Chapter 12 Gaussian Elimination (II)

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Reference book: David Kincaid, Numerical Analysis

Reference lecture note: Wen-wei Lin, chapter 2, matrix computation

http://math.ntnu.edu.tw/~min/matrix_computation.html

OutLine

- Weakness of *A=LU*
  - erroneous judgment: A is invertible but A=LU does not exist
  - unstable, A is far from LU
- *A=LU* versus *PA=LU*
- Pivoting strategy
- Implementation of *PA=LU*
- *MATLAB usage*

Fail of *LU*: singular of leading principal minor

$$-\frac{1}{0} = -\infty \qquad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \infty & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & ? \end{pmatrix} \qquad a_{11} = 0,\ \ LU \ \ \text{cannot continue}$$

Question: How about interchanging row 1 and row 2?

$$P\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = P\begin{pmatrix} 1 \\ 2 \end{pmatrix} \xrightarrow[P=(2,1)]{} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

unstable of *LU*: near singular of leading principal minor [1]

$$
\begin{tikzcd}
\begin{pmatrix} \boxed{\varepsilon} & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \arrow[r] \arrow[d, "\varepsilon \to 0"] & \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \dfrac{1}{\varepsilon - 1}\begin{pmatrix} 1 & -1 \\ -1 & \varepsilon \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \dfrac{1}{1-\varepsilon}\begin{pmatrix} 1 \\ 1-2\varepsilon \end{pmatrix} \arrow[d, "\varepsilon \to 0", "?"'] \\
\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \arrow[r] & \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\end{tikzcd}
$$

Theoretical: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left(1 + \varepsilon + O\left(\varepsilon^2\right)\right)\begin{pmatrix} 1 \\ 1-2\varepsilon \end{pmatrix} = \begin{pmatrix} 1+\varepsilon \\ 1-\varepsilon \end{pmatrix} + O\left(\varepsilon^2\right) \xrightarrow{\varepsilon \to 0} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, why?

Numerical:

$$A = \begin{pmatrix} \varepsilon & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \boxed{\dfrac{1}{\varepsilon}} & 1 \end{pmatrix}\begin{pmatrix} \varepsilon & 1 \\ 0 & \boxed{1 - \dfrac{1}{\varepsilon}} \end{pmatrix}$$

Problem: for double-precision, we only have 16 digit-accuracy, we cannot accept $\pm\infty$

if $\varepsilon \to 0$ then $L_{21} \nearrow \infty,\ U_{22} \searrow -\infty$

unstable of *LU*: near singular of leading principal minor [2]

$$A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \longrightarrow U\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = L^{-1}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \longrightarrow \begin{pmatrix} \varepsilon & 1 \\ 0 & 1-\frac{1}{\varepsilon} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{\varepsilon} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2-\frac{1}{\varepsilon} \end{pmatrix}$$

backward substitution $\longrightarrow$

step 1: $\left(1-\frac{1}{\varepsilon}\right)x_2 = 2-\frac{1}{\varepsilon} \longrightarrow x_2 = \frac{2-1/\varepsilon}{1-1/\varepsilon} \xrightarrow{\varepsilon\to 0} 1$

step 2: $\varepsilon x_1 + x_2 = 1 \longrightarrow x_1 = \frac{1-x_2}{\varepsilon} \rightarrow 0$

Question: why not $x_2 = \frac{2-1/\varepsilon}{1-1/\varepsilon} = \frac{-1/\varepsilon}{-1/\varepsilon}\frac{1-2\varepsilon}{1-\varepsilon} = 1-\varepsilon+O\left(\varepsilon^2\right)$

$1-1/\varepsilon \xrightarrow{\varepsilon\to 0} -1/\varepsilon$ due to rounding error

$2-1/\varepsilon \xrightarrow{\varepsilon\to 0} -1/\varepsilon$ due to rounding error

unstable of *LU*: near singular of leading principal minor [3]

Case 1: $\varepsilon = 10^{-14}$

```
>> [L, U, isLU_succ] = lu_nopivot_lsc(A)

L =

  1.0e+014 *

   0.00000000000001                  0
   1.00000000000000   0.00000000000001


U =

  1.0e+013 *

   0.00000000000000   0.00000000000010
                  0  -9.99999999999990


isLU_succ =

     0

>> U\(L\[1;2])
Warning: Matrix is close to singular or badly scaled.
         Results may be inaccurate. RCOND = 1.000000e-028.

ans =

   0.99920072216264
   0.99999999999999
```

Case 2: $\varepsilon = 10^{-18}$

```
>> [L, U, isLU_succ] = lu_nopivot_lsc(A)

L =

  1.0e+017 *

   0.00000000000000                  0
  10.00000000000000   0.00000000000000


U =

  1.0e+017 *

   0.00000000000000   0.00000000000000
                  0 -10.00000000000000


isLU_succ =

     0

>> U\(L\[1;2])
Warning: Matrix is close to singular or badly scaled.
         Results may be inaccurate. RCOND = 1.000000e-036.

ans =

     0
     1
```

wrong solution

unstable of *LU*: near singular of leading principal minor [4]

Question: How about interchanging row 1 and row 2?

$$-\varepsilon \quad \begin{pmatrix} 1 & 1 \\ \varepsilon & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \xrightarrow{\text{LU-factorization}} A = \begin{pmatrix} 1 & 1 \\ \varepsilon & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1-\varepsilon \end{pmatrix}$$

$$A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \longrightarrow U\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = L^{-1}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1-\varepsilon \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\varepsilon & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1-2\varepsilon \end{pmatrix}$$

backward substitution $\longrightarrow$

step 1: $(1-\varepsilon)x_2 = 1-2\varepsilon \longrightarrow x_2 = \dfrac{1-2\varepsilon}{1-\varepsilon} \xrightarrow{\varepsilon\to 0} 1$

step 2: $x_1 + x_2 = 2 \longrightarrow x_1 = 2 - x_2 \to 1$

Key observation:

without pivoting

$$x_2 = \frac{2-1/\varepsilon}{1-1/\varepsilon} \xrightarrow{\varepsilon\to 0} 1$$ (rounding normal number)

pivoting

$$x_2 = \frac{1-2\varepsilon}{1-\varepsilon} \xrightarrow{\varepsilon\to 0} 1$$ (rounding error does not occur for normal number)

unstable of *LU*: near singular of leading principal minor [5]

Case 1: $\varepsilon = 10^{-14}$

```
>> [L, U, isLU_succ] = lu_nopivot_lsc(A)

L =

   1.00000000000000                  0
   0.00000000000001   1.00000000000000


U =

   1.00000000000000   1.00000000000000
                  0   0.99999999999999


isLU_succ =

     0

>> U\(L\[2 ; 1])

ans =

   1.00000000000001
   0.99999999999999
```

Why not 1?

Case 2: $\varepsilon = 10^{-18}$

```
>> [L, U, isLU_succ] = lu_nopivot_lsc(A)

L =

   1.00000000000000                  0
   0.00000000000000   1.00000000000000


U =

     1     1
     0     1


isLU_succ =

     0

>> U\(L\[2 ; 1])

ans =

     1
     1
```

unstable of *LU*: cause and controllability

define $|A|_\infty = \max_{1\le i,j\le n} |A_{ij}| =$ largest component of matrix A

without pivoting $\varepsilon = 10^{-18}$

```
>> [L, U, isLU_succ] = lu_nopivot_lsc(A)

L =

  1.0e+017 *

   0.00000000000000                  0
  10.00000000000000   0.00000000000000


U =

  1.0e+017 *

   0.00000000000000   0.00000000000000
                  0 -10.00000000000000
```

$|L|_\infty = |U|_\infty = 10^{16}$ (**NOT** stable)

pivoting $\varepsilon = 10^{-18}$

```
>> [L, U, isLU_succ] = lu_nopivot_lsc(A)

L =

   1.00000000000000                  0
   0.00000000000000   1.00000000000000


U =

     1     1
     0     1
```

$|L|_\infty = |U|_\infty = 1$ (stable)

Objective: control growth rate of $|L|_\infty, |U|_\infty$ $\longrightarrow$ control multiplier

OutLine

- Weakness of *A=LU*
- *A=LU* versus *PA=LU*
  - controllability of lower triangle matrix L
- Pivoting strategy
- Implementation of *PA=LU*
- *MATLAB usage*

Recall *LU* example in chapter 11 [1]

$$-\frac{1}{6}\begin{pmatrix}12\\3\\-6\end{pmatrix}\qquad \overset{A=A^{(1)}}{\begin{pmatrix}6 & -2 & 2 & 4\\12 & -8 & 6 & 10\\3 & -13 & 9 & 3\\-6 & 4 & 1 & -18\end{pmatrix}} = \overset{L^{(1)}}{\begin{pmatrix}1 & & & \\2 & 1 & & \\0.5 & & 1 & \\-1 & & & 1\end{pmatrix}}\overset{A^{(2)}}{\begin{pmatrix}6 & -2 & 2 & 4\\0 & -4 & 2 & 2\\0 & -12 & 8 & 1\\0 & 2 & 3 & -14\end{pmatrix}}$$

$L_{2,1}^{(1)} = 2 > 1$ since $A_{11}^{(1)} = 6$ is not maximum among $A(1:4,1)$

$$-\frac{1}{-4}\begin{pmatrix}-12\\2\end{pmatrix}\qquad \overset{A^{(2)}}{\begin{pmatrix}6 & -2 & 2 & 4\\0 & -4 & 2 & 2\\0 & -12 & 8 & 1\\0 & 2 & 3 & -14\end{pmatrix}} = \overset{L^{(2)}}{\begin{pmatrix}1 & & & \\ & 1 & & \\ & 3 & 1 & \\ & -0.5 & & 1\end{pmatrix}}\overset{A^{(3)}}{\begin{pmatrix}6 & -2 & 2 & 4\\0 & -4 & 2 & 2\\0 & 0 & 2 & -5\\0 & 0 & 4 & -13\end{pmatrix}}$$

$L_{3,2}^{(2)} = 3 > 1$ since $A_{22}^{(2)} = -4$ is not maximum among $A^{(2)}(2:4,2)$

Recall *LU* example in chapter 11 [2]

$$
-\frac{4}{2}\quad
\overset{A^{(3)}}{\begin{pmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 4 & -13 \end{pmatrix}}
=
\overset{L^{(3)}}{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 2 & 1 \end{pmatrix}}
\overset{A^{(4)}}{\begin{pmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{pmatrix}}
$$

$L_{4,3}^{(3)} = 2 > 1$ since $A_{33}^{(3)} = 2$ is not maximum among $A^{(3)}(3:4,3)$

$$
\overset{A}{\begin{pmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{pmatrix}}
=
\overset{L}{\begin{pmatrix} 1 & & & \\ 2 & 1 & & \\ 0.5 & 3 & 1 & \\ -1 & -0.5 & 2 & 1 \end{pmatrix}}
\overset{U}{\begin{pmatrix} 6 & -2 & 2 & 4 \\ & -4 & 2 & 2 \\ & & 2 & -5 \\ & & & -3 \end{pmatrix}}
$$

$|A|_\infty = 18 \qquad |L|_\infty = 3 \qquad |U|_\infty = 6$

Question: How can we control $|L|_\infty$, is uniform bound possible?

$PA = LU$ in MATLAB

```
>> help lu

 LU    LU factorization.
    [L,U] = LU(X) stores an upper triangular matrix in U and a
    "psychologically lower triangular matrix" (i.e. a product
    of lower triangular and permutation matrices) in L, so
    that X = L*U. X can be rectangular.

    [L,U,P] = LU(X) returns lower triangular matrix L, upper
    triangular matrix U, and permutation matrix P so that
    P*X = L*U.
```

$$A = \begin{pmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{pmatrix} \qquad PA = \begin{pmatrix} 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \\ 6 & -2 & 2 & 4 \end{pmatrix}$$

$$P = (2,3,4,1) = \begin{pmatrix} e_2^T \\ e_3^T \\ e_4^T \\ e_1^T \end{pmatrix}$$

```
>> [L, U, P] = lu(A)

L =

    1.0000         0         0         0
    0.2500    1.0000         0         0
   -0.5000         0    1.0000         0
    0.5000   -0.1818    0.0909    1.0000


U =

   12.0000   -8.0000    6.0000   10.0000
         0  -11.0000    7.5000    0.5000
         0         0    4.0000  -13.0000
         0         0         0    0.2727


P =

     0     1     0     0
     0     0     1     0
     0     0     0     1
     1     0     0     0
```

PA = LU: assume P is given [1]

$$-\frac{1}{12}\begin{pmatrix}3\\-6\\6\end{pmatrix} \qquad \underbrace{\begin{pmatrix}12 & -8 & 6 & 10\\ 3 & -13 & 9 & 3\\ -6 & 4 & 1 & -18\\ 6 & -2 & 2 & 4\end{pmatrix}}_{PA \equiv A^{(1)}} = \underbrace{\begin{pmatrix}1 & & & \\ 0.25 & 1 & & \\ -0.5 & & 1 & \\ 0.5 & & & 1\end{pmatrix}}_{L^{(1)}} \underbrace{\begin{pmatrix}12 & -8 & 6 & 10\\ 0 & -11 & 7.5 & 0.5\\ 0 & 0 & 4 & -13\\ 0 & 2 & -1 & -1\end{pmatrix}}_{A^{(2)}}$$

$\left|L^{(1)}(2:4,1)\right| < \vec{1}$ since $A_{11}^{(1)} = 12$ is maximum among $A^{(1)}(1:4,1)$

$$-\frac{1}{-11}\begin{pmatrix}0\\2\end{pmatrix} \qquad \underbrace{\begin{pmatrix}12 & -8 & 6 & 10\\ 0 & -11 & 7.5 & 0.5\\ 0 & 0 & 4 & -13\\ 0 & 2 & -1 & -1\end{pmatrix}}_{A^{(2)}} = \underbrace{\begin{pmatrix}1 & & & \\ & 1 & & \\ & 0 & 1 & \\ & -2/11 & & 1\end{pmatrix}}_{L^{(2)}} \underbrace{\begin{pmatrix}12 & -8 & 6 & 10\\ 0 & -11 & 7.5 & 0.5\\ 0 & 0 & 4 & -13\\ 0 & 0 & 4/11 & -10/11\end{pmatrix}}_{A^{(3)}}$$

$\left|L^{(2)}(3:4,2)\right| < \vec{1}$ since $A_{22}^{(2)} = -11$ is maximum among $A^{(2)}(2:4,2)$

$PA = LU$: assume P is given [2]

$$-\frac{1}{4}\times\frac{4}{11}$$

$$\overset{A^{(3)}}{\begin{pmatrix} 12 & -8 & 6 & 10 \\ 0 & -11 & 7.5 & 0.5 \\ 0 & 0 & 4 & -13 \\ 0 & 0 & 4/11 & -10/11 \end{pmatrix}} = \overset{L^{(3)}}{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1/11 & 1 \end{pmatrix}} \overset{A^{(4)}}{\begin{pmatrix} 12 & -8 & 6 & 10 \\ 0 & -11 & 7.5 & 0.5 \\ 0 & 0 & 4 & -13 \\ 0 & 0 & 0 & 3/11 \end{pmatrix}}$$

$L^{(3)} = \frac{1}{11} < 1$ since $A^{(3)}_{33} = 4$ is maximum among $A^{(3)}(3:4,3)$

$$\overset{PA}{\begin{pmatrix} 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \\ 6 & -2 & 2 & 4 \end{pmatrix}} = \overset{L}{\begin{pmatrix} 1 & & & \\ 0.25 & 1 & & \\ -0.5 & 0 & 1 & \\ 0.5 & -2/11 & 1/11 & 1 \end{pmatrix}} \overset{U}{\begin{pmatrix} 12 & -8 & 6 & 10 \\ & -11 & 7.5 & 0.5 \\ & & 4 & -13 \\ & & & 3/11 \end{pmatrix}}$$

$|A|_\infty = 18$ $\quad$ $|L|_\infty = 0.5 < 1$ $\quad$ $|U|_\infty = 13$

Observation: though proper permutation, we can control $|L|_\infty < 1$

Question: in fact, we cannot know permutation matrix in advance, how can we do?

Sequence of matrices during LU-decomposition

$$A = A^{(1)} = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} \\ \vdots & \cdots & \ddots & \vdots \\ a_{n1}^{(1)} & a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} \end{pmatrix} \xrightarrow{A = L^{(1)} A^{(2)}} A^{(2)} = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & \cdots & a_{3n}^{(2)} \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ 0 & a_{n2}^{(2)} & a_{n3}^{(2)} & \cdots & a_{nn}^{(2)} \end{pmatrix}$$

$$\cdots \xrightarrow{A = L^{(1)} L^{(2)} \cdots L^{(k-1)} A^{(k)}} A^{(k)} = \begin{pmatrix} a_{11}^{(1)} & \cdots & \cdots & \cdots & \cdots & a_{1n}^{(1)} \\ 0 & \ddots & & & & \\ \vdots & & a_{k-1,k-1}^{(k-1)} & \cdots & \cdots & a_{k-1,n}^{(k-1)} \\ \vdots & & 0 & a_{k,k}^{(k)} & \cdots & a_{k,n}^{(k)} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & a_{nk}^{(k)} & \cdots & a_{nn}^{(k)} \end{pmatrix}$$

$$L^{(1)} L^{(2)} \cdots L^{(k-1)} = \begin{array}{cc} \left(\begin{array}{c|c|c} \times & & \\ \hline \times & 1 & \\ \cline{2-3} \times & 0 & I \end{array}\right) & \begin{array}{l} \\ \longleftarrow k-row \\ \\ \end{array} \\ \begin{array}{c} \qquad\uparrow \\ \qquad k-col \end{array} & \end{array}$$

OutLine

- Weakness of *A=LU*
- *A=LU* versus *PA=LU*
- Pivoting strategy
  - partial pivoting (we adopt this formulation)
  - complete pivoting
- Implementation of *PA=LU*
- *MATLAB usage*

Partial pivoting and complete pivoting

Partial pivoting

(1) find a $p \in \{k, k+1, \cdots, n\}$ such that $\left|A_{pk}^{(k)}\right| = \max\left|A^{(k)}(k:n,k)\right|$

(2) swap row $A^{(k)}(k,1:n)$ and row $A^{(k)}(p,1:n)$

then $PA = LU$ with $|L|_\infty \le 1$

complete pivoting

(1) find a $p, q \in \{k, k+1, \cdots, n\}$ such that $\left|A_{pq}^{(k)}\right| = \max\left|A^{(k)}(k:n,k:n)\right|$

(2) swap row $A^{(k)}(k,1:n)$ and row $A^{(k)}(p,1:n)$

(2) swap column $A^{(k)}(1:n,k)$ and column $A^{(k)}(1:n,q)$

then $PA\Pi = LU$ with $|L|_\infty \le 1$

Recall permutation matrix

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ Define permutation matrix $P = (1,3,2) \equiv \begin{pmatrix} e_1^T \\ e_3^T \\ e_2^T \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

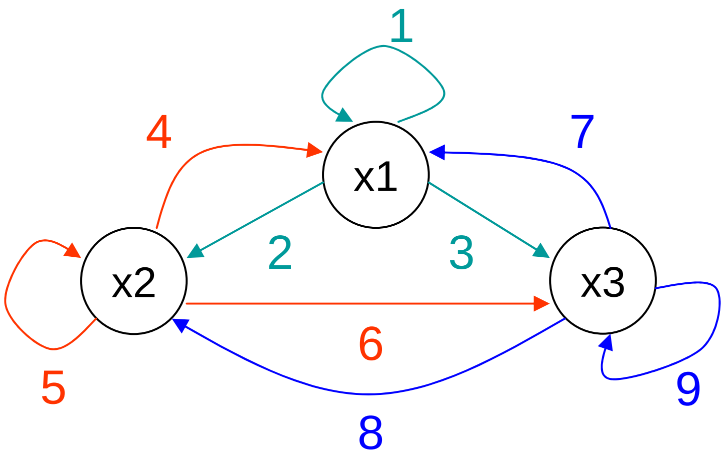
$$Px = \begin{pmatrix} e_1^T \\ e_3^T \\ e_2^T \end{pmatrix} x = \begin{pmatrix} e_1^T x \\ e_3^T x \\ e_2^T x \end{pmatrix} = \begin{pmatrix} x_1 \\ x_3 \\ x_2 \end{pmatrix} \xrightarrow{\text{interchange row 2, 3}} PA = \begin{pmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{pmatrix}$$

$$x^T P^T = (Px)^T = \begin{pmatrix} x_1 \\ x_3 \\ x_2 \end{pmatrix}^T = \begin{pmatrix} x_1 & x_3 & x_2 \end{pmatrix} \xrightarrow{\text{interchange column 2, 3}} AP^T = \begin{pmatrix} 1 & 3 & 2 \\ 4 & 6 & 5 \\ 7 & 9 & 8 \end{pmatrix}$$

Symmetric permutation: $A \to PAP^T$

$$\underset{A}{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}} \xrightarrow[\text{row 2, 3}]{\text{interchange}} \underset{PA}{\begin{pmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{pmatrix}} \xrightarrow[\text{column 2, 3}]{\text{interchange}} \underset{(PA)P^T}{\begin{pmatrix} 1 & 3 & 2 \\ 7 & 9 & 8 \\ 4 & 6 & 5 \end{pmatrix}}$$

Meaning of matrix coefficient



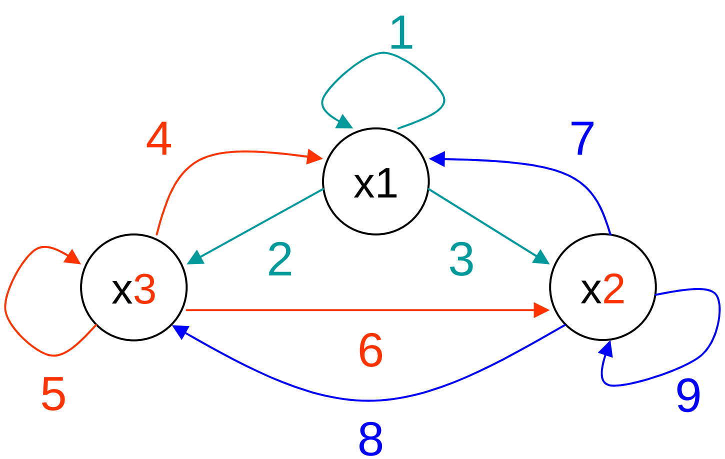
A_{ij} = relationship between node i and node j

$\sum_{j=1}^{3} A_{ij} x_j = b_i$: constraint on node i

node i is named x_i

$$
\begin{array}{c c}
 & \begin{array}{ccc} x1 & x2 & x3 \end{array} \\
\begin{array}{c} x1 \\ x2 \\ x3 \end{array} & \begin{pmatrix} 1 & \mathbf{2} & 3 \\ 4 & \mathbf{5} & 6 \\ 7 & \mathbf{8} & 9 \end{pmatrix}
\end{array}
$$

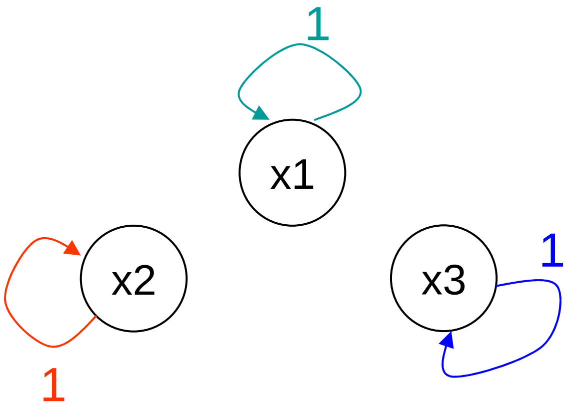
$P=(1,3,2)$ change node 2 to node 3 and node 3 to node 2



write constraint for now node index

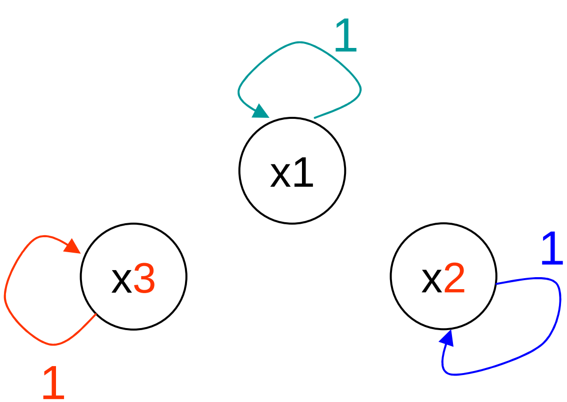
$$
\begin{array}{c c}
 & \begin{array}{ccc} x1 & x2 & x3 \end{array} \\
\begin{array}{c} x1 \\ x2 \\ x3 \end{array} & \begin{pmatrix} 1 & 3 & \mathbf{2} \\ \mathbf{7} & 9 & \mathbf{8} \\ 4 & 6 & \mathbf{5} \end{pmatrix}
\end{array}
$$

Identity matrix is invariant under symmetric permutation



$$\begin{array}{c c} & \begin{array}{ccc} x1 & x2 & x3 \end{array} \\ \begin{array}{c} x1 \\ x2 \\ x3 \end{array} & \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \end{array}$$

$P = (1, 3, 2)$ change node 2 to node 3 and node 3 to node 2



$$\begin{array}{c c} & \begin{array}{ccc} x1 & x2 & x3 \end{array} \\ \begin{array}{c} x1 \\ x2 \\ x3 \end{array} & \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \end{array}$$

$PA = LU$: partial pivoting [1]

$$A = A^{(1)} = \begin{pmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{pmatrix} \xrightarrow[P_1 = (2,1,3,4)]{\left|A^{(1)}_{21}\right| = \max\left|A^{(1)}(1:4,1)\right|} P_1 A^{(1)} = \tilde{A}^{(1)} = \begin{pmatrix} 12 & -8 & 6 & 10 \\ 6 & -2 & 2 & 4 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{pmatrix}$$

$$-\frac{1}{12}\begin{pmatrix} 6 \\ 3 \\ -6 \end{pmatrix}$$

$$\tilde{A}^{(1)} = \begin{pmatrix} 12 & -8 & 6 & 10 \\ 6 & -2 & 2 & 4 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{pmatrix} = L^{(1)} = \begin{pmatrix} 1 & & & \\ 0.5 & 1 & & \\ 0.25 & & 1 & \\ -0.5 & & & 1 \end{pmatrix} \quad A^{(2)} = \begin{pmatrix} 12 & -8 & 6 & 10 \\ 0 & 2 & -1 & -1 \\ 0 & -11 & 7.5 & 0.5 \\ 0 & 0 & 4 & -13 \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} 12 & -8 & 6 & 10 \\ 0 & 2 & -1 & -1 \\ 0 & -11 & 7.5 & 0.5 \\ 0 & 0 & 4 & -13 \end{pmatrix} \xrightarrow[P_2 = (1,3,2,4)]{\left|A^{(2)}_{32}\right| = \max\left|A^{(2)}(2:4,2)\right|} P_2 A^{(2)} = \tilde{A}^{(2)} = \begin{pmatrix} 12 & -8 & 6 & 10 \\ 0 & -11 & 7.5 & 0.5 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 4 & -13 \end{pmatrix}$$

$PA = LU$: partial pivoting [2]

$$P_1 A = \tilde{A}^{(1)} = L^{(1)} A^{(2)} = L^{(1)} \left(P_2^T P_2\right) A^{(2)} = L^{(1)} P_2^T \left(P_2 A^{(2)}\right) = L^{(1)} P_2^T \tilde{A}^{(2)}$$

$$P_2 P_1 A = \left(P_2 L^{(1)} P_2^T\right) \tilde{A}^{(2)} = \tilde{L}^{(1)} \tilde{A}^{(2)} \quad \text{where} \quad P_2 P_1 = (1,3,2,4)(2,1,3,4) = (2,3,1,4)$$

$$L^{(1)} = \begin{pmatrix} 1 & & & \\ 0.5 & 1 & & \\ 0.25 & & 1 & \\ -0.5 & & & 1 \end{pmatrix} \xrightarrow{L^{(1)} P_2^T} L^{(1)} P_2^T = \begin{pmatrix} 1 & & & \\ 0.5 & & 1 & \\ 0.25 & 1 & & \\ -0.5 & & & 1 \end{pmatrix} \xrightarrow{P_2\left(L^{(1)} P_2^T\right)} P_2 L^{(1)} P_2^T = \begin{pmatrix} 1 & & & \\ 0.25 & 1 & & \\ 0.5 & & 1 & \\ -0.5 & & & 1 \end{pmatrix}$$

Interchange columns 2, 3

Interchange rows 2, 3

$$P_2 = (1,3,2,4)$$

verify

$$P_2 P_1 A = \begin{pmatrix} 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ 6 & -2 & 2 & 4 \\ -6 & 4 & 1 & -18 \end{pmatrix} = \tilde{L}^{(1)} \tilde{A}^{(2)} = \begin{pmatrix} 1 & & & \\ 0.25 & 1 & & \\ 0.5 & & 1 & \\ -0.5 & & & 1 \end{pmatrix} \begin{pmatrix} 12 & -8 & 6 & 10 \\ 0 & -11 & 7.5 & 0.5 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 4 & -13 \end{pmatrix}$$

PA = LU: partial pivoting [3]

$$-\frac{1}{-11}\begin{pmatrix}2\\0\end{pmatrix} \qquad \overset{\tilde{A}^{(2)}}{\begin{pmatrix}12 & -8 & 6 & 10\\ 0 & -11 & 7.5 & 0.5\\ 0 & 2 & -1 & -1\\ 0 & 0 & 4 & -13\end{pmatrix}} = \overset{L^{(2)}}{\begin{pmatrix}1 & & & \\ & 1 & & \\ & -2/11 & 1 & \\ & 0 & & 1\end{pmatrix}} \overset{A^{(3)}}{\begin{pmatrix}12 & -8 & 6 & 10\\ 0 & -11 & 7.5 & 0.5\\ 0 & 0 & 4/11 & -10/11\\ 0 & 0 & 4 & -13\end{pmatrix}}$$

$$\overset{A^{(3)}}{\begin{pmatrix}12 & -8 & 6 & 10\\ 0 & -11 & 7.5 & 0.5\\ 0 & 0 & 4/11 & -10/11\\ 0 & 0 & 4 & -13\end{pmatrix}} \xrightarrow[P_3=(1,2,4,3)]{\left|A_{43}^{(3)}\right| = \max\left|A^{(3)}(3:4,3)\right|} \overset{P_3 A^{(3)} = \tilde{A}^{(3)}}{\begin{pmatrix}12 & -8 & 6 & 10\\ 0 & -11 & 7.5 & 0.5\\ 0 & 0 & 4 & -13\\ 0 & 0 & 4/11 & -10/11\end{pmatrix}}$$

$$P_2 P_1 A = \tilde{L}^{(1)} \tilde{A}^{(2)} = \tilde{L}^{(1)} L^{(2)} A^{(3)} = \tilde{L}^{(1)} L^{(2)} \left(P_3^T P_3\right) A^{(3)} = \tilde{L}^{(1)} L^{(2)} P_3^T \left(P_3 A^{(3)}\right) = \tilde{L}^{(1)} L^{(2)} P_3^T \tilde{A}^{(3)}$$

$$P_3 P_2 P_1 A = \left(P_3 \tilde{L}^{(1)} L^{(2)} P_3^T\right) \tilde{A}^{(3)} = \tilde{L}^{(2)} \tilde{A}^{(3)}$$ where $P_3\left(P_2 P_1\right) = (1,2,4,3)(2,3,1,4) = (2,3,4,1)$

PA = LU: partial pivoting [4]

$$\tilde{L}^{(1)}L^{(2)} = \begin{pmatrix} 1 & & & \\ 0.25 & 1 & & \\ 0.5 & -2/11 & 1 & \\ -0.5 & & & 1 \end{pmatrix}$$

$$\xrightarrow{\tilde{L}^{(1)}L^{(2)}P_3^T}$$

Interchange columns 3,4

$$P_3 = (1, 2, 4, 3)$$

$$\tilde{L}^{(1)}L^{(2)}P_3^T = \begin{pmatrix} 1 & & & \\ 0.25 & 1 & & \\ 0.5 & -2/11 & & 1 \\ -0.5 & 0 & 1 & \end{pmatrix}$$

$$\xrightarrow{P_3\left(\tilde{L}^{(1)}L^{(2)}P_3^T\right)}$$

Interchange rows 3.4

$$P_3\left(\tilde{L}^{(1)}L^{(2)}P_3^T\right) = \begin{pmatrix} 1 & & & \\ 0.25 & 1 & & \\ -0.5 & 0 & 1 & \\ 0.5 & -2/11 & & 1 \end{pmatrix}$$

verify

$$P_3P_2P_1A = \begin{pmatrix} 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \\ 6 & -2 & 2 & 4 \end{pmatrix} = \tilde{L}^{(2)}\tilde{A}^{(3)} = \begin{pmatrix} 1 & & & \\ 0.25 & 1 & & \\ -0.5 & 0 & 1 & \\ 0.5 & -2/11 & & 1 \end{pmatrix} \begin{pmatrix} 12 & -8 & 6 & 10 \\ 0 & -11 & 7.5 & 0.5 \\ 0 & 0 & 4 & -13 \\ 0 & 0 & 4/11 & -10/11 \end{pmatrix}$$

PA = LU: partial pivoting [5]

$$
-\frac{1}{4}\times\frac{4}{11}
\qquad
\overset{\tilde{A}^{(3)}}{\begin{pmatrix} 12 & -8 & 6 & 10 \\ 0 & -11 & 7.5 & 0.5 \\ 0 & 0 & 4 & -13 \\ 0 & 0 & 4/11 & -10/11 \end{pmatrix}}
=
\overset{L^{(3)}}{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1/11 & 1 \end{pmatrix}}
\overset{A^{(4)}}{\begin{pmatrix} 12 & -8 & 6 & 10 \\ 0 & -11 & 7.5 & 0.5 \\ 0 & 0 & 4 & -13 \\ 0 & 0 & 0 & 3/11 \end{pmatrix}}
$$

---

we have $P_3 P_2 P_1 A = \tilde{L}^{(2)} \tilde{A}^{(3)} = \tilde{L}^{(2)} L^{(3)} A^{(4)} \longrightarrow PA = LU$

where $P = P_3 P_2 P_1 = (2,3,4,1)$

$$
L = \tilde{L}^{(2)} L^{(3)} = \begin{pmatrix} 1 & & & \\ 0.25 & 1 & & \\ -0.5 & 0 & 1 & \\ 0.5 & -2/11 & 1/11 & 1 \end{pmatrix}
\qquad
U = A^{(4)} = \begin{pmatrix} 12 & -8 & 6 & 10 \\ & -11 & 7.5 & 0.5 \\ & & 4 & -13 \\ & & & 3/11 \end{pmatrix}
$$

$$\boxed{\left| L \right|_{\infty} = 0.5 < 1}$$

OutLine

- Weakness of *A=LU*
- *A=LU* versus *PA=LU*
- Pivoting strategy
- Implementation of *PA=LU*
  - commutability of GE matrix and permutation
  - recursive formula
- *MATLAB usage*

Implementation issue: commutability of GE matrix and permutation [1]

$$P_2 P_1 A = \left(P_2 L^{(1)} P_2^T \right) \tilde{A}^{(2)} = \tilde{L}^{(1)} \tilde{A}^{(2)}$$

$$P_2 L^{(1)} = \tilde{L}^{(1)} P_2$$

$$L^{(1)} = \begin{pmatrix} 1 & & & \\ 0.5 & 1 & & \\ 0.25 & & 1 & \\ -0.5 & & & 1 \end{pmatrix} \xrightarrow{P_2 L^{(1)} P_2^T} \tilde{L}^{(1)} = \begin{pmatrix} 1 & & & \\ 0.25 & 1 & & \\ 0.5 & & 1 & \\ -0.5 & & & 1 \end{pmatrix}$$

Interchange rows and columns 2, 3

$$P_2 = \left(1, \boxed{3, 2,}\ 4\right)$$

$$P_3 P_2 P_1 A = \left(P_3 \tilde{L}^{(1)} L^{(2)} P_3^T \right) \tilde{A}^{(3)} = \tilde{L}^{(2)} \tilde{A}^{(3)}$$

$$P_2 \left(\tilde{L}^{(1)} L^{(2)} \right) = \tilde{L}^{(2)} P_2$$

$$\tilde{L}^{(1)} L^{(2)} = \begin{pmatrix} 1 & & & \\ 0.25 & 1 & & \\ 0.5 & -2/11 & 1 & \\ -0.5 & & & 1 \end{pmatrix} \xrightarrow{P_3 \left(\tilde{L}^{(1)} L^{(2)} P_3^T \right)} \tilde{L}^{(2)} = \begin{pmatrix} 1 & & & \\ 0.25 & 1 & & \\ -0.5 & 0 & 1 & \\ 0.5 & -2/11 & & 1 \end{pmatrix}$$

Interchange rows and columns 3, 4

$$P_3 = \left(1, 2, \boxed{4, 3}\right)$$

Implementation issue: commutability of GE matrix and permutation [2]

Observation: If we define $\tilde{L}^{(0)} = I$, then $\tilde{L}^{(k-2)} L^{(k-1)} \rightarrow P_k \left(\tilde{L}^{(k-2)} L^{(k-1)} \right) P_k^T \equiv \tilde{L}^{(k-1)}$

where

$$\tilde{L}^{(k-2)} L^{(k-1)} = \begin{array}{c} \overbrace{}^{k-1} \\ \left(\begin{array}{c|c} W & O \\ \hline M & I \end{array} \right) \end{array} \begin{array}{l} \left.\right\} k-1 \\ \end{array} \qquad W = \begin{pmatrix} 1 & & \\ \times & \ddots & \\ \times & \times & 1 \end{pmatrix} \in R^{(k-1)\times(k-1)}$$

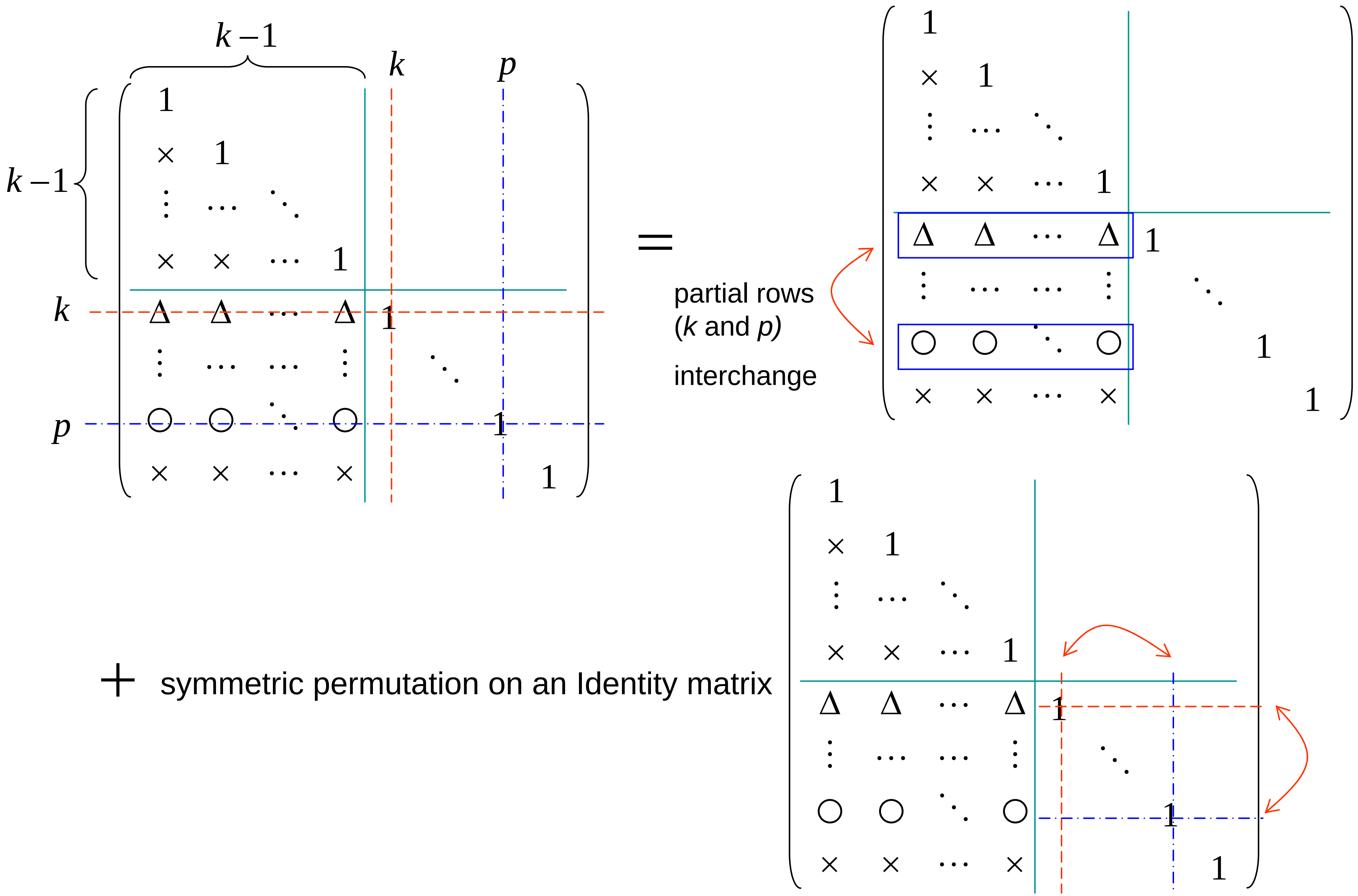
$$P_k = \left(1, 2, \cdots k-1, \underset{\substack{\uparrow \\ k-th}}{p}, k+1, \cdots, p-1, \underset{\substack{\uparrow \\ p-th}}{k}, p+1, \cdots, n\right)$$

interchanges rows k, p or columns k, p and $p \notin \{1, 2, 3, \cdots, k\}$

$$\begin{array}{c} \\ k-1 \left\{ \begin{array}{c} \\ \\ \\ \\ \end{array} \right. \\ k \\ \\ p \\ \\ \end{array} \begin{array}{c} \begin{array}{cc} \overbrace{\qquad\qquad\qquad\qquad}^{k-1} & \quad k \qquad\quad p \qquad\quad \end{array} \\ \left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ \times & 1 & & & & & & \\ \vdots & \cdots & \ddots & & & & & \\ \times & \times & \cdots & 1 & & & & \\ \hline \Delta & \Delta & \cdots & \Delta & 1 & & & \\ \vdots & \cdots & \cdots & \vdots & & \ddots & & \\ \bigcirc & \bigcirc & \ddots & \bigcirc & & & 1 & \\ \times & \times & \cdots & \times & & & & 1 \end{array} \right) \end{array} \xrightarrow{P_k \left(\tilde{L}^{(k-2)} L^{(k-1)} \right) P_k^T} \left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ \times & 1 & & & & & & \\ \vdots & \cdots & \ddots & & & & & \\ \times & \times & \cdots & 1 & & & & \\ \hline \bigcirc & \bigcirc & \cdots & \bigcirc & 1 & & & \\ \vdots & \cdots & \cdots & \vdots & & 1 & & \\ \Delta & \Delta & \ddots & \Delta & & & \ddots & \\ \times & \times & \cdots & \times & & & & 1 \end{array} \right)$$

interchange

Implementation issue: commutability of GE matrix and permutation [3]



Algorithm (PA = LU) [1]

Given full matrix $A \in R^{n\times n}$, construct initial lower triangle matrix $L = I$

use permutation vector P to record permutation matrix $P^{(k)}$

let $A^{(1)} := A$, $\tilde{L}^{(-1)} = L^{(0)} = I$ and $P^{(0)} = (1, 2, 3, \cdots, n)$

for $k = 1 : n-1$

we have compute $P^{(k-1)} A = \tilde{L}^{(k-2)} L^{(k-1)} A^{(k)}$, $P^{(k-1)} = P_{k-1} P_{k-2} \cdots P_2 P_1$

$$A^{(k)} = \left(\begin{array}{ccc|ccc} a_{11}^{(1)} & \cdots & \cdots & \cdots & \cdots & a_{1n}^{(1)} \\ 0 & \ddots & & & & \\ \vdots & & a_{k-1,k-1}^{(k-1)} & \cdots & \cdots & a_{k-1,n}^{(k-1)} \\ \hline \vdots & & 0 & a_{k,k}^{(k)} & \cdots & a_{k,n}^{(k)} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & a_{nk}^{(k)} & \cdots & a_{nn}^{(k)} \end{array}\right)$$

update original matrix A

$$\tilde{L}^{(k-2)} L^{(k-1)} = \begin{array}{c} \overbrace{}^{k-1} \\ \left(\begin{array}{c|c} W & O \\ \hline M & I \end{array}\right) \end{array} \begin{array}{l} \}\, k-1 \\ \end{array}$$

stores in lower triangle matrix L

Algorithm (PA = LU) *[2]*

1 find a $p \in \{k, k+1, \cdots, n\}$ such that $\left|A_{pk}^{(k)}\right| = \max\left|A^{(k)}(k:n, k)\right|$

2 swap row $A^{(k)}(k, k:n)$ and row $A^{(k)}(p, k:n)$

define permutation matrix $P_k = (1, 2, \cdots k-1, p, k+1, \cdots, p-1, k, p+1, \cdots, n)$

then after swapping rows k, p, we have $\tilde{A}^{(k)} = P_k A^{(k)}$

$$\begin{array}{cc} & \begin{array}{ccc} & k-col & \\ & \downarrow & \end{array} \\ \tilde{A}^{(k)} = & \begin{pmatrix} \times & \times & \times \\ 0 & \xi & d^T \\ 0 & c & B \end{pmatrix} \begin{array}{l} \\ \longleftarrow k-row \\ \\ \end{array} \end{array} \qquad \left(\ |\xi| \geq \max|c| \ \right)$$

3 compute $P^{(k)} \leftarrow P_k P^{(k-1)}$ for $P^{(0)} = (1, 2, 3, \cdots, n)$

if $k > 1$ *then*

4 compute $\tilde{L}^{(k-1)} \leftarrow P_k \left(\tilde{L}^{(k-2)} L^{(k-1)}\right) P_k^T$ by swapping $L(k, 1:k-1)$ and $L(p, 1:k-1)$

then $P^{(k-1)} A = \tilde{L}^{(k-2)} L^{(k-1)} A^{(k)} \longrightarrow P^{(k)} A = \tilde{L}^{(k-1)} \tilde{A}^{(k)}$

endif

Algorithm (PA = LU) *[3]*

5 decompose $\tilde{A}^{(k)} = L^{(k)} A^{(k+1)}$ where

$$L^{(k)} = \begin{array}{cc} \begin{array}{ccc} & k-col & \\ & \downarrow & \end{array} & \\ \left(\begin{array}{c|c|c} I & & \\ \hline 0 & 1 & \\ \hline 0 & c/\xi & I \end{array}\right) & \begin{array}{l} \\ \leftarrow k-row \\ \\ \end{array} \end{array}$$

by updating $L(k+1:n,k) \leftarrow c/\xi$

$$A^{(k+1)} = \left(\begin{array}{c|c|c} \times & \times & \times \\ \hline 0 & \xi & d^T \\ \hline 0 & 0 & \tilde{B} \end{array}\right), \quad \tilde{B} = B - \frac{cd^T}{\xi}$$

by updating matrix $A(k+1:n,k+1:n) -= \dfrac{cd^T}{\xi}$

then $P^{(k-1)}A = \tilde{L}^{(k-2)} L^{(k-1)} A^{(k)} \longrightarrow P^{(k)}A = \tilde{L}^{(k-1)} L^{(k)} A^{(k+1)}$ (recursion is done)

endfor

Question: recursion implementation

normal

- Initialization
  - check algorithm holds for $k=1$
- Recursion formula
  - check algorithm holds for k, if $k-1$ is true
- Termination condition
  - check algorithm holds for $k=n$

abnormal

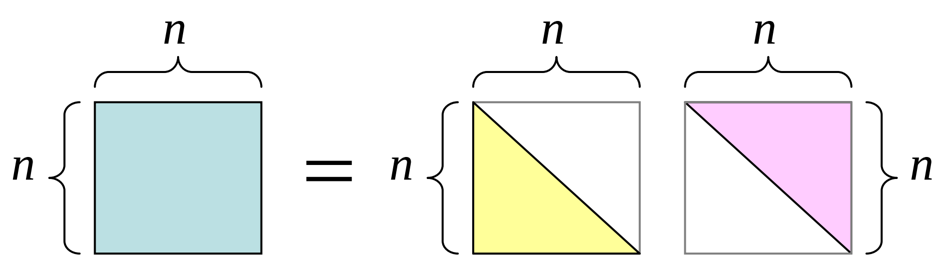
- Breakdown of algorithm
  - check which condition $PA=LU$ fails

Extension of algorithm

- Exception: algorithm works only for square matrix?

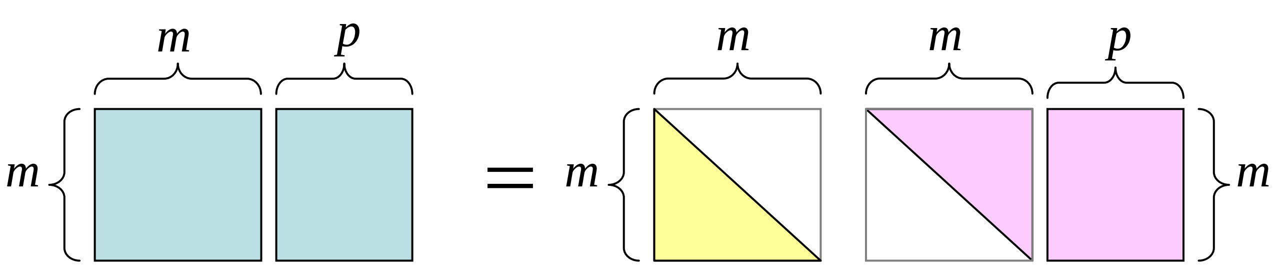
Exception: algorithm works only for square matrix? [1]

Case 1: $m = n$



$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & & \\ 4 & 1 & \\ 7 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ & -3 & -6 \\ & & 0 \end{pmatrix}$$

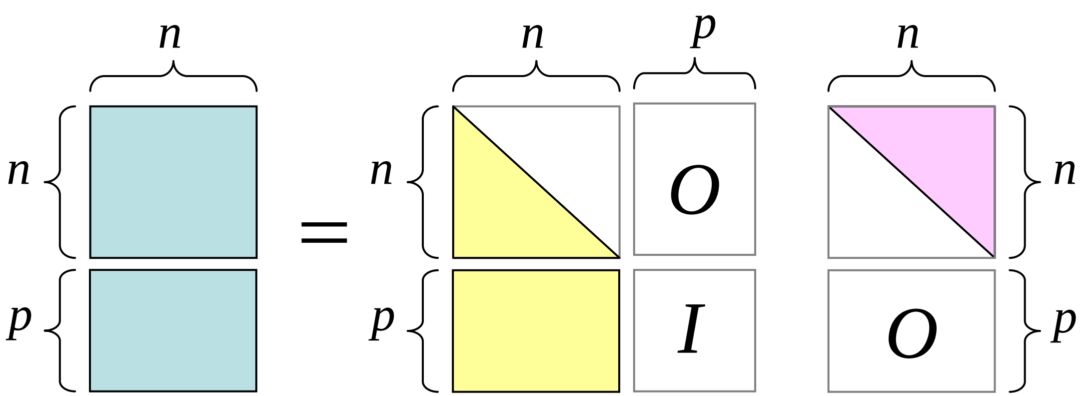
Case 2: $n = m + p > m$



$$A = \left(\begin{array}{ccc|cc} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 1 & 19 & 13 & 14 & 17 \end{array}\right) = \begin{pmatrix} 1 & & \\ 6 & 1 & \\ 1 & -3.4 & 1 \end{pmatrix} \left(\begin{array}{ccc|cc} 1 & 2 & 3 & 4 & 5 \\ & -5 & -10 & -15 & -20 \\ & & -24 & -41 & -56 \end{array}\right)$$

Exception: algorithm works only for square matrix? [2]

Case 3: $m = n + p > n$



$$A = \left(\begin{array}{ccc} 1 & 6 & 1 \\ 2 & 7 & 19 \\ 3 & 8 & 13 \\ \hline 4 & 9 & 14 \\ 5 & 10 & 17 \end{array}\right) = \left(\begin{array}{ccc|cc} 1 & & & & \\ 2 & 1 & & & \\ 3 & 2 & 1 & & \\ \hline 4 & 3 & \sim 1.7 & 1 & \\ 5 & 4 & \sim 2.3 & & 1 \end{array}\right) \left(\begin{array}{ccc} 1 & 6 & 1 \\ & -5 & 17 \\ & & -24 \\ \hline & & \\ & & \end{array}\right)$$

we can simplify it as $A = \begin{pmatrix} B \\ C \end{pmatrix} = \begin{pmatrix} L & O \\ M & I \end{pmatrix} \begin{pmatrix} U \\ O \end{pmatrix} = \begin{pmatrix} LU \\ MU \end{pmatrix} = \begin{pmatrix} L \\ M \end{pmatrix} U$

Question: what is termination condition of *case 3* ?

OutLine

- Weakness of $A=LU$
- $A=LU$ versus $PA=LU$
- Pivoting strategy
- Implementation of $PA=LU$
- *MATLAB usage*
  - *abs*
  - *max*

Command "*abs*": take absolute value

In *PA=LU* algorithm, we need to take absolute value of one column vector $\left(\left|A_{pk}^{(k)}\right| = \max\left|A^{(k)}(k:n,k)\right|\right)$ for pivoting

```
>> help abs

 ABS    Absolute value.
    ABS(X) is the absolute value of the elements of X. When
    X is complex, ABS(X) is the complex modulus (magnitude) of
    the elements of X.

    See also SIGN, ANGLE, UNWRAP.

 Overloaded methods
    help sym/abs.m

>> A = [-3 -4 -5]

A =

    -3    -4    -5

>> abs(A)

ans =

     3     4     5
```

abs also works for matrix

```
>> A = [-1 2 3; -4 -5 -6 ; 7 8 9]

A =

    -1     2     3
    -4    -5    -6
     7     8     9

>> abs(A)

ans =

     1     2     3
     4     5     6
     7     8     9
```

Command “*max*”: take maximum value

```
>> help max

 MAX    Largest component.
    For vectors, MAX(X) is the largest element in X. For matrices,
    MAX(X) is a row vector containing the maximum element from each
    column. For N-D arrays, MAX(X) operates along the first
    non-singleton dimension.

    [Y,I] = MAX(X) returns the indices of the maximum values in vector I.
    If the values along the first non-singleton dimension contain more
    than one maximal element, the index of the first one is returned.
```

```
>> A = [-1 2 3; -4 -5 -6 ; 7 8 9]

A =

    -1     2     3
    -4    -5    -6
     7     8     9

>> max(A)

ans =

     7     8     9

>> max( max(A) )

ans =

     9
```

```
>> A = [-3 -4 -5]

A =

    -3    -4    -5

>> [y,i] = max(abs(A))

y =

     5


i =

     3

>> [y,i] = max(A)

y =

    -3


i =

     1
```

$$5 = |A(3)| = \max\{|A(1:3)|\}$$

$$-3 = A(1) = \max\{A(1:3)\}$$