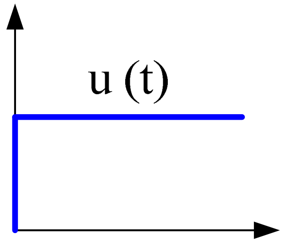
5.5 Unit Step Function

In applications, systems are often subjected to discontinuous forcing functions. (show real power of Laplace transform)

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 \le t \end{cases}$$



u (t-a)

$$u(t-a) = \begin{cases} 0 & 0 \le t \le a \\ 1 & a \le t \end{cases}$$

a

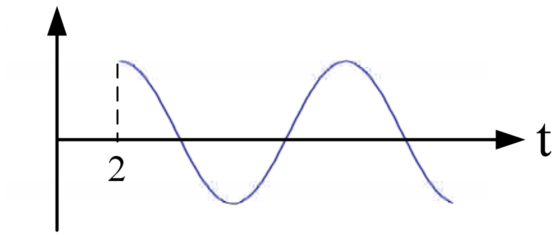
為 typical “engineering function”

如 electrical or mechanical driving force off/on.

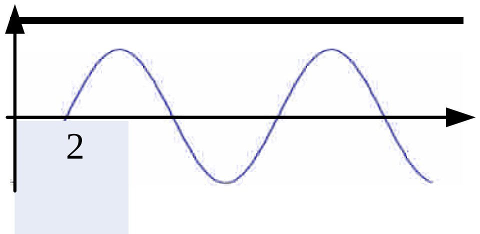
- 把 f(t) 乘上 u(t-a) 可 produce all sorts of effects.

如

$$f(t) = 5\sin t$$

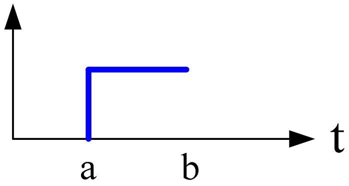


$f(t)u(t-2)$ 會把 $f(t)$ 在 $t<2$ 之部分關掉

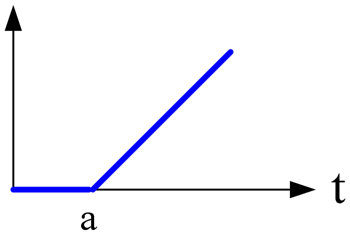


$f(t-2)u(t-2)$ 則把 $f(t)$ shift right by 2 (t-shifting)

而

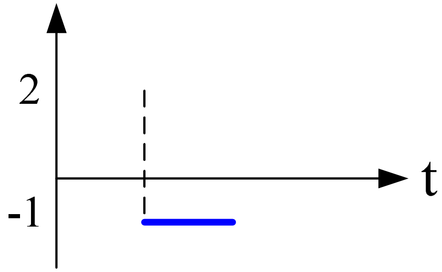


$$u(t-a) - u(t-b)$$



$$f(t) = \begin{cases} 0 & 0 < t < a \\ t-a & a \le t < \infty \end{cases}$$

$$= (t-a)u(t-a)$$



$$f(t) = 2 - 3u(t-2) + 2u(t-3)$$

而

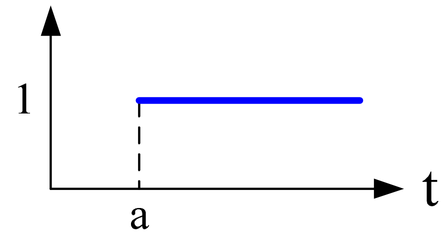
$$f(t) = \begin{cases} g(t) & 0 \le t < a \\ h(t) & t \ge a \end{cases}$$

$$f(t) = g(t) - g(t)u(t-a) + h(t)u(t-a)$$

$$f(t) = \begin{cases} 0 & 0 \le t < a \\ g(t) & a \le t < b \\ 0 & t \ge b \end{cases}$$

$$f(t) = g(t)\left[u(t-a) - u(t-b)\right]$$

1)　若要解之微分方程 右邊為



$$f(t)=\begin{cases}0 & 0\le t<a\\ 1 & a\le t\end{cases}$$

即　$f(t)=u(t-a)$

$$y''+a\,y'+b\,y(t)=\quad f(t)$$

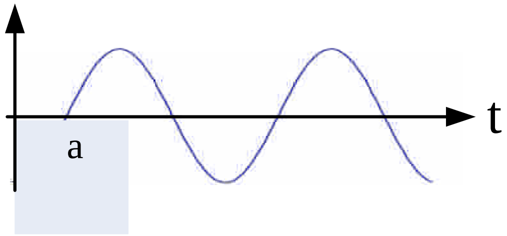
$$\downarrow$$

$$L\{f(t)\}=?$$

即　$L\{u(t-a)\}=?$

‖　$L\{u(t-a)\}=\int_0^\infty u(t-a)\,e^{-st}dt=\int_a^\infty e^{-st}dt=\dfrac{e^{-as}}{s}$

2) 若要解之微分方程 右邊為



$$f(t)=\begin{cases}0 & 0\le t<a\\ \sin(t-a) & a\le t\end{cases}$$

即 $f(t)=\sin(t-a)u(t-a)$

$$y''+a\,y'+b\,y(t)=\underset{\displaystyle L\{f(t)\}=?}{\underset{\downarrow}{f(t)}}$$

即 $L\{\sin(t-a)u(t-a)\}=?$

$$=\int_a^\infty \sin(t-a)e^{-st}dt \qquad \text{令} \quad v=t-a$$

$$=\int_a^\infty \sin v\, e^{-(v+a)s}dv=e^{-as}\int_0^\infty \sin v\, e^{-sv}dv$$

$$=e^{-as}L\{\sin v\} \quad or \quad e^{-as}L\{\sin t\}$$

即 $L\{\sin(t-a)u(t-a)\}=e^{-as}L\{\sin t\}$

又 $L\{f(t-a)u(t-a)\}=e^{-as}L\{f(t)\}$

即
$$\begin{cases}\text{若 } f(t)\xrightarrow{L}F(s)\\ \text{則 } f(t-a)u(t-a)\xrightarrow{L}e^{-as}F(s)\end{cases}$$

$$f(t) \to F(s)$$

$$f(t-a)u(t-a) \to e^{-as}F(s)$$

證：

$$L\{f(t-a)u(t-a)\} = \int_0^\infty f(t-a)u(t-a)e^{-st}dt = \int_a^\infty f(t-a)e^{-st}dt$$

令 $v = t-a$

$$= \int_0^\infty f(v)e^{-(v+a)s}dv = e^{-as}\int_0^\infty f(v)e^{-sv}dv = e^{-as}F(s)$$

$f(t)$	$F(s)$
1	$\frac{1}{s}$
$\sin t$	$\frac{1}{s^2+1}$

$u(t-a)$	$e^{-as}\cdot\frac{1}{s}$
$\sin(t-a)u(t-a)$	$e^{-as}\cdot\frac{1}{s^2+1}$
$f(t-a)u(t-a)$	$e^{-as}F(s)$

也可反過來證　$e^{-as}F(s)\xrightarrow{L^{-1}} f(t-a)u(t-a)$

proof：　$e^{-as}F(s)=e^{-as}\int_0^{\infty} f(\tau)e^{-s\tau}d\tau=\int_0^{\infty} f(\tau)e^{-s(\tau+a)}d\tau$

令　$t=\tau+a$　　$\tau：a\to\infty$　　$\Rightarrow t：a\to\infty$　　$d\tau=dt$

$\therefore$　$e^{-as}F(s)=\int_a^{\infty} f(t-a)e^{-st}dt$

$=\int_a^{\infty} f(t-a)u(t-a)e^{-st}dt$

把下限換為0
只要得證 integrand 在 0 - a 間 = 0
即用 $u(t-a)$

即　$e^{-as}F(s)\xrightarrow{L^{-1}} f(t-a)u(t-a)$　　得證

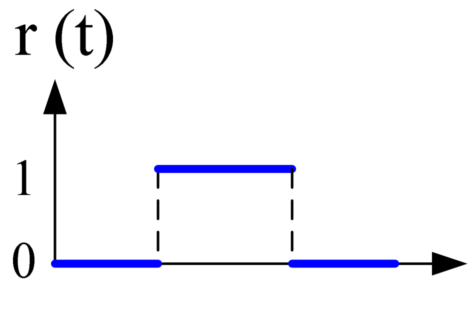
Kreyszig P. 282 例 4.

$$y'' + 3y' + 2y(t) = r(t)$$

$$y(0) = y'(0) = 0$$

求 $y(t)$

$$r(t) = \begin{cases} 1 & 1 < t < 2 \\ 0 & \text{otherwise} \end{cases}$$



用 Laplace transform

$$\left[s^2 + 3s + 2\right] Y(s) = R(s)$$

$$Y(s) = \frac{R(s)}{s^2 + 3s + 2} = Q(s)R(s)$$

$$Q(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$q(t) = e^{-t} - e^{-2t}$$

$$y(t) = L^{-1}\left[Q(s)R(s)\right] = q(t) * R(t)$$

看所要 t 之值在哪個範圍

Case 1. if $t<1$ $\quad y(t)=\int_0^{t<1} \underset{\underset{0}{\parallel}}{r(\tau)}\, q(t-\tau)\,d\tau$

Case 2. if $1<t<2$ $\quad y(t)=\int_0^{t} r(\tau)\, q(t-\tau)\,d\tau$

$$=\int_1^{t} r(\tau)\, q(t-\tau)\,d\tau \qquad \text{此時} \quad r(\tau)=1$$

$$=\int_1^{t} \left[e^{-(t-\tau)} - e^{-2(t-\tau)} \right] d\tau$$

$$=\int_1^{t} e^{\tau-t}\, d(\tau-t) - \int_1^{t} e^{2(\tau-t)}\, d(\tau-t)$$

$$=e^{\tau-t}\Big|_1^{t} - \frac{1}{2} e^{2(\tau-t)}\Big|_1^{t}$$

$$=\left[e^{0} - e^{1-t} \right] - \frac{1}{2} e^{2(1-t)} = \frac{1}{2} - e^{1-t} + \frac{1}{2} e^{2(1-t)}$$

$$y(t)=\frac{1}{2} - e^{1-t} + \frac{1}{2} e^{2(1-t)}$$

Case 3. if $t > 2$

則 $\int_0^t \rightarrow \int_1^2$ $\because$ 其他區 $r(\tau) = 0$

$$y(t) = \int_1^t q(t-\tau)\, d\tau$$

$$= \int_1^t \left[e^{-(t-\tau)} - e^{-2(t-\tau)} \right] d\tau$$

$$= \left\{ e^{-(t-\tau)} - \frac{1}{2} e^{-2(t-\tau)} \right\} \Bigg|_1^2$$

$$y(t) = \left[e^{-(t-2)} - e^{-(t-1)} \right] - \frac{1}{2} \left[e^{-2(t-2)} - e^{-2(t-1)} \right]$$

例： $y'' + 3y' + 2y = r(t)$
$= u(t-1)$

$$r(t) = \begin{cases} 1 & 1 < t \\ 0 & \text{otherwise} \end{cases} \qquad y(0) = y'(0) = 0$$

$$L[u(t-1)] = \frac{e^{-s}}{s}$$

$$\left(s^2 + 3s + 2\right)Y(s) = \frac{e^{-s}}{s}$$

$$Y(s) = \frac{e^{-s}}{s(s+1)(s+2)}$$

先找 $L^{-1}\left\{\frac{1}{s(s+1)(s+2)}\right\}$ 然後用 $L^{-1}\left\{e^{-s}F(s)\right\} = u(t-1)f(t-1)$

可用 convolution

$$\frac{1}{s} \xrightarrow{L^{-1}} 1$$

$$\frac{1}{s+1} \xrightarrow{L^{-1}} e^{-t}$$

$$\frac{1}{s+2} \xrightarrow{L^{-1}} e^{-2t}$$

也可用 partial fraction

$$\frac{1}{s(s+1)(s+2)} = \frac{1/2}{s} - \frac{1}{s+1} + \frac{1/2}{s+2}$$

$$L^{-1}\left[F(s)G(s)V(s)\right] = f * g * v = (f * g) * v$$

$$f * g = \int_0^t e^{-\tau} d\tau = 1 - e^{-t}$$

$$(f * g) * v = \int_0^t \left(1 - e^{-\tau}\right) e^{-2(t-\tau)} d\tau$$

$$= e^{-2t} \int_0^t \left(e^{2\tau} - e^{\tau}\right) d\tau$$

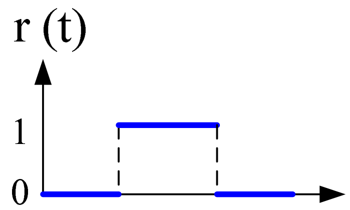
$$= e^{-2t}\left[\frac{e^{2t}}{2} - e^{t} + \frac{1}{2}\right] = \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t}$$

$$Y(s) = u(t-1)\left[\frac{1}{2} - e^{-(t-1)} + \frac{1}{2} e^{-2(t-1)}\right]$$

例： $y'' + 3y' + 2y = r(t) \qquad y(0) = y'(0) = 0$

$(A) \quad r(t) = u(t-1) - u(t-2)$

$(B) \quad r(t) = \delta(t-1)$



Sol： $(A) \quad s^2 Y(s) + 3sY(s) + 2Y(s) = \dfrac{e^{-s}}{s} - \dfrac{e^{-2s}}{s}$

$$Y(s) = \frac{e^{-s}}{s(s^2+3s+2)}\left[1 - e^{-s}\right] = \frac{1}{s(s+1)(s+2)}\left[e^{-s} - e^{-2s}\right]$$

(另法) 令 $F(s) = \dfrac{1}{s(s+1)(s+2)}$

$$= \frac{1/2}{s} - \frac{1}{s+1} + \frac{1/2}{s+2}$$

$$\downarrow L^{-1} \quad \downarrow L^{-1} \quad \downarrow L^{-1}$$

$$f(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$

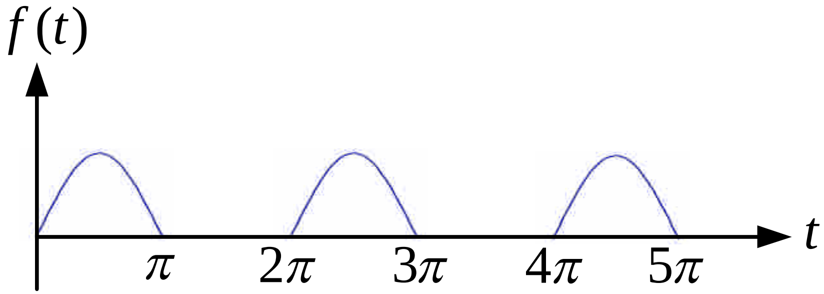
$$y(t) = \begin{cases} 0 & t < 1 \\ \dfrac{1}{2} - e^{-(t-2)} + \dfrac{1}{2}e^{-2(t-1)} & 1 \le t < 2 \\ e^{-(t-1)} + e^{-(t-2)} + \dfrac{1}{2}e^{-2(t-1)} + \dfrac{1}{2}e^{-2(t-2)} & \\ \text{即} \quad (e^2 - e)e^{-t} - \dfrac{1}{2}(e^4 - e^2)e^{-2t} & 2 < t < \infty \end{cases}$$

$$L^{-1}\left\{F(s)e^{-s}\right\} = f(t-1)u(t-1)$$

$$\therefore \quad L^{-1}\{Y(s)\} = L^{-1}\left\{F(s)e^{-s}\right\} - L^{-1}\left\{F(s)e^{-2s}\right\} = f(t-1)u(t-1) - f(t-2)u(t-2)$$

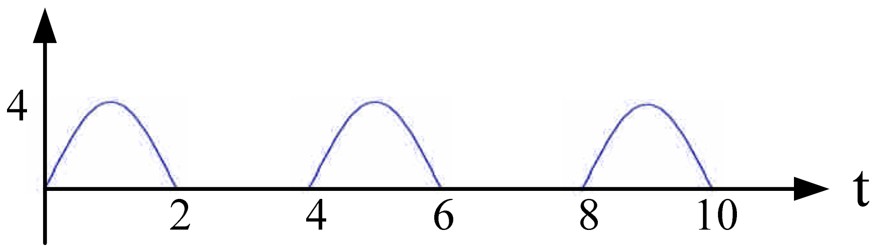
$$= \left[\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\right]u(t-1) - \left[\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}\right]u(t-2)$$

1. 寫出 $f(t)$

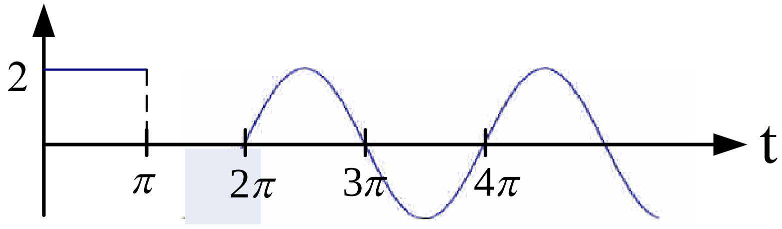


$$f(t) = \sin t - \sin t \cdot u(t-\pi) + \sin t \cdot u(t-2\pi) - \sin t \cdot u(t-3\pi) + \cdots$$
$$= \sin t\left[\, u(t) - u(t-\pi) + u(t-2\pi) - u(t-3\pi) + \cdots \right]$$

2. 寫出 $f(t)$



$$f(t) = 4\sin\left(\frac{1}{2}\pi t\right)\left[\, u(t) - u(t-2) + u(t-4) - u(t-6) + \cdots \right]$$



$$f(t)=\begin{cases}2 & 0\le t<\pi\\ 0 & \pi\le t\le 2\pi\\ \sin t & 2\pi\le t\end{cases}$$

求 $f(t)$ 之 Laplace transform $F(s)$

解： step 1. 先寫出 $f(t)$ in terms of unit step function

$$f(t)=2u(t)-2u(t-\pi)+u(t-2\pi)\sin(t-2\pi)$$

$$=2u(t)-2u(t-\pi)+u(t-2\pi)\sin t \qquad (\text{不一定要把 } t-2\pi \text{ 變為 } t)$$

step 2.

$$L\{f(t)\}=2L\{u(t)\}-2L\{u(t-\pi)\}+L\{u(t-2\pi)\sin(t-2\pi)\}$$

$$=2\cdot\frac{1}{s}-2\cdot\frac{e^{-\pi s}}{s}+e^{-2\pi s}L\{\sin t\}$$

$$=2\cdot\frac{1}{s}-2\cdot\frac{e^{-\pi s}}{s}+\frac{1}{s^2+1}e^{-2\pi s}$$

反之　有 $F(s)$　求 $f(t)$

例：　$F(s)=\dfrac{2}{s^2}-\dfrac{2}{s^2}e^{-2s}-\dfrac{4}{s}e^{-2s}+\dfrac{s}{s^2+1}e^{-\pi s}$

$$L^{-1}\{F(s)\}=L^{-1}\left\{\frac{2}{s^2}\right\}-L^{-1}\left\{\frac{2}{s^2}e^{-2s}\right\}-4L^{-1}\left\{\frac{e^{-2s}}{s}\right\}+L^{-1}\left\{\frac{s}{s^2+1}e^{-\pi s}\right\}$$

若沒有e項　即 $F(s)=\dfrac{2}{s^2}-\dfrac{2}{s^2}-\dfrac{4}{s}+\dfrac{s}{s^2+1}$

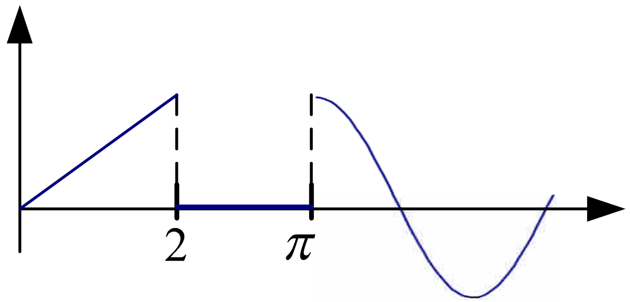
$$f(t)=2t-2t-4-\cos t$$

用 $e^{-as}F(s)\to f(t-a)u(t-a)$

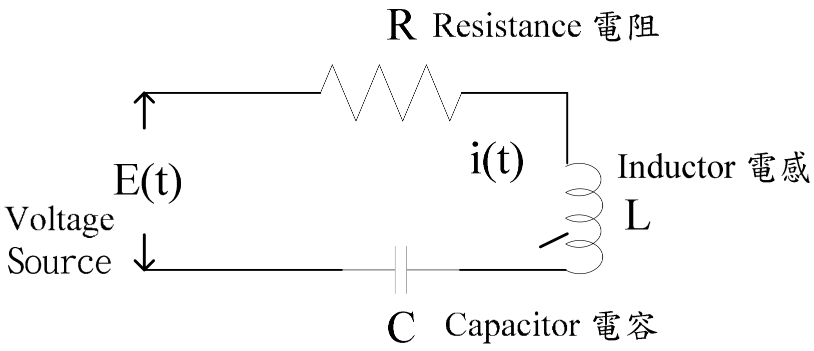
$\therefore$ now $L^{-1}\{F(s)\}=2t-2(t-2)u(t-2)-4u(t-2)-\cos(t-\pi)u(t-\pi)$

$$=2t-2(t-2)u(t-2)-\cos t\,u(t-\pi)$$

即　$f(t)=\begin{cases}2t & 0\le t<2\\ 0 & 2\le t<\pi\\ -\cos t & \pi\le t\end{cases}$



Ex. 3 of P. 271 LC circuit

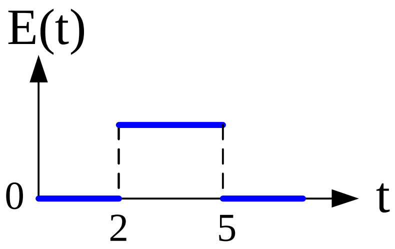


$$R\,i(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt = E(t)$$

$$i(t) = \frac{dQ(t)}{dt}$$

$$\Rightarrow \quad L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q(t) = E(t)$$

Now, E(t) 如圖：



$$E(t) = \begin{cases} 0 & 0 \le t < 2 \\ 1 & 2 \le t < 5 \\ 0 & 5 \le t \end{cases}$$

本題 $R = 0$, $Q(0) = Q_0$, $Q'(0) = 0$, 求 $Q(t)$

$\Downarrow$

($i(0) = 0$, switch 在 t = 0 之前為 open)

Solve：

step1. $E(t)=E_0\left[u(t-2)-u(t-5)\right]$ 微分為 $LQ''+\frac{1}{C}Q(t)=E(t)$

用Laplace transform

$$L\left(s^2\overline{Q}(s)-s\underbrace{Q(0)}_{=Q_0}-\underbrace{Q'(0)}_{=0}\right)+\frac{1}{C}\overline{Q}(s)=E_0\left(\frac{1}{s}e^{-2s}-\frac{1}{s}e^{-5s}\right) \qquad \omega^2=\frac{1}{LC}$$

$$\left(Ls^2+\frac{1}{C}\right)\overline{Q}(s)=LsQ_0+E_0\left(\frac{1}{s}e^{-2s}-\frac{1}{s}e^{-5s}\right)$$

$$\overline{Q}(s)=\frac{Q_0 s}{s^2+1/LC}+\frac{E_0/L}{s\left(s^2+1/LC\right)}\underbrace{\left(e^{-2s}-e^{-5s}\right)}_{\text{暫不看}}$$

$$Q(t)=Q_0\cos\omega t+E_0\ ?$$

Use convolution：

$$L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+\omega^2}\right\}$$
$$=L^{-1}\left\{F(s)G(s)\right\}$$
$$=\int_0^t f(\tau)g(t-\tau)\,d\tau$$
$$=\int_0^t \sin\omega\tau\cdot 1\,d\tau$$

$$L^{-1}\left\{\frac{1}{s(s^2+\omega^2)}\right\}=\frac{1}{\omega}\int_0^t \sin\omega\tau\,d\tau=\frac{1}{\omega^2}(1-\cos\omega t)$$

$$\frac{1/L}{s(s^2+\omega^2)}\left(e^{-2s}-e^{-5s}\right)\xrightarrow{L^{-1}}\frac{1}{\underbrace{L\omega^2}_{=C}}\left\{u(t-2)\left[1-\cos\omega(t-2)\right]+u(t-5)\left[1-\cos\omega(t-5)\right]\right\}$$

$$\therefore\quad Q(t)=Q_0\cos\omega t+E_0C\left\{u(t-2)\left[1-\cos\omega(t-2)\right]+u(t-5)\left[1-\cos\omega(t-5)\right]\right\}$$

用 $u(t-a)$ 及 Laplace Transform 可解 $Q(t)$ on entire domain $(0<t<\infty)$

若用Chap.3之方法需分成二段

$$0 \rightarrow 2$$
$$2 \rightarrow 5$$
$$5 \rightarrow \infty$$

比較麻煩

用 $u(t-a)$ 及 Laplace Transform 可解 $Q(t)$ on entire domain $(0 < t < \infty)$

若用Chap.3之方法需分成二段

$$0 \to 2$$
$$2 \to 5$$
$$5 \to \infty$$

比較麻煩

也可用 $L\{f(t)u(t-a)\}=\int_a^{\infty} e^{-st} f(t)dt \qquad let \quad v=t-a$

$$=\int_0^{\infty} e^{-s(v+a)} f(v+a)dv$$

$$=e^{-as}\int_0^{\infty} e^{-st} f(t+a)dt$$

$$=e^{-as}L\{ f(t+a)\}$$

即 $f(t)u(t-a)=e^{-as}L\{ f(t+a)\}$

回到例： $L\{t^2u(t-2)\}=?$

已知 $L\{t^2\}\xrightarrow{L}\frac{2}{s^3}$

$$L\{t^2u(t-2)\}=e^{-2s}L\{(t+2)^2\}=e^{-2s}L\{t^2+4t+4\}$$

$$=e^{-2s}\left[\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}\right]$$ 一樣

例： $L\{\sin t\,u(t-\pi)\}=e^{-\pi s}L\{\sin(t+\pi)\}=e^{-\pi s}L\{-\sin t\}=e^{-\pi s}\left(-\frac{1}{s^2+1}\right)$

或 $L\{-\sin(t-\pi)u(t-\pi)\}=-e^{-\pi s}L\{\sin t\}=e^{-\pi s}\left(-\frac{1}{s^2+1}\right)$