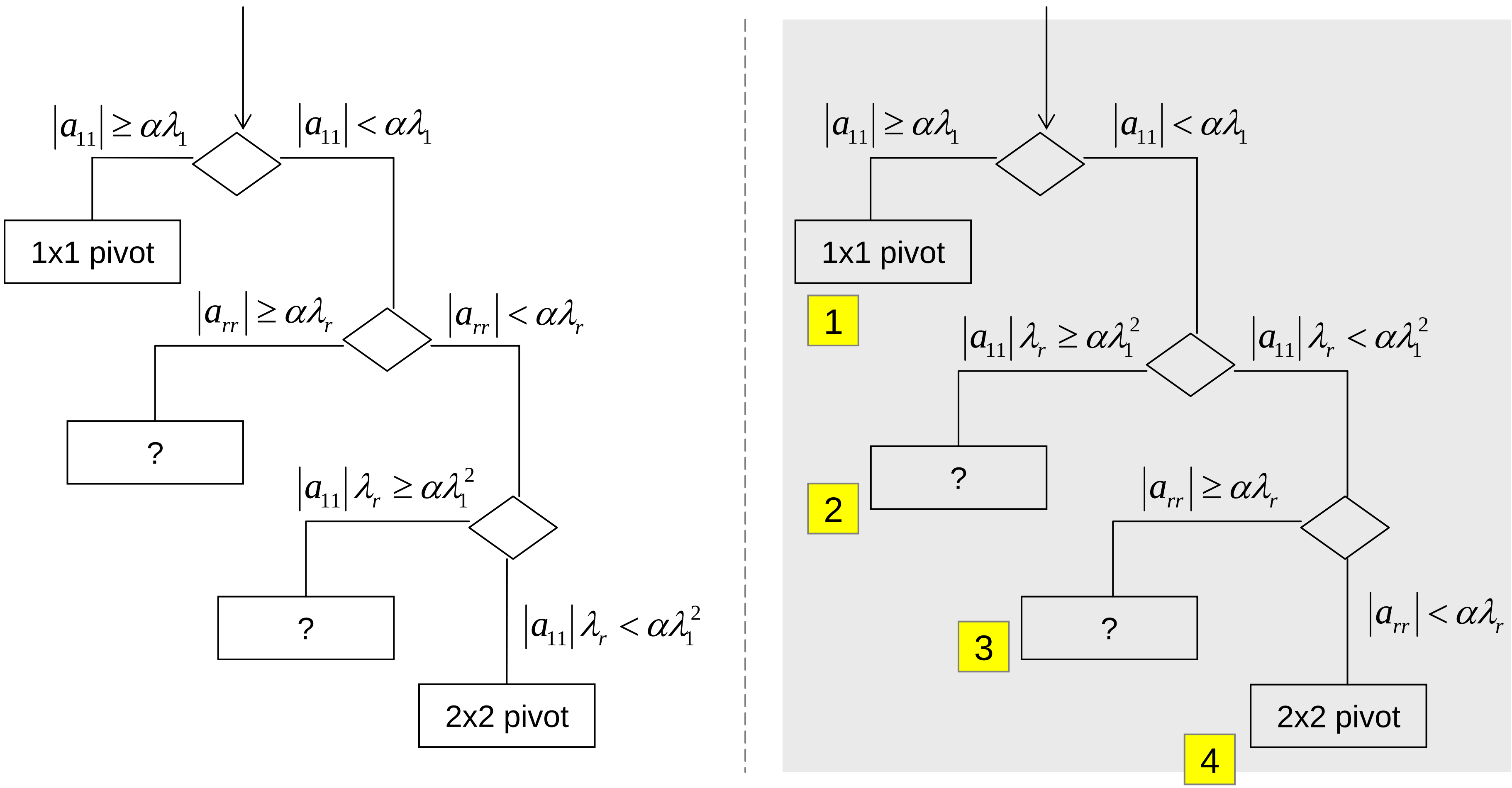
$$\longrightarrow \quad \left|\tilde{A}^{(3)}\right|_{\infty} \leq \mu_0 + \frac{\left(1+\alpha\right)\lambda_r\lambda_1^2 + \left(\lambda_1^2 + \alpha\lambda_1^2\right)\lambda_r}{\left|\det \tilde{E}\right|}$$

(1) $\left|a_{11}\right| < \alpha\lambda_1$

$$\longrightarrow \quad \left|\tilde{A}^{(3)}\right|_{\infty} \leq \mu_0 + \frac{2\left(1+\alpha\right)\lambda_r\lambda_1^2}{\left(1-\alpha^2\right)\lambda_1^2} = \mu_0 + \frac{2\lambda_r}{1-\alpha} \leq \mu_0\left(1 + \frac{2}{1-\alpha}\right)$$

Partial pivoting: basic idea [8]

So far, we deal with two cases in the following decision-making tree



Question: how to deal with the other two cases and what is the order of decision-making, briefly speaking which tree is correct, left or right?

Partial pivoting: basic idea [9]

$A = \begin{pmatrix} E & c^T \\ c & B \end{pmatrix}$, $E = a_{11}$ and $\lambda_1 = |a_{r1}| = \max|c| =$ maximum of off-diagonal elements in column 1

$\mu_0 = \max_{1 \le i, j \le n} |A_{ij}| =$ maximum elements of A, we do not compute it explicitly.

1 $|a_{11}| \ge \alpha\lambda_1$ 1×1 pivot

$$A = \begin{pmatrix} 1 & \\ cE^{-1} & I \end{pmatrix} \begin{pmatrix} E & \\ & A^{(2)} \end{pmatrix} L^T, \quad c = A(2:n,1) = \begin{pmatrix} a_{21} \\ \vdots \\ a_{r1} \\ \vdots \\ a_{n1} \end{pmatrix}, \quad B = A(2:n,2:n) = \begin{pmatrix} a_{22} & & & \\ a_{31} & a_{22} & & \\ \vdots & & \ddots & \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix}$$

$$cE^{-1} = \frac{c}{a_{11}} \to \left|cE^{-1}\right|_\infty \le \max_{j \ne 1} \left|\frac{a_{j1}}{a_{11}}\right| \le \frac{1}{|a_{11}|} \max_{j\ne 1} |a_{j1}| \le \frac{\lambda_1}{|a_{11}|} \le \frac{1}{\alpha} \longrightarrow \boxed{|L|_\infty = \max\left(1, \left|cE^{-1}\right|_\infty\right) \le \frac{1}{\alpha}}$$

$$A^{(2)} = B - cE^{-1}c^T = A(2:n,2:n) - L(2:n,1)\left[A(2:n,1)\right]^T$$

$$A_{ij}^{(2)} = A_{i+1,j+1} - L_{i+1,1}A_{j+1,1} \longrightarrow \boxed{\left|A^{(2)}\right|_\infty \le \mu_0 + |L|_\infty \lambda_1 \le \left(1 + \frac{1}{\alpha}\right)\mu_0}$$

Partial pivoting: basic idea [10]

define $\lambda_r \equiv \max_{j \neq r} \left| a_{j,r} \right| =$ maximum of off-diagonal elements in column ***r***

$$
\begin{pmatrix}
a_{11} & & & & & & & \\
a_{2,1} & \times & & & & & & \\
\vdots & \cdots & \ddots & & & & & \\
a_{r-1,1} & \times & \cdots & \times & & & & \\
a_{r,1} & a_{r,2} & \cdots & a_{r,r-1} & a_{rr} & & & \\
\Box & \cdots & \cdots & \vdots & a_{r+1,r} & \ddots & & \\
\vdots & \cdots & \ddots & \times & \vdots & \cdots & \times & \\
\Box & \times & \cdots & \times & a_{n,r} & \times & \cdots & \times
\end{pmatrix}
$$

$$
\lambda_r = \max_{j \neq r} \left| A_{j,r} \right| = \max \left(\left(\left| a_{r,1} \right| \quad \cdots \quad \left| a_{r,r-1} \right| \right), \begin{pmatrix} \left| a_{r+1,r} \right| \\ \vdots \\ \left| a_{n,r} \right| \end{pmatrix} \right)
$$

$$
\lambda_r \geq \left| a_{r1} \right| = \lambda_1
$$

Contiguous memory block, fast

NOT contiguous memory block, ***slowly.***

Remark: when *case 1* is not satisfied, we must compute λ_r for remaining decision-making.

Remark: In general, λ_r may be more larger than λ_1 , that is why *case 2* holds

2 $\left| a_{11} \right| \lambda_r \geq \alpha \lambda_1^2$

Partial pivoting: basic idea [11]

2 $|a_{11}|\lambda_r \geq \alpha\lambda_1^2$ $\left(\text{under } |a_{11}| < \alpha\lambda_1\right)$ **1×1 pivot**

$$A = \begin{pmatrix} 1 & \\ cE^{-1} & I \end{pmatrix}\begin{pmatrix} E & \\ & A^{(2)} \end{pmatrix} L^T, \qquad c = A(2:n,1) = \begin{pmatrix} a_{21} \\ \vdots \\ a_{r1} \\ \vdots \\ a_{n1} \end{pmatrix}, \qquad B = A(2:n,2:n) = \begin{pmatrix} a_{22} & & & \\ a_{31} & a_{22} & & \\ \vdots & & \ddots & \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix}$$

$$cE^{-1} = \frac{c}{a_{11}} \rightarrow \left|cE^{-1}\right|_\infty \leq \max_{j\neq 1}\left|\frac{a_{j1}}{a_{11}}\right| \leq \frac{\lambda_1}{|a_{11}|} = \frac{\lambda_1\lambda_r}{|a_{11}|\lambda_r} \leq \frac{\lambda_r}{\lambda_1}\frac{1}{\alpha} \longrightarrow \boxed{|L|_\infty = \max\left(1, \left|cE^{-1}\right|_\infty\right) \leq \frac{\lambda_r}{\lambda_1}\frac{1}{\alpha}}$$

$$A^{(2)} = B - cE^{-1}c^T = A(2:n,2:n) - L(2:n,1)\left[A(2:n,1)\right]^T$$

$$A_{ij}^{(2)} = A_{i+1,j+1} - L_{i+1,1}A_{j+1,1} \longrightarrow \boxed{\left|A^{(2)}\right|_\infty \leq \mu_0 + |L|_\infty \lambda_1 \leq \mu_0 + \left(\frac{\lambda_r}{\lambda_1}\frac{1}{\alpha}\right)\lambda_1 \leq \left(1 + \frac{1}{\alpha}\right)\mu_0}$$

Remark: bound on lower triangle matrix is NOT good since it is proportional to $\frac{\lambda_r}{\lambda_1}$

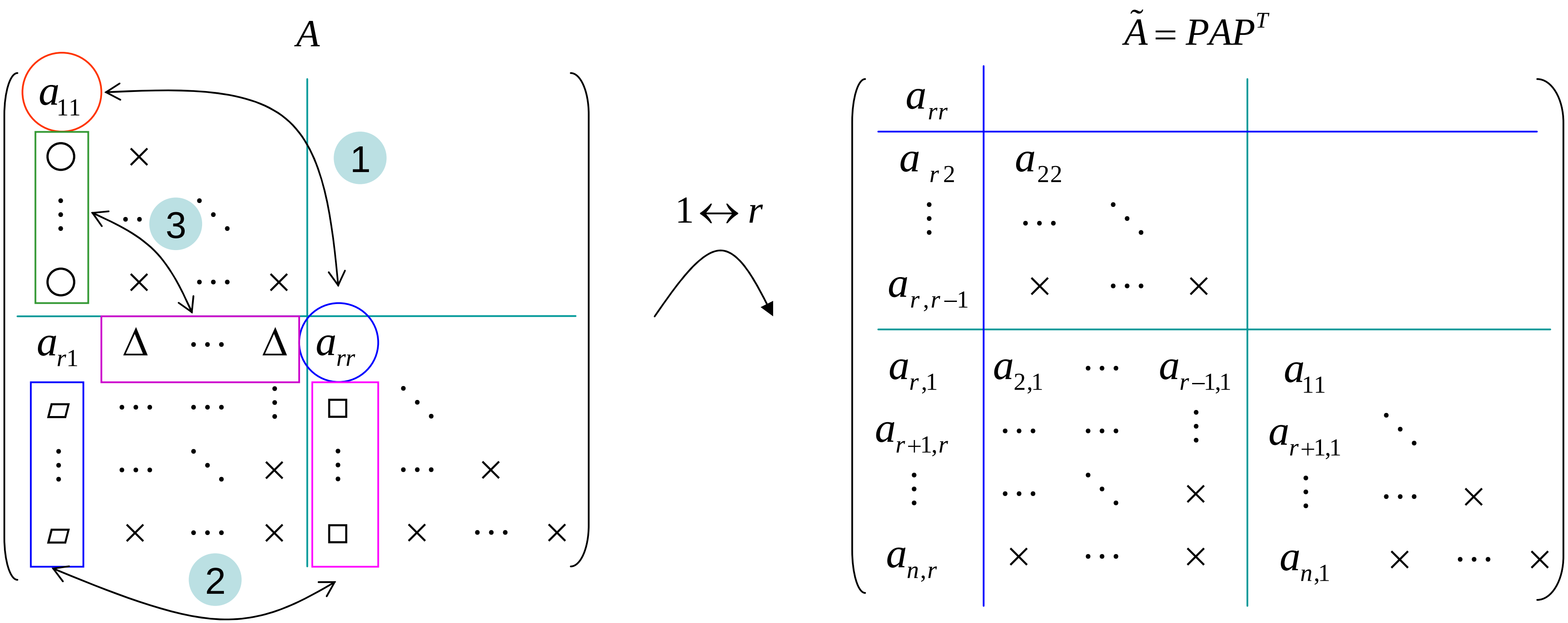
however, the bound of reduced matrix $A^{(2)}$ is good.

Partial pivoting: basic idea [12]

3 $|a_{rr}| \geq \alpha\lambda_r$ $\left(\text{under } |a_{11}| < \alpha\lambda_1,\ |a_{11}|\lambda_r < \alpha\lambda_1^2 \right)$ 1×1 pivot

Clearly, this is the same as case 1 if we interchange row/column $1 \leftrightarrow r$

define permutation $P = (r, 2:r-1, 2, r+1:n)$, change a_{rr} to a_{11}



Partial pivoting: basic idea [13]

Then do LDL^T on $\tilde{A}$ with 1×1 pivot, $\tilde{A}=\begin{pmatrix} E & c^T \\ c & B\end{pmatrix}=\begin{pmatrix} 1 & \\ cE^{-1} & I\end{pmatrix}\begin{pmatrix} E & \\ & A^{(2)}\end{pmatrix}L^T$, $E=a_{rr}$

$$c=\begin{pmatrix} A(r,2:r-1) \\ a_{r,1} \\ A(r+1:n,r)\end{pmatrix}=\begin{pmatrix} a_{r,2} \\ \vdots \\ a_{r,r-1} \\ a_{r,1} \\ a_{r+1,r} \\ \vdots \\ a_{n,r}\end{pmatrix}=\begin{pmatrix} a_{2,r} \\ \vdots \\ a_{r-1,r} \\ a_{1,r} \\ a_{r+1,r} \\ \vdots \\ a_{n,r}\end{pmatrix} \qquad \begin{pmatrix} a_{1,r} \\ a_{2,r} \\ \vdots \\ a_{r-1,r} \\ a_{r,r} \\ a_{r+1,r} \\ \vdots \\ a_{n,r}\end{pmatrix}$$

$$B=\tilde{A}(2:n,2:n),\ A^{(2)}=B-cE^{-1}c^T$$

$$cE^{-1}=\frac{c}{a_{rr}} \rightarrow \left|cE^{-1}\right|_\infty \le \max_{j\ne r}\left|\frac{a_{jr}}{a_{rr}}\right| \le \frac{1}{|a_{rr}|}\max_{j\ne r}\left|a_{jr}\right| \le \frac{\lambda_r}{|a_{rr}|} \le \frac{1}{\alpha} \longrightarrow \boxed{|L|_\infty=\max\left(1,\left|cE^{-1}\right|_\infty\right)\le\frac{1}{\alpha}}$$

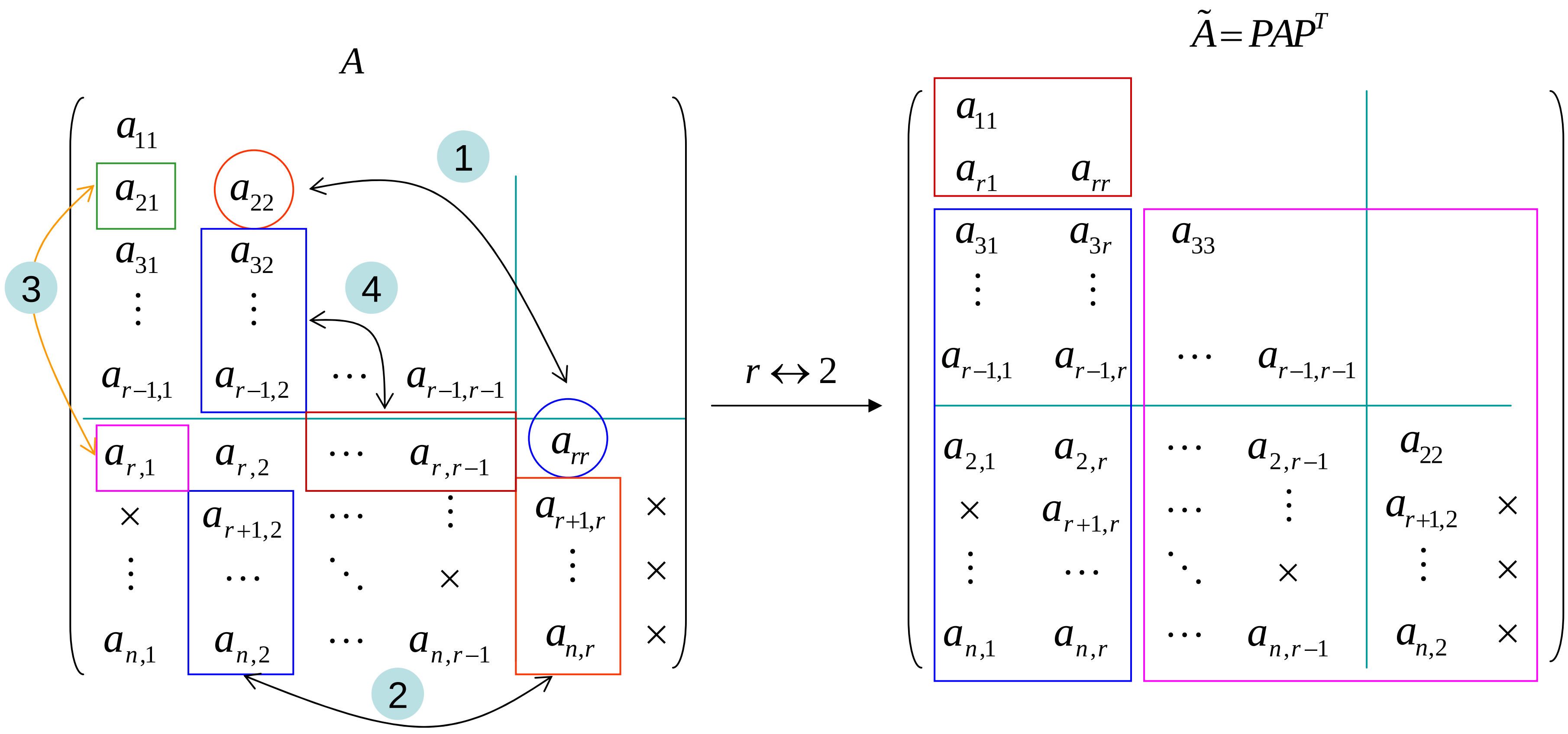
$$A^{(2)}=B-cE^{-1}c^T=\tilde{A}(2:n,2:n)-L(2:n,1)\left[\tilde{A}(2:n,1)\right]^T$$

$$\boxed{\left|A^{(2)}\right|_\infty \le \mu_0+|L|_\infty|c|_\infty \le \mu_0+\frac{1}{\alpha}\lambda_r \le \left(1+\frac{1}{\alpha}\right)\mu_0}$$

Partial pivoting: basic idea [14]

4 $|a_{rr}| < \alpha\lambda_r$ $\left(\text{under } |a_{11}| < \alpha\lambda_1, \ |a_{11}|\lambda_r < \alpha\lambda_1^2 \right)$ 2×2 pivot

define permutation $P = (1, r, 3 : r-1, 2, r+1 : n)$, change a_{r1} to a_{21}



Partial pivoting: basic idea [15]

Then do LDL^T on $\tilde{A}$ with 2×2 pivot, $\tilde{A}=\begin{pmatrix} E & c^T \\ c & B \end{pmatrix}=\begin{pmatrix} 1 & \\ cE^{-1} & I \end{pmatrix}\begin{pmatrix} E & \\ & A^{(3)} \end{pmatrix}L^T$

$$E=\begin{pmatrix} a_{11} & a_{r1} \\ a_{r1} & a_{rr} \end{pmatrix}, \quad \boxed{\left|\det E\right| = \left|a_{11}a_{rr}-a_{r1}^2\right| \ge \left(1-\alpha^2\right)\lambda_1^2} \qquad L=\begin{pmatrix} I_2 & \\ (l_1 \mid l_r) & I_{n-2} \end{pmatrix}$$

$$\forall\ i\ne 1,r \quad \begin{cases} l_{i1}=\dfrac{1}{\det \tilde{E}}\left(a_{i,1}a_{rr}-a_{i,r}a_{r1}\right) \\ l_{ir}=\dfrac{1}{\det \tilde{E}}\left(-a_{i,1}a_{r1}+a_{i,r}a_{11}\right) \end{cases} \longrightarrow \begin{cases} \left|l_{i1}\right| \le \dfrac{\lambda_r}{\lambda_1}\dfrac{1}{1-\alpha} \\ \left|l_{ir}\right| \le \dfrac{1}{1-\alpha} \end{cases} \longrightarrow \boxed{\left|L\right|_\infty \le \dfrac{\lambda_r}{\lambda_1}\dfrac{1}{1-\alpha}}$$

$$\forall\ i+2\ne 1,r \text{ and } j+2\ne 1,r \qquad A_{ij}^{(3)}=a_{i+2,j+2}-\frac{\left(a_{i+2,1}a_{rr}-a_{i+2,r}a_{r1}\right)a_{j+2,1}+\left(-a_{i+2,1}a_{r,1}+a_{i+2,r}a_{11}\right)a_{j+2,r}}{\det \tilde{E}}$$

$$\boxed{\left|A^{(3)}\right|_\infty \le \mu_0\left(1+\frac{2}{1-\alpha}\right)}$$

Remark: bound on lower triangle matrix is NOT good since it is proportional to $\dfrac{\lambda_r}{\lambda_1}$

however, the bound of reduced matrix $A^{(3)}$ is good.

Partial pivoting: basic idea [16]

To sum up, $A = \begin{pmatrix} E & c^T \\ c & B \end{pmatrix} = \begin{pmatrix} 1 & \\ cE^{-1} & I \end{pmatrix} \begin{pmatrix} E & \\ & A^{(2,3)} \end{pmatrix} L^T$

$\lambda_1 = |a_{r1}| = \max |c|$ = maximum of off-diagonal elements in column 1

$\lambda_r \equiv \max_{j \neq r} |a_{j,r}|$ = maximum of off-diagonal elements in column ***r***

1	$\lvert a_{11}\rvert \geq \alpha \lambda_1$	1×1 pivot	$\lvert L \rvert_\infty \leq \frac{1}{\alpha}$	$\lvert A^{(2)} \rvert_\infty \leq \left(1 + \frac{1}{\alpha}\right) \mu_0$
2	$\lvert a_{11}\rvert \lambda_r \geq \alpha \lambda_1^2$	1×1 pivot	$\lvert L \rvert_\infty \leq \frac{\lambda_r}{\lambda_1} \frac{1}{\alpha}$	$\lvert A^{(2)} \rvert_\infty \leq \left(1 + \frac{1}{\alpha}\right) \mu_0$
3	$\lvert a_{rr}\rvert \geq \alpha \lambda_r$, $P = (r, 2:r-1, 2, r+1:n)$	1×1 pivot	$\lvert L \rvert_\infty \leq \frac{1}{\alpha}$	$\lvert A^{(2)} \rvert_\infty \leq \left(1 + \frac{1}{\alpha}\right) \mu_0$
4	$\lvert a_{rr}\rvert < \alpha \lambda_r$, $P = (1, r, 3:r-1, 2, r+1:n)$	2×2 pivot	$\lvert L \rvert_\infty \leq \frac{\lambda_r}{\lambda_1} \frac{1}{1-\alpha}$	$\lvert A^{(3)} \rvert_\infty \leq \left(1 + \frac{2}{1-\alpha}\right) \mu_0$

Partial pivoting: basic idea [17]

worst case analysis : choose $0 < \alpha < 1$ such that reduced matrix $A^{(2)}, A^{(3)}$ satisfies

$$\left[\text{growth rate of } 1\times1 \text{ pivot} + 1\times1 \text{ pivot}\right] \approx \left[\text{growth rate of } 2\times2 \text{ pivot}\right]$$

or equivalently $\left[\text{growth rate of } 1\times1 \text{ pivot}\right] \approx \sqrt{\left[\text{growth rate of } 2\times2 \text{ pivot}\right]}$

$$\left|A^{(2)}\right|_\infty \le \left(1+\frac{1}{\alpha}\right)\mu_0 \qquad \left|A^{(3)}\right|_\infty \le \left(1+\frac{2}{1-\alpha}\right)\mu_0$$

Define $B(\alpha) = \max\left(1+\frac{1}{\alpha}, \sqrt{1+\frac{2}{1-\alpha}}\right)$

$$\min_{0<\alpha<1} B(\alpha) = B(\alpha_0) = \frac{1+\sqrt{17}}{2} \approx 2.5616 < 2.57$$

where $\alpha_0 = \frac{1+\sqrt{17}}{8} \approx 0.6404$ satisfies $1+\frac{1}{\alpha_0} = \sqrt{1+\frac{2}{1-\alpha_0}}$

Remark: this optimal value is the same as we have in complete pivoting.

however, the bound of lower triangle matrix is NOT good.

Complete pivoting versus partial pivoting

$PAP^T = LDL^T$, growth rate of reduce matrix $A^{(k)}$ contributes to final block diagonal matrix D

complete pivoting

$$|L|_\infty \le \max\left(\frac{1}{\alpha_0}, \frac{1}{1-\alpha_0}\right) = \max(1.56, 2.78) = 2.78$$

$$|D|_\infty \le B(\alpha_0)^{n-1}\mu_0 \le 2.57^{n-1}\mu_0$$

partial pivoting

$$|L|_\infty \le \max_k\left(\frac{\lambda_r^{(k)}}{\lambda_1^{(k)}} \max\left(\frac{1}{\alpha_0}, \frac{1}{1-\alpha_0}\right)\right) = 2.78 \cdot \max_k\left(\frac{\lambda_r^{(k)}}{\lambda_1^{(k)}}\right)$$

$$|D|_\infty \le B(\alpha_0)^{n-1}\mu_0 \le 2.57^{n-1}\mu_0$$

Remark: Bunch in [1] only deal with controllability of reduced matrix $A^{(k)}$

However Ashcraft in [2] point out the importance of growth rate of L

Reference: [1] James R. *Bunch* and Linda *Kaufman*, Some Stable Methods for Calculating Inertia and Solving Symmetric Linear Systems, Mathematics of Computation, volume 31, number 137, January 1977, page 163-179

[2] Cleve Ashcraft, Roger G. Grimes, and John G. Lewis, Accurate Symmetric Indefinite Linear Equation Solvers, SIAM J. MATRIX ANAL. APPL. Vol. 20, No. 2, 1998, pp. 513-561

OutLine

- Review Bunch-Parlett diagonal pivoting
- Partial pivoting: basic idea
- Implementation of partial pivoting
- Example

Algorithm (PAP' = LDL', partial pivot) [1]

Given symmetric indefinite matrix $A \in R^{n \times n}$, construct initial lower triangle matrix $L = I$

use permutation vector P to record permutation matrix $P^{(k)}$

let $A^{(1)} := A$, $L^{(0)} = I$, $P^{(0)} = (1, 2, 3, \cdots, n)$ and $pivot = zero(n)$, $\alpha = \frac{1+\sqrt{17}}{8} \approx 0.6404$

$k = 1$

while $k \le (n-1)$

we have compute $P^{(k-1)} A \left(P^{(k-1)}\right)^T = L^{(k-1)} A^{(k)} \left(L^{(k-1)}\right)^T$

$$A^{(k)} = \left(\begin{array}{ccc|ccc} D_1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \ddots & & & & \\ \vdots & & D_s & \cdots & \cdots & 0 \\ \hline \vdots & & 0 & a_{k,k}^{(k)} & \cdots & a_{n,k}^{(k)} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & a_{n,k}^{(k)} & \cdots & a_{nn}^{(k)} \end{array}\right)$$

update original matrix A , where $D_i : 1 \times 1$ or 2×2

$$L^{(k-1)} = \left(\begin{array}{c|c} W & O \\ \hline M & I \end{array}\right) \quad \begin{array}{l} \overbrace{\quad}^{k-1} \text{ columns of } W \\ \} k-1 \text{ rows} \end{array}$$

combines all lower triangle matrix and store in L

Algorithm (PAP' = LDL', partial pivot) [2]

1 compute $\lambda_1 = \max_{k+1 \le j \le n} |A_{j1}| = |A_{r1}|, \quad r \ge k$

Case 1: $|a_{kk}| \ge \alpha \lambda_1$ **1×1 pivot** no interchange

2 do 1x1 pivot : $A^{(k)} = \begin{pmatrix} D_{k-1} & & \\ & a_{kk}^{(k)} & c^T \\ & c & B \end{pmatrix} = \begin{pmatrix} I & & \\ & 1 & \\ & c / a_{kk}^{(k)} & I \end{pmatrix} \begin{pmatrix} D_{k-1} & & \\ & a_{kk}^{(k)} & \\ & & B - cc^T / a_{kk}^{(k)} \end{pmatrix} \left(L^{(k)} \right)^T$

$$\begin{cases} L(k+1:n,k) \leftarrow c / a_{kk}^{(k)} \\ A(k+1:n,k+1:n) -= cc^T / a_{kk}^{(k)} \end{cases}$$

then $A = L^{(k-1)} L^{(k)} A^{(k+1)} L^{(k)} \left(L^{(k-1)} \right)^T$

3 $k \leftarrow k+1$ and $pivot(k) = 1$

When *case 1* is not satisfied, we compute

$\lambda_r \equiv \max_{j \ne r} |a_{j,r}| =$ maximum of off-diagonal elements in column ***r***

under lower triangular part of matrix A is used.

Algorithm (PAP' = LDL', partial pivot) [3]

4 $\lambda_r = \max\left(\max\left|A(r,k:r-1)\right|, \max\left|A(r+1:n,r)\right|\right)$

$\lambda_r \geq \lambda_1 > 0?$

$$\begin{pmatrix} a_{kk} & & & & & & & \\ a_{2,k} & \times & & & & & & \\ \vdots & \cdots & \ddots & & & & & \\ a_{r-1,k} & \times & \cdots & \times & & & & \\ a_{r,k} & a_{r,k+2} & \cdots & a_{r,r-1} & a_{rr} & & & \\ \square & \cdots & \cdots & \vdots & a_{r+1,r} & \ddots & & \\ \vdots & \cdots & \ddots & \times & \vdots & \cdots & \times & \\ \square & \times & \cdots & \times & a_{n,r} & \times & \cdots & \times \end{pmatrix}$$

Case 2: $|a_{kk}|\lambda_r \geq \alpha\lambda_1^2$ 1×1 pivot   no interchange

5 do 1x1 pivot : $A^{(k)} = \begin{pmatrix} D_{k-1} & & \\ & a_{kk}^{(k)} & c^T \\ & c & B \end{pmatrix} = \begin{pmatrix} I & & \\ & 1 & \\ & c/a_{kk}^{(k)} & I \end{pmatrix} \begin{pmatrix} D_{k-1} & & \\ & a_{kk}^{(k)} & \\ & & B - cc^T/a_{kk}^{(k)} \end{pmatrix} \left(L^{(k)}\right)^T$

$$\begin{cases} L(k+1:n,k) \leftarrow c/a_{kk}^{(k)} \\ A(k+1:n,k+1:n) -= cc^T/a_{kk}^{(k)} \end{cases}$$

then $A = L^{(k-1)} L^{(k)} A^{(k+1)} \left(L^{(k)}\right)^T \left(L^{(k-1)}\right)^T$

6 $k \leftarrow k+1$ and $pivot(k) = 1$

The same as code in case 1

Algorithm (PAP' = LDL', partial pivot) [4]

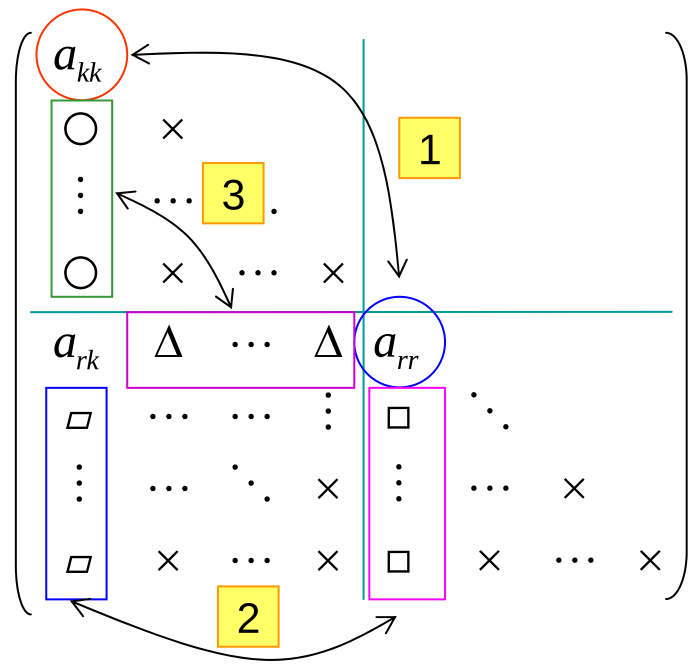
Case 3: $|a_{rr}| \geq \alpha\lambda_r$ 1×1 pivot   do interchange

define permutation $P = (1:k-1, r, k+1:r-1, k, r+1:n)$ to do symmetric permutation

7 $P(k) \leftrightarrow P(r)$

To compute $\tilde{A}^{(k)} = P_k A^{(k)} P_k^T$, we only update lower triangle of $A^{(k)}$

8
- 1 $A(k,k) \leftrightarrow A(r,r)$
- 2 $A(r+1:n,k) \leftrightarrow A(r+1:n,r)$
- 3 $A(k+1:r-1,k) \leftrightarrow A(r,k+1:r-1)$



then $$\tilde{A}^{(k)} = \begin{pmatrix} D_{k-1} & & \\ & a_{kk}^{(k)} & c^T \\ & c & B \end{pmatrix}, \quad a_{kk}^{(k)} := a_{rr}$$

Algorithm (PAP' = LDL', partial pivot)   [5]

To compute $\tilde{L}^{(k-1)} = P_k L^{(k-1)} P_k^T$

9 We only update lower triangle matrix L

$$L(k,1:k-1) \leftrightarrow L(r,1:k-1)$$

then

$$P^{(k)} A \left(P^{(k)}\right)^T = \tilde{L}^{(k-1)} \tilde{A}^{(k)} \left(\tilde{L}^{(k-1)}\right)^T$$

$$L^{(k-1)} = \begin{pmatrix} 1 & & & & & & & \\ \times & 1 & & & & & & \\ \vdots & \cdots & \ddots & & & & & \\ \times & \times & \cdots & 1 & & & & \\ l_{k,1} & l_{k,2} & \cdots & l_{k,k-1} & 1 & & & \\ \vdots & \cdots & \cdots & \vdots & & 1 & & \\ l_{r,1} & l_{r,2} & \ddots & l_{r,k-1} & & & \ddots & \\ \times & \times & \cdots & \times & & & & 1 \end{pmatrix}$$

10 do 1x1 pivot : $\tilde{A}^{(k)} = \begin{pmatrix} D_{k-1} & & \\ & a_{kk}^{(k)} & c^T \\ & c & B \end{pmatrix} = \begin{pmatrix} I & & \\ & 1 & \\ & c/a_{kk}^{(k)} & I \end{pmatrix} \begin{pmatrix} D_{k-1} & & \\ & a_{kk}^{(k)} & \\ & & B - cc^T/a_{kk}^{(k)} \end{pmatrix} \left(L^{(k)}\right)^T$

$$\begin{cases} L(k+1:n,k) \leftarrow c/a_{kk}^{(k)} \\ A(k+1:n,k+1:n) -= cc^T/a_{kk}^{(k)} \end{cases}$$

then $P^{(k)} A \left(P^{(k)}\right)^T = \tilde{L}^{(k-1)} L^{(k)} A^{(k+1)} \left(L^{(k)}\right)^T \left(\tilde{L}^{(k-1)}\right)^T$

11 $k \leftarrow k+1$ and $pivot(k) = 1$

Algorithm (PAP' = LDL', partial pivot) [6]

Case 4: $|a_{rr}| < \alpha\lambda_r$ 2×2 pivot do interchange

define permutation $P = (1:k, r, k+2:r-1, k+1, r+1:n)$, change a_{rk} to $a_{k+1,k}$

12 $P(k+1) \leftrightarrow P(r)$

To compute $\tilde{A}^{(k)} = P_k A^{(k)} P_k^T$, we only update lower triangle of $A^{(k)}$

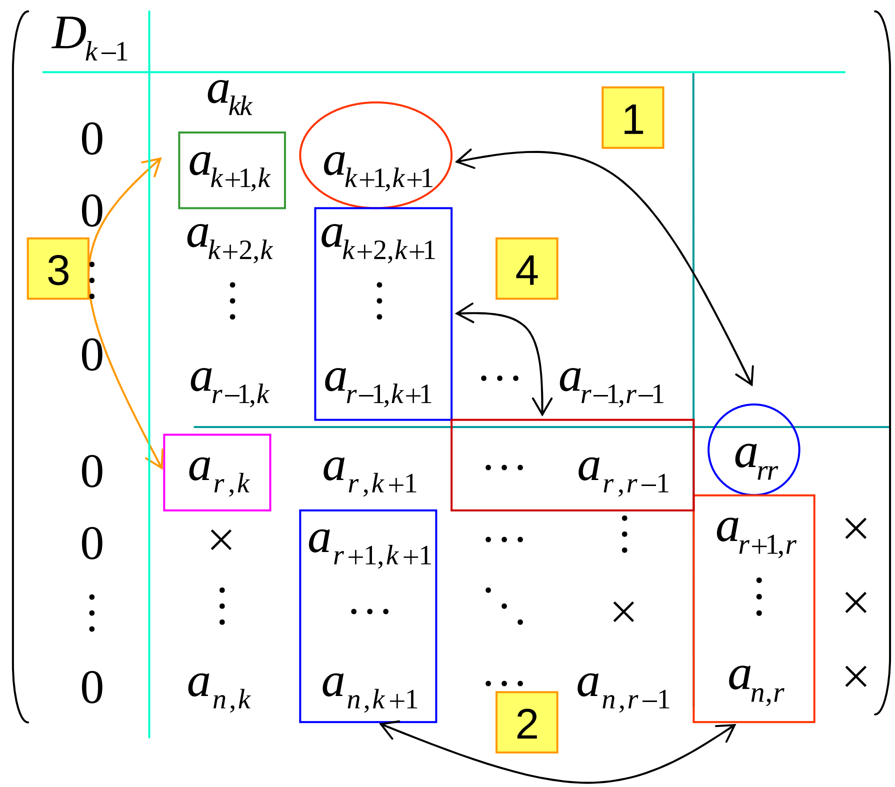
13 do interchange row/col $k+1 \leftrightarrow r$

1 $A(k+1,k+1) \leftrightarrow A(r,r)$

2 $A(r+1:n,k+1) \leftrightarrow A(r+1:n,r)$

3 $A(k+1,k) \leftrightarrow A(r,k)$

4 $A(k+2:r-1:k+1) \leftrightarrow A(r,k+2:r-1)$



then $\tilde{A}^{(k)} = \begin{pmatrix} D_{k-1} & & \\ & E & c^T \\ & c & B \end{pmatrix}$, $E = \begin{pmatrix} a_{kk}^{(k)} & a_{r,k}^{(k)} \\ a_{r,k}^{(k)} & a_{rr}^{(k)} \end{pmatrix}$

Algorithm (PAP' = LDL', partial pivot) *[7]*

14 do interchange row $k+1 \leftrightarrow r$

$$L(k+1,1:k-1) \leftrightarrow L(r,1:k-1)$$

then

$$P^{(k)} A \left(P^{(k)}\right)^T = \tilde{L}^{(k-1)} \tilde{A}^{(k)} \left(\tilde{L}^{(k-1)}\right)^T$$

$$\begin{pmatrix} 1 & & & & & & & \\ \times & 1 & & & & & & \\ \vdots & \cdots & \ddots & & & & & \\ l_{k,1} & l_{k,2} & \cdots & 1 & & & & \\ l_{k+1,1} & \cdots & l_{k+1,k-1} & 0 & 1 & & & \\ \vdots & \cdots & \cdots & \vdots & & 1 & & \\ l_{r,1} & \cdots & l_{r,k-1} & 0 & & & \ddots & \\ \times & \times & \cdots & 0 & & & & 1 \end{pmatrix}$$

15 do 2x2 pivot : $\tilde{A}^{(k)} = \begin{pmatrix} D_{k-1} & & \\ & E & c^T \\ & c & B \end{pmatrix} = \begin{pmatrix} I & & \\ & I & \\ & cE^{-1} & I \end{pmatrix} \begin{pmatrix} D_{k-1} & & \\ & E & \\ & & B - cE^{-1}c^T \end{pmatrix} \left(L^{(k)}\right)^T$

$$\begin{cases} L(k+2:n, k:k+1) \leftarrow cE^{-1} \\ A(k+2:n, k+2:n) -= cE^{-1}c^T \end{cases}$$

then $P^{(k)} A \left(P^{(k)}\right)^T = \tilde{L}^{(k-1)} L^{(k)} \; A^{(k+2)} \; \left(L^{(k)}\right)^T \left(\tilde{L}^{(k-1)}\right)^T$

16 $k \leftarrow k+2$ and $pivot(k) = 2$

endwhile

Question: recursion implementation

normal

- Initialization
  - check algorithm holds for $k=1$
- Recursion formula
  - check algorithm holds for k *or* $k+1$, if $k-1$ is true
- Termination condition
  - check algorithm holds for $k=n-1$

abnormal

- Breakdown of algorithm
  - check which condition $PAP'=LDL'$ fails

Extension of algorithm

- No extension: algorithm works only for square, symmetric indefinite matrix.

OutLine

- Review Bunch-Parlett diagonal pivoting
- Partial pivoting: basic idea
- Implementation of partial pivoting
- Example

Example (partial pivoting) [1]

$\alpha \approx 0.6404$

iteration $1: k=1$

$\lambda_1 = \max |A(2:4,1)| = |A_{21}| = 12$ $r=2$ $\lambda_r = \max |(12,-13,4)| = 13$

$A^{(1)} = A$

6	12	3	-6
12	-8	-13	4
3	-13	-7	1
-6	4	1	6

1. $6 = |a_{11}| \geq \alpha\lambda_1 = 7.6847$ invalid
2. $78 = |a_{11}|\lambda_r \geq \alpha\lambda_1^2 = 92.2159$ invalid
3. $8 = |a_{rr}| \geq \alpha\lambda_r = 8.325$ invalid
4. $|a_{rr}| < \alpha\lambda_r$ 2×2 pivot

12 $P(k+1) \leftrightarrow P(r)$,since *k +1* = *r* , we don't need permutation

13 do interchange row/col $k+1 \leftrightarrow r$, since *k +1* = *r* , we don't need permutation

1. $A(k+1,k+1) \leftrightarrow A(r,r)$
2. $A(r+1:n,k+1) \leftrightarrow A(r+1:n,r)$
3. $A(k+1,k) \leftrightarrow A(r,k)$
4. $A(k+2:r-1:k+1) \leftrightarrow A(r,k+2:r-1)$

Example (partial pivoting) [2]

14 do interchange row $k+1 \leftrightarrow r$, since *k +1 = r* , we don't need permutation

$$L(k+1, 1:k-1) \leftrightarrow L(r, 1:k-1)$$

15 do 2x2 pivot :

$$\tilde{A}^{(k)} = \begin{pmatrix} D_{k-1} & & \\ & E & c^T \\ & c & B \end{pmatrix} = \begin{pmatrix} I & & \\ & I & \\ & cE^{-1} & I \end{pmatrix} \begin{pmatrix} D_{k-1} & & \\ & E & \\ & & B - cE^{-1}c^T \end{pmatrix} \left(L^{(k)}\right)^T$$

$A^{(1)}$

6	12	3	-6
12	-8	-13	4
3	-13	-7	1
-6	4	1	6

=

$L^{(1)}$

1			
	1		
-0.6875	0.5938	1	
0	-0.5		1

$A^{(3)}$

6	12		
12	-8		
		2.7813	-5.5
		-5.5	8

$\left(L^{(1)}\right)^T$

16 $k \leftarrow k+2$ and $pivot(k) = 2$

$k = 3$

$pivot =$

2	0	0	0

Example (partial pivoting) [3]

iteration $2 : k = 3$

$A^{(3)}$

6	12		
12	-8		
		2.7813	-5.5
		-5.5	8

$\lambda_1 = \max\left|A(3:4,3)\right| = \left|A_{43}\right| = 5.5$ $\quad r = 4 \quad$ $\lambda_r = \max\left|(-5.5)\right| = 5.5$

1. $2.7813 = \left|a_{33}\right| \geq \alpha\lambda_1 = 3.5221$ invalid
2. $15.2971 = \left|a_{11}\right|\lambda_r \geq \alpha\lambda_1^2 = 19.3717$ invalid
3. $8 = \left|a_{rr}\right| \geq \alpha\lambda_r = 3.5221$ 1×1 pivot

7 $\quad P(k) \leftrightarrow P(r)$ $\qquad$ $P = (1,2,3,4) \xrightarrow{3\leftrightarrow4} P = (1,2,4,3)$

Compute $\tilde{A}^{(k)} = P_k A^{(k)} P_k^T$

8
1. $A(k,k) \leftrightarrow A(r,r)$
2. $A(r+1:n,k) \leftrightarrow A(r+1:n,r)$
3. $A(k+1:r-1,k) \leftrightarrow A(r,k+1:r-1)$

$\tilde{A}^{(3)}$

6	12		
12	-8		
		8	-5.5
		-5.5	2.7813

Example (partial pivoting) [4]

Compute $\tilde{L}^{(k-1)} = P_k L^{(k-1)} P_k^T$ (9) $L(k,1:k-1) \leftrightarrow L(r,1:k-1)$

$$L^{(1)} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ -0.6875 & 0.5938 & 1 & \\ 0 & -0.5 & & 1 \end{pmatrix} \xrightarrow{3 \leftrightarrow 4} \tilde{L}^{(1)} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ 0 & -0.5 & 1 & \\ -0.6875 & 0.5938 & & 1 \end{pmatrix}$$

(10) do 1x1 pivot : $\tilde{A}^{(k)} = \begin{pmatrix} D_{k-1} & & \\ & a_{kk}^{(k)} & c^T \\ & c & B \end{pmatrix} = \begin{pmatrix} I & & \\ & 1 & \\ & c / a_{kk}^{(k)} & I \end{pmatrix} \begin{pmatrix} D_{k-1} & & \\ & a_{kk}^{(k)} & \\ & & B - cc^T / a_{kk}^{(k)} \end{pmatrix} \left(L^{(k)}\right)^T$

$$\tilde{A}^{(3)} = \begin{pmatrix} 6 & 12 & & \\ 12 & -8 & & \\ & & 8 & -5.5 \\ & & -5.5 & 2.7813 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -0.6875 & 1 \end{pmatrix}}_{L^{(3)}} \underbrace{\begin{pmatrix} 6 & 12 & & \\ 12 & -8 & & \\ & & 8 & \\ & & & -1 \end{pmatrix}}_{A^{(4)}} \left(L^{(3)}\right)^T$$

Example (partial pivoting) [5]

11 $k \leftarrow k+1$ and $pivot(k)=1$

$k=4$ $pivot =$

2	0	1	0

$\text{iteration } 3: k=4$

issue proper termination condition.

To sum up

$PAP^T = LDL^T$

$P =$

1	2	4	3

$pivot =$

2	0	1	1

$L = \tilde{L}^{(1)} L^{(3)} =$

1			
	1		
0	-0.5	1	
-0.6875	0.5938	-0.6875	1

$D = A^{(4)} =$

6	12		
12	-8		
		8	
			-1

Example: complete pivot versus partial pivot

A

6	12	3	-6
12	-8	-13	4
3	-13	-7	1
-6	4	1	6

$$\alpha = \frac{1+\sqrt{17}}{8}$$

Partial pivot

$P =$

1	2	4	3

$pivot =$

2	0	1	1

$L =$

1			
	1		
0	-0.5	1	
-0.6875	0.5938	-0.6875	1

$D =$

6			
12	-8		
		8	
			-1

Complete pivot

$P =$

2	3	4	1

$pivot =$

2	0	1	1

$L =$

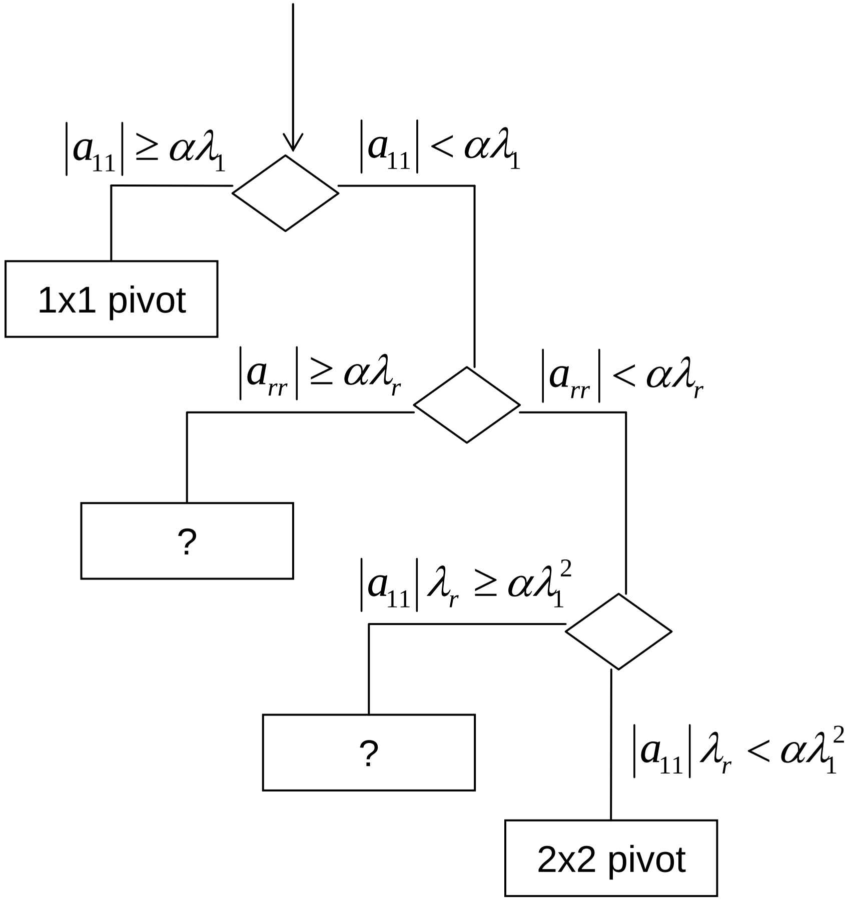
1			
	1		
0.1327	-0.3984	1	
0.3982	-1.1681	-1.0967	1

$D =$

-8			
-13	-7		
		5.8584	
			-2.3202

Exercise

How about flow-chart of left figure?



Bunch-Kaufman proposed flow chart