Right Triangles, unit circle, radians, memorize important sine/cosine values... see "Library of functions" above.

Show work: unit circle, reference right triangle, reference angle's trig value, +-... Corrected at start of class... no late work!

Eratosthenes, Syene, Alexandri

5th	6.1 (P. 474) # 2,	14,	20,	24,	26, 28, 29	34,	42,	46,	51,	55,✓	59,	61,✓	66,	67,✓	72,	85.
6th	(P.440) # 4	16	21	25	28 sketch 30	36	43	48	53	57	61	62	68	70	74 81	87

Bonus	#88.	CW	#1	7	23	29	33	39	49	65	80
	90		3	9	26	32	35	42	52	67	84

	6.2 (P. 484) #3,	7,	9,	11,	13,	16,	18,	22,✓	26✓ 23~27,	35,	36,	✓ 39,	43,	44,	47,	55,	60.
(P448) Concepts	#6	10	11	14	16	17	20	24	25~29	38	37	42	46	45	50	55	59

Bonus	# 61~65.	CW #	2	8	10	12	14	15	17	21	28	33	34	63	64
	61~65		4	9	12,	13.	18	15	19	23	30	35	36	64	63

astronomy (63 64) ← What's measured? Gravity G, g R_E ⇒ M_E ρ_E → core clue

Sun, black hole

6.3 (P. 495) No calculator # 2,	6,	10~50	even,	54,	55,	✓ 58,	65,	66,	70.
6.3 (P. 459)	4	8	12~52	even	56	58	59	68	67	72

5.1 (pg. 400)	# 1~17	odd,	19,	21,✓	29,✓	31,✓	33,	35,✓	~~39~~,	92✓ 42, 24✓ 50 43, 21✓ 44 51~54.
5.1 (Pg. 375)	3~19	odd	21	23	32	33	35	38	~~41~~	43, 51, 45, 46, 53~56

5.2 (pg. 416) No calculator. Show reference angle.

5.2	#1,✓	7,	11,	16,	19,✓	21,✓	22~27,	29,	33,	35,	45~77 odd.
(P.384)	3	10	14	18	21	24	23, 25~28	32	36	37	39~79 odd Use protractor

----

transformations

5th edition →

Graph 5 points

5.3 (pg. 429)	#5,	13,	21,	22,	25,	35,	38,	39,	41~47 odd,	59,	67,	68,	75,	76,	80.
6th edition → 5.3 (396)	8	16	24	23	28	38	40	42	43~49 odd	62	70	69	78	77	82

5.4 (pg. 441)	# 1~6,	12,	24,	46,	51,	52,	55.	Bonus: #56.
5.4 (405)	3~8	13	25	47	54	53	57	58

Remember to graph one period; label the 5 points and amplitude.

Bonus:

a) Explain why f(x)=sinx/x has thex-axis as an asymptote.

b) Use the limit of the difference quotient and the fact that f(x)--> 1 as x-->0 to show that the derivative of sinx is cosx. Hint: You will need a trig identity to expand; also, use a calculator to graph (cosx-1)/x to figure out a limit that will appear.

c) Do the same thing to find the derivative of cosx.

Setting up the sinusoid (word problems)

5.5 (pg. 451)	# 9,	13,	17,	25,	31,	33,	35,	37,	39,	40.
5.6 (P420)	~~Concepts~~ and # 12	15	19	28	33	34	38	40	42	41

inverses

7.4 (pg. 557) # 3, 5, 6, 7, 9, 11, 12, 13, 14, 17, 18, 19, 20, 21, 25, 26, 27, 30,

6.4 (467) 5.5 (411) 3, 6, 7, 9, 10 15 19 21 24 23 34 30 31 33 38 42 41 $\tan^{-1}(2\sin\frac{\pi}{3})$ 28

31, 33, 34, 37, 39, 40, 41, 43, 44, 47, 48, 53, 58, 59.

33 30 29 34 35 36 40 45 44

CW CW

$\sin(2\cos^{-1}\frac{3}{5})$

see 6.4 ex7 Soln 2

CW $\cos(\sin^{-1}\frac{3}{5} - \cos^{-1}\frac{3}{5})$

$\sin(\sin^{-1}\frac{1}{2} + \cos^{-1}\frac{1}{2})$

$\sin(\tan^{-1}x - \sin^{-1}x)$ ← ☆☆ Use formulas first!

CW

$\cos(\cos^{-1}x + \sin^{-1}x)$

① Unit circle

6.1 (P.474)

CW # 1.7, 23, 29, 33, 39, 49, 65, 80

HW # 2, 14, 20, 24, $\underbrace{26, 28, 29}_{\text{Sketch}}$, 34, 42, 46
51, 55, 59, 61, 65, 66, 67, 72, 85 *88

6.2 (P.484)

HW # 3, 7, 9, 11, 13, 16, 18, 22, 23~27, 35, 36,
39, 43, 44, 47, 55, 60. BONUS 61~65. Your own Poster w/Triangle-height or force

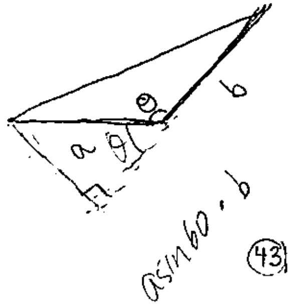
CW # 2, 8, 10, 12, 14, 15, 17, 21, 28, 33, 34, 63, 64

6.3 (P.495)
No calculator

HW # $\underbrace{2, 6, 10\sim50 \text{ even}}_{\text{in class}}$, 54, 55, 58, 65, 66, 70.

ref ∠ * Draw unit circle

(CW : #5, 9~49 odd)



⑤ Trig IDs

7.1 Fund. IDs (P.533)
2, 6, ⑧ ⑩, 14, 18, ㉓, 27, 30, 33, 36, 38, ㊸, 46
50, 53, 57, 61, (62), 72, 77, (78), 91, 93, 100 *, (94)

7.2 (P539) ± Formulas
④ 7, 8, ⑨, 11, ⑭, 15, 17, 19, 27, ㉙, 30, 36, ㊲,
㊸ ㊹ 46 ㊼, 48, ㊾, ㊿, 54, 55, 56*, 57*

7.3 (P.548) more IDs # 2, 3, 5, ⑦, ⑪ 12, 23, ㉔, 25, 27, 29, ㉚, 31, 32, ㊱, 38,
41, ㊹, 45, ㊻, 47, ㊿, 51, 55, (56), 58, 59, 62,
65, 73, (74), 77, 80, 81, 88*, 91, 93

BONUS Prove half-angle formula

7.5 Eqns. ex1~6 #, 4, 6, 8, 15, 17, 19, 24, 27, 54, 67,
ex7~11 # 56, 58, 61, 64, 66, 68, 72, 75, 79, 80, 81, 82 66, 68 79, 80, 81

1, 4, 6, 8, 9, 13, 17, 19, 27, 29, 30, 37, 38, 39, 43, 45, 47, 55, 54, 61, 67, 71.

② Unit Circle

5.1 (P.400)

HW # ~~19, 20, 21, 22, 29, 31, 33, 36, 39, 42, 43~~,
#1~17 odd, 19, 21, 29, 31, 33, 35, 39, 42, 43, 51~54

* Memorize formulas over winter break

CW #2.8, 12, 14, 20, 28, 32, 36, 47, 52

5.2 (P.416) No calculator. Show ref triangle

HW # 1, 7, 11, 16, 19, 21, 22, 23~26, 27, 29, 33, 35,
45~77 odd.

sinx(2sinx+1) max 1(3) min -1 can't tell w/o calc

③ Graph/Function - see SAT Guide Pg24

5.3 (P.429) # 5, ⑬, 21, ㉒ sin, 25, 35, ㊳ sin, 39, 41~47 odd, 59, 67, 68,
75, 76, (80)

CW : (Partner on board) # 36 cos, 34 sin,
5 8 9 13 15 17 25 41 46 41 43 59 67

5.4 (P.441) #1~6 why, ⑫ cot, ㉔ csc, ㊻ sec, (51) tan, 52 csc, 55 Bonus 56

CW : (Partner on board) ㊹ sec, 43 csc, 42 tan, 41 cot
1~6, 13, 14, 17, 19, 37, 38, 41, 45, 46, 49

5.5 Pg451 In class : Pairs go to board write answers :
(9 13, 17, ㉕, 31, 33, 35, 37, 39, ㊵)

④ 7.4 Inverse Trig fn.

HW Pg 557 #
& CW ③ 5, 6, 7, 9, 11, 12, ⑬ 14, ⑰ 18, 19, 20, 21, ㉕ 26, 27, $\underbrace{31, 33, 37}_{\text{triangle}}$, ㉞, 39, ㊵, 41, 43, ㊹ 47, ㊽, 53, (58, 59)

⑥ Triangles

6.4

6.5

① Ch.7 Review (P.571) CW : Tue & TH. Quiz from here

Check 7, 8, 9
Exercise 2, 5, 6, 7, 9, ⑮, 20, 23, 26, 31, 32, 35, ㊲, ㊴, ㊶, ㊷ 47, 51, 54~57,
(59), (61), (64), (72), (74), 75, 76.

MTC Pg.

P1

② Discovery Project Pg 560

③ Standing waves (Pg 575)

6.4 Law of Sines (P. 506)

\# ⑥, 10, 13, 21, $\overset{25}{\text{㉒}}$, 26, 27, 30, 39, 43

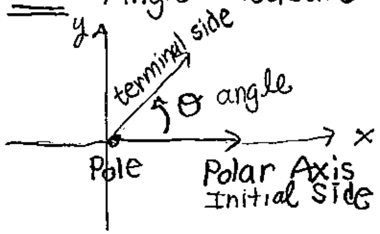
\# 2, 3, 5, 6, 9, 13, 15, 21, 22, 25, 26, 27, 30, 37, 39

(Pg 519) #57 ~62 } don't solve. Just say Law of Sine or Cosine?
521 #1~14 }

6.5 Law of Cosines (P. 513)

\# 3, 7, 8, 15, 16, 17, 18, 19, 20, 21, 27, 30, 33, *35, 37, 42, 43,
44, 47, 49

6.1 Angle Measure



Standard Position

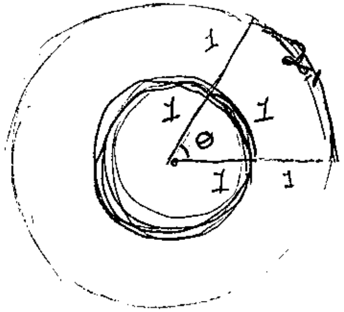
360° Radians?

↶ Positive Angle

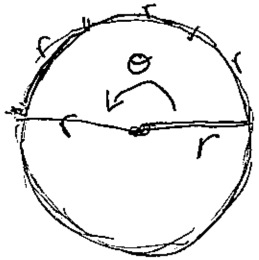
↷ Negative Angle

★ Unless angle stated with 30° sign
Assume it's Radians

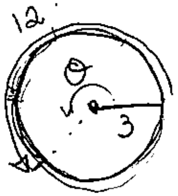
① **Radian** ≡ measure of angle s.t. arc that subtends angle is 1 radius long



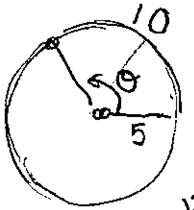
θ = "1 radian"
≈ 60°
57.3°



θ ≈ 3 rad
"radiuses"
≈ π



θ = 4 rad

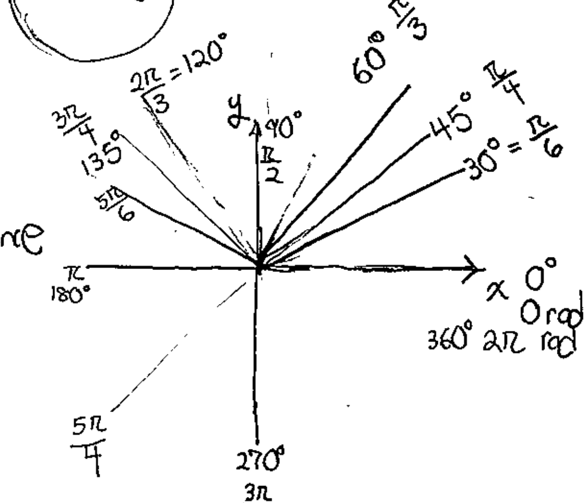


θ = 2 rad

θ = # radiuses
in circumference
= 2π

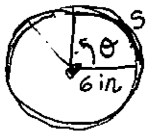
$\overset{2}{120°} \times \boxed{\frac{\pi}{\cancel{180°}}} = \frac{2\pi}{3}$ rad

$\frac{5\pi}{6} \times \boxed{\frac{\overset{3}{180°}}{\pi}} = 150°$



★ $\boxed{\text{Coterminal}}$ → see examples $\boxed{2 \,\&\, 3}$

② Pizza - Logically



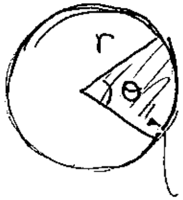
Slice's angle is 90° = θ

• **Arc Length** = $\textcircled{\tfrac14} \times 2\pi 6 = 3\pi$ inches

• sector **Area** = $\textcircled{\tfrac14} \times \pi 6^2 = 9\pi \text{ in}^2$

$\frac{90°}{360°} = \frac{A}{\pi r^2}$

$\underbrace{\frac{90°}{360°}}_{\text{Ratio of angle}} = \underbrace{\frac{s}{2\pi r}}_{\text{Ratio of arc length}}$



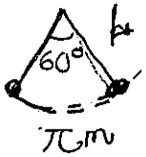
Arc Length $s = \left(\frac{\theta_{deg}}{360°}\right) \times 2\pi r = \frac{\theta_{Rad}}{\cancel{2\pi}} \times \cancel{2\pi} r$

$\boxed{s = r\theta_{Rad}}$

Sector Area $A = \frac{\theta_{deg}}{360°} \cdot \pi r^2 = \frac{\theta_{Rad}}{2\cancel{\pi}} \cdot \cancel{\pi} r^2$

$\boxed{A = \frac{1}{2} r^2 \theta_{Rad}}$

ex5

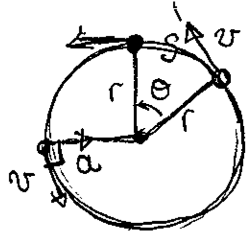


a) Length of pendulum? $\cancel{\pi} = \frac{60°}{360°} \times 2\cancel{\pi} L \Rightarrow \boxed{L = 3m}$

b) area swepth thru? $A = \frac{60°}{360°} \times \pi L^2 = \frac{1}{6}\pi 3^2 = \boxed{\frac{3\pi}{2} m^2}$

faster $A = \frac{1}{2}r^2\theta = \frac{1}{2}r(r\theta) = \frac{1}{2}rs = \boxed{\frac{1}{2} \times 3 \times \pi}$

Uniform Circular Motion



- speed is constant
- Velocity changes (direction)
- acceleration in same direction as force (centripetal)

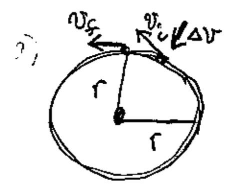
Linear speed $v = \frac{\Delta s}{\Delta t} = \omega r \because \frac{\Delta s}{\Delta t} = \frac{r\Delta\theta}{\Delta t} = r\omega$

Angular speed $\omega = \frac{\Delta\theta}{\Delta t}$

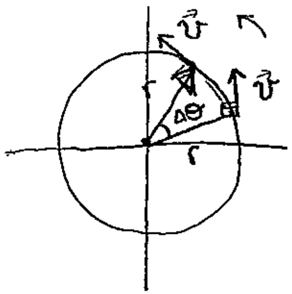
① Picture
② Calculus

centripetal Acceleration $= \frac{v^2}{r} = \omega^2 r \leftarrow \left(\frac{v}{r}\right)^2 r$

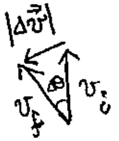
Centripetal Acceleration - Prove by Calculus
-By Trigonometry



$\theta=\omega t$



①



$\vec{v}_f = \vec{v}_i + \Delta\vec{v}$

$|\Delta\vec{v}| \neq \underbrace{\Delta|\vec{v}|}_{0}$

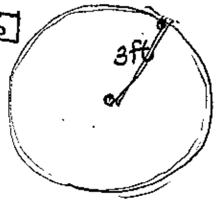
$$\frac{|\Delta\vec{v}|}{v} = \frac{s}{r} = \frac{\cancel{r}\Delta\theta}{\cancel{r}}$$

$$a = \frac{|\Delta\vec{v}|}{\Delta t} = v\frac{\Delta\theta}{\Delta t} = v\omega$$
$$= (r\omega)\omega$$
$$= \boxed{r\omega^2}$$
$$= r\left(\frac{v}{r}\right)^2$$
$$= \boxed{\frac{v^2}{r}}$$

②
$$\vec{r} = \langle r\cos\omega t, r\sin\omega t\rangle$$
$$\vec{v} = \frac{d\vec{r}}{dt} = \omega r\langle -\sin\omega t, \cos\omega t\rangle$$
$$\vec{a} = \frac{d\vec{v}}{dt} = -r\omega^2\langle\cos\omega t, \sin\omega t\rangle$$
$$= -r\omega^2\hat{r}$$

$\begin{cases}\text{direction to center}\\ a = r\omega^2\end{cases}$

ex6



$$\omega = \frac{15\text{ rev}}{10\text{ s}} = \frac{15\times 2\pi\text{ rad}}{10\text{s}} = 3\pi\text{ rad/S}$$

$$v = \frac{15\text{rev}\cdot 2\pi 3\text{ ft}}{10\text{s}} = 9\pi\text{ ft/S}$$

OR $v = \omega r = 3\pi\times 3 = 9\pi\text{ ft/S}$

ex7

wheel

$\downarrow$ 125 rev/min $= \omega$

12in=1ft
5280ft

How fast is the bike traveling? in miles per hour

$$v = \omega r = \frac{125\times 2\pi\text{ rad}}{\cancel{60}\,1\,\cancel{\text{min}}}\times 13\,\cancel{\text{in}}\cdot\frac{1\cancel{\text{ft}}}{12\cancel{\text{in}}}\cdot\frac{1\text{mi}}{5280\cancel{\text{ft}}}\cdot\frac{60\cancel{\text{min}}}{1\text{hr}}$$

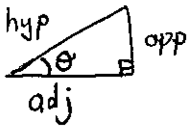
$$\approx\boxed{9.7\text{ mi/h}}$$

CW

$$\underset{\text{mile}}{r}\ \underset{\text{rad/hr}}{\omega}\quad v = r\omega = \left(13\cancel{\text{in}}\times\left(\overset{12}{5280}\right)^{-1}\text{miles}\right)\left(125\times\frac{2\pi\text{ rad}}{\text{min}}\cdot\frac{60\text{min}}{1\text{hr}}\right)$$
$$= 9.7\text{ mph}$$

6.2 Right Triangle Review ↓ Q&A

①



$\sin\theta = \frac{opp}{hyp}$

$\cos\theta = \frac{adj}{hyp}$

$\tan\theta = \frac{opp}{adj} = \frac{\sin\theta}{\cos\theta}$

$\csc\theta = \frac{1}{\sin\theta} = \frac{hyp}{opp}$

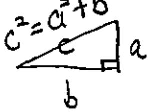
$\sec\theta = \frac{1}{\cos\theta} = \frac{hyp}{adj}$

$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = \frac{adj}{opp}$

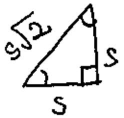
②

Pythagorean Thm (Right Triangles)

$c^2=a^2+b^2$

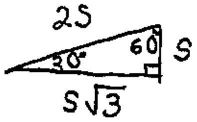


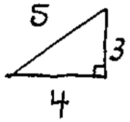
Special Right Triangles

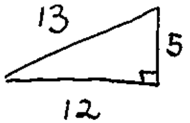


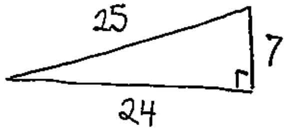
45-45-90°

Right isosceles

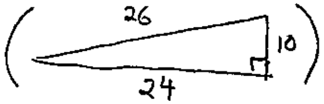




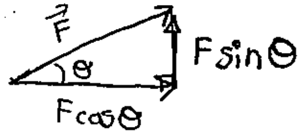


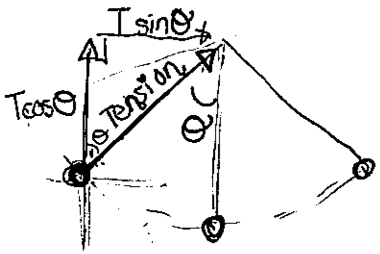


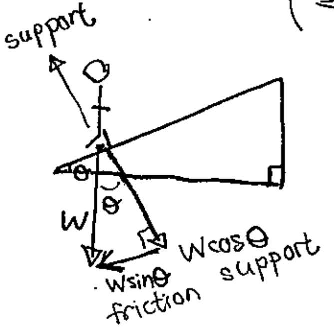
Show java list Pythagorean Triplets?

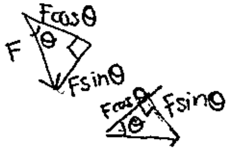


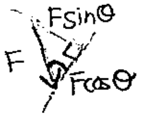
③



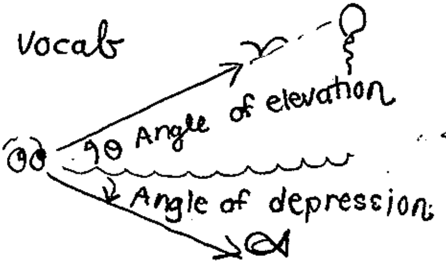






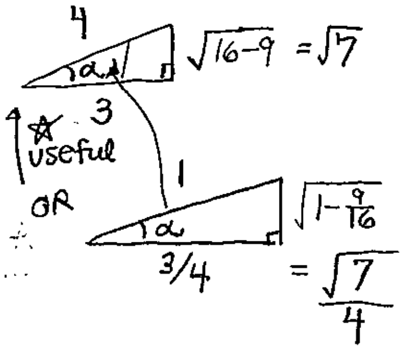


④ Vocab



See torque

ex2 $\cos\alpha = \frac{3}{4}$. Sketch & find other trig ratios of α



★ useful

OR

$\sin\alpha = \frac{\sqrt{7}}{4}$

$\tan\alpha = \frac{\sqrt{7}}{3}$

same

$\csc\alpha = \frac{4}{\sqrt{7}} = \frac{4\sqrt{7}}{7}$

$\sec\alpha = \frac{4}{3}$

$\cot\alpha = \frac{3}{\sqrt{7}} = \frac{3\sqrt{7}}{7}$

ex3 CALC mode

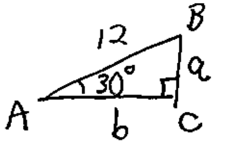
$\sin 17° \approx 0.29237$

$\sec 88° \approx \frac{1}{\cos 88°} \approx 28.65371$

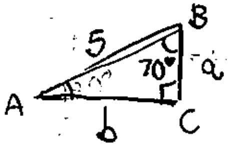
$\cos 1.2 \approx 0.36236$

$\cot 1.54 \approx \frac{1}{\tan 1.54} \approx 0.03081$

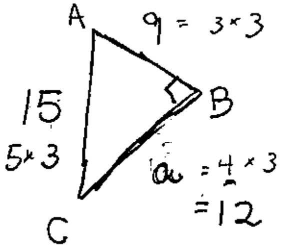
ex4 Solve a Right Triangle (Get all Lengths & Angles)



$\begin{cases} \angle B = 60° \\ a = 6 \\ b = 6\sqrt{3} \end{cases}$

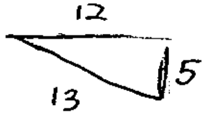


$\begin{cases} a = 5\cos 70° \\ b = 5\sin 70° \\ \angle A = 20° \end{cases}$

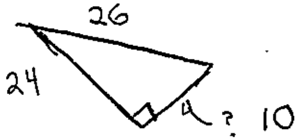


$\angle A = \tan^{-1}(\frac{12}{9})$

$\angle C = \tan^{-1}(\frac{3}{4})$



← Must be a Right Triangle



(5-12-13)

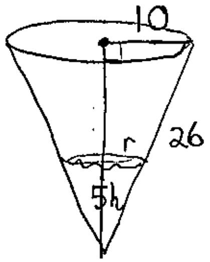
Torque

$\tau = |\vec{r} \times \vec{F}| = rF\sin\phi$

$= (F\sin\phi) \cdot r$

$= F \times (r\sin\phi)$

ex5



Conical Tank

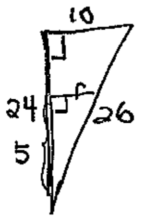
a) Radius when height is 5 feet ?

b) $V(h) = ?$

volume water height

Soln:

a) 5-12-13



Similar Triangles

$$\frac{r}{10} = \frac{5}{24}$$

$$r = \frac{50}{24} = \boxed{\frac{25}{12} \text{ ft}}$$

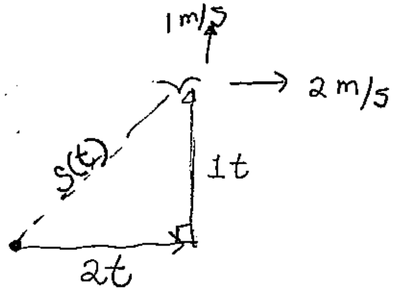
b)

$$V = \frac{1}{3}\pi r^2 h \qquad \frac{r}{10} = \frac{h}{24}$$

$$= \frac{1}{3}\pi\left(\frac{5h}{12}\right)^2 h \qquad r = \frac{h}{24} \cdot 10 = \frac{5h}{12}$$

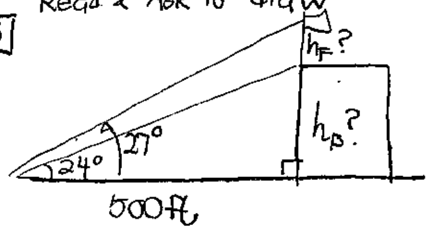
$$= \boxed{\frac{25\pi}{432} h^3}$$

Ⓠ



$$S = \sqrt{(2t)^2 + (1t)^2} = t\sqrt{5} \quad m$$

ex6 Read & Ask to draw



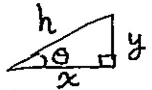
$$h_B = (\tan 24°)500 = 223 \text{ ft}$$

$$h_F + h_B = 500 \tan 27° = 255 \text{ ft}$$

$$h_F = 255 - 223 = 32 \text{ ft}$$

6.3 Trigonometric Functions of Angles

extension

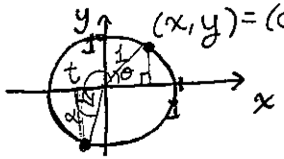


$\sin\theta \equiv \frac{y}{h}$

$0 \leq \theta \leq 90°$

① Unit Circle

$(x, y) = (\cos\theta, \sin\theta)$

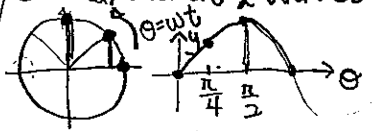


$\sin\theta \equiv y$ $\quad \csc\theta \equiv \frac{1}{y}$

$\cos\theta \equiv x$ $\quad \sec\theta \equiv \frac{1}{x}$

$\tan\theta \equiv \frac{y}{x}$ $\quad \cot\theta \equiv \frac{x}{y}$

(Circular Motion & waves



Find a video)

simple harmonic motion
(anything w/net restoring force)

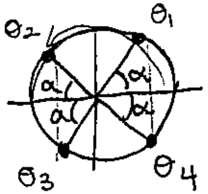

Hooke


Pendulum

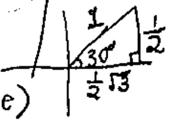

incline

② Reference Angle

x & y all same magnitude



$\sin\alpha = |\sin\theta_1| = |\sin\theta_2| = |\sin\theta_3| = |\sin\theta_4|$
cos
csc
⋮

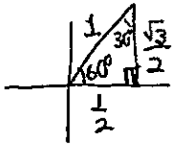


$|\sin 30°| = |\sin 150°| = |\sin 210°| = |\sin 330°|$

$\frac{1}{2}$ $\quad \frac{1}{2}$ $\quad -\frac{1}{2}$ $\quad -\frac{1}{2}$

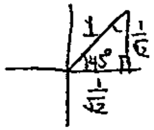
$\cos 30°$ $\quad \cos 150°$

$\frac{\sqrt{3}}{2}$ $\quad -\frac{\sqrt{3}}{2}$ $\quad -\frac{\sqrt{3}}{2}$ $\quad \frac{\sqrt{3}}{2}$



$|\sin 60°| = |\sin 120°| = |\sin 240°| = |\sin 300°|$

$\frac{\sqrt{3}}{2}$ $\quad \frac{\sqrt{3}}{2}$ $\quad -\frac{\sqrt{3}}{2}$ $\quad -\frac{\sqrt{3}}{2}$



$|\cos 45°| = |\cos 135°| = |\cos 225°| = |\cos 315°|$

$\frac{\sqrt{2}}{2}$ $\quad -\frac{\sqrt{2}}{2}$ $\quad -\frac{\sqrt{2}}{2}$ $\quad \frac{\sqrt{2}}{2}$

★ Just find reference angle's value
★ Then ± by Quadrant

③

QII
Students
$\sin\theta$, $\csc\theta$ > 0
$(y>0)$

QI
All
$(x \,\&\, y > 0)$

Take
QIII
tan, cot > 0
$(x<0,\ y<0)$

Calculus
QIV
cos, sec > 0
$(x>0)$

- Solve

$\sec\theta \tan\theta > 0$

$\frac{1}{x}$ $\quad \frac{y}{x}$

$(0, \pi)$

@ ⋆Find SAT examples

Solve

$\sin\theta\cos\theta > 0$

y & x>0 or y & x<0

$(0, \frac{\pi}{2}) \cup (\pi, 3\pi/2)$

- Solve

$\underbrace{\csc\theta}\ \underbrace{\cot\theta} > 0$

$\frac{1}{y} \cdot \frac{x}{y} = \frac{x}{y^2} > 0$

$x > 0$

$(-\frac{\pi}{2}, \frac{\pi}{2})$

- Solve

$\frac{\sec\theta}{\csc\theta} > 0$

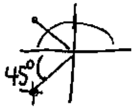
$\frac{1/\cos\theta}{1/\sin\theta}$ $\quad \frac{1/x}{1/y} = \frac{y}{x} = \tan\theta$

$(0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2})$ ← x & y<0 or x & y>0

④

	θ	0° 0	30° π/6	45° π/4	60° π/3	90° π/2	180° π	270° 3π/2	360° 2π
memorize	sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
	cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1
	tan θ	0	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	dne	0	dne	0
reciprocal	csc θ	dne	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1	dne	-1	dne
	sec θ	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	dne	-1	dne	1
	cot θ	dne	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	dne	0	dne

ex1 a) $\cos 135^\circ = -\cos 45^\circ = -\frac{\sqrt{2}}{2}$



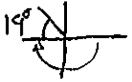
b) $\tan 390^\circ = \tan 30^\circ = \frac{\sqrt{3}}{3}$

Reference Angle Practice

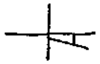
④

a) $99^\circ \rightarrow 81^\circ$

b) $-199^\circ \rightarrow 19^\circ$



c) $359^\circ \rightarrow 1^\circ$

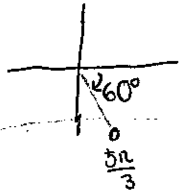


ex2 Reference Angle?

a) $\theta = \frac{5\pi}{3} \quad \alpha = \frac{\pi}{3}$

$= 2\pi - \frac{\pi}{3}$

$\frac{6\pi - \pi}{3}$

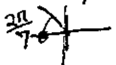


$\Rightarrow \sin \frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$

$\pi \approx 3.14$

$\frac{\pi}{2} \approx 1.5 \sim$

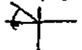
⑦ a) $\frac{5\pi}{7} \Rightarrow \frac{7\pi - 2\pi}{7} \rightarrow \frac{2\pi}{7}$



c) 1.4

$1.4 = \alpha$

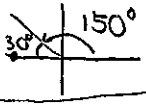
b) $-1.4\pi = -\pi - 0.4\pi \rightarrow 0.4\pi$



b) $\theta = 870^\circ$

-720°

150°

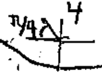


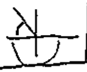
$\alpha = 30^\circ \Rightarrow \tan 870^\circ = -\tan 30^\circ$

$= -\frac{\sqrt{3}}{3}$

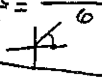
⑤

ex $\frac{11\pi}{4} = \frac{12\pi - \pi}{4} = \cancel{\frac{16\pi - 5\pi}{4}} \Rightarrow -\pi - \frac{\pi}{4}$ 3π $\pi/4$

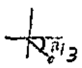




$-\frac{11\pi}{6} = \frac{-12\pi + \pi}{6} \rightarrow \frac{\pi}{6}$



$\frac{11\pi}{3} = \frac{12\pi - \pi}{3} \Rightarrow \frac{\pi}{3}$



ex3

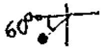
a) $\sin 240^\circ = -\sin 60^\circ = -\frac{\sqrt{3}}{2}$

d) $\sec\left(-\frac{\pi}{4}\right)$

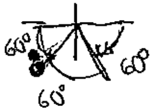
$= \frac{1}{\cos\frac{\pi}{4}} = \sqrt{2}$

b) $\sin \frac{16\pi}{3} = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$

$\frac{15\pi + \pi}{3} = 5\pi + \frac{\pi}{3}$



$\frac{18\pi - 2\pi}{3} = 6\pi - \frac{2\pi}{3}$

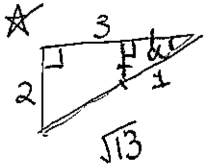


c) $\cot 495^\circ = -\cot 45^\circ = -1$

-360°

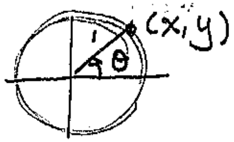
135°

★



⑤ Reciprocal Identities. By definition

$\frac{1}{y} = \csc\theta = \frac{1}{\sin\theta}$ $\quad \tan\theta = \frac{\sin\theta}{\cos\theta}$



$\sec\theta = \frac{1}{\cos\theta}$ $\quad \cot\theta = \frac{\cos\theta}{\sin\theta}$

$\cot\theta = \frac{1}{\tan\theta}$

Pythagorean Identities

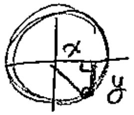
$$\boxed{\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \sec^2\theta &= 1 + \tan^2\theta \\ \csc^2\theta &= 1 + \cot^2\theta \end{aligned}}$$

$\div \cos^2\theta$

$\div \sin^2\theta$

Proof: $\sin^2\theta + \cos^2\theta = 1$

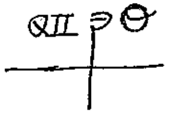
$y^2 + x^2 = 1^2$ ∵ Pythagorean Thm



ex5 a) $\sin\theta$ in terms of $\cos\theta$

$\sin\theta = \pm\sqrt{1 - \cos^2\theta}$

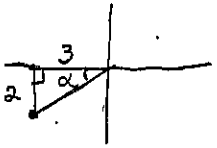
b) $\tan\theta$ in terms of $\sin\theta$ $\quad$ QII $\Rightarrow \theta$



$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\pm\sqrt{1 - \sin^2\theta}} = \frac{\sin\theta}{-\sqrt{1 - \sin^2\theta}}$

ex6 $\begin{cases} \tan\theta = \frac{2}{3} & \theta \in QIII \\ \cos\theta = ? \end{cases}$

★ Find SAT examples



$\cos\theta = -\cos\alpha = -\frac{3}{\sqrt{13}} = -\frac{3\sqrt{13}}{13}$

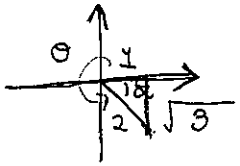
same ratio

similar triangles

ex7 Evaluating Trigonometric Functions

$2 = \sec\theta$ is in QIV

find the other trigonometric functions



① Draw terminal side of θ in Quadrant
② Reference Triangle
③ Find trig fn. Value

$\sin\theta = -\frac{\sqrt{3}}{2}$ $\qquad \csc\theta = -\frac{2\sqrt{3}}{3}$

$\cos\theta = \frac{1}{2}$

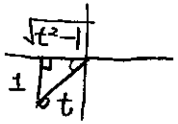
$\tan\theta = -\frac{\sqrt{3}}{1}$ $\qquad \cot\theta = -\frac{\sqrt{3}}{3}$

$\alpha = 60^\circ$
$\pi/3$

Ⓠ

$\csc\theta = t^{<0}$

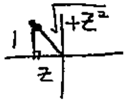
$\pi < \theta < \frac{3\pi}{2}$



?
$$\begin{cases} \tan\theta = \frac{1}{\sqrt{t^2-1}} & >0 \\ \cos\theta = \underset{-}{} & \frac{+\sqrt{t^2-1}}{t^{<0}} \end{cases}$$

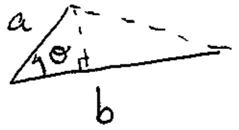
Ⓠ $\cot\theta = z \quad <0$

$\frac{\pi}{2} < \theta < \pi$



$\sec\theta = ? \quad \frac{\sqrt{1+z^2}}{\textcircled{z}} \quad < 0$

⑥ Area of a triangle

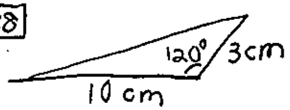


$$\boxed{A = \frac{1}{2}ab\sin\theta}$$

$\left(\begin{matrix} \theta \in [0,\pi] \\ \sin\theta \geq 0 \end{matrix} \right)$

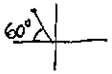
$\frac{1}{2}bh = \frac{1}{2}b(a\sin\theta)$

ex8



$A = \frac{1}{2}(3cm)(10cm)\sin 120^\circ$

$= 15 \cdot \frac{\sqrt{3}}{2}\ cm^2$



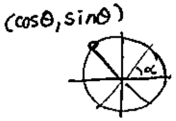
$\approx 13cm^2$

Ⓠ Solve $\csc\theta = \frac{-2}{\sqrt{3}}$

$\sin\theta = -\frac{\sqrt{3}}{2}$ ←

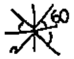
$\alpha = 30^\circ \Rightarrow \boxed{\theta = \frac{7\pi}{6}, \frac{11\pi}{6}}$
$\frac{\pi}{6}$

5.1 Unit Circle Review



solve $|\sec\theta| = 2$

$\cos\theta = \frac{1}{2}$

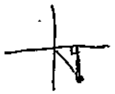


$\theta = 60^\circ, 120^\circ, 240^\circ, 300^\circ$

ex1 $P\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3}\right)$ on unit circle?

$x^2 + y^2 = 1$ yes

ex2 $P(\sqrt{3}/2, y)$ on unit circle in QIV

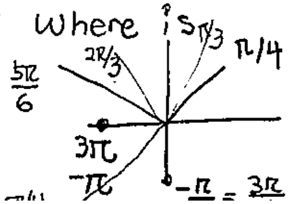


$\cos\theta = \frac{\sqrt{3}}{2}$

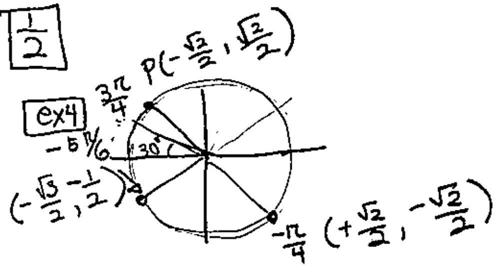
$\theta = 30^\circ$

$\sin\theta = y = \boxed{\frac{1}{2}}$

ex3 Where is



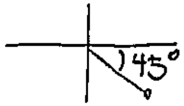
ex4



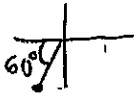
ex5 Reference Angle

a) $t=\frac{5\pi}{6}=\frac{6\pi-\pi}{6}\rightarrow\pi-\boxed{\frac{\pi}{6}}$

b) $\frac{7\pi}{4}=\frac{8\pi-\pi}{4}=2\pi-\boxed{\frac{\pi}{4}}$

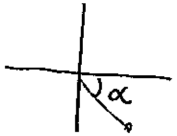


c) $-\frac{2\pi}{3}=\frac{-3\pi+\pi}{3}=-\pi+\boxed{\frac{\pi}{3}}$



d) $5.80\Rightarrow 2\pi-5.80=\alpha$

≈ 6 $\quad 0.48$



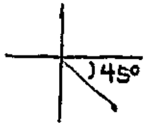
ex6 Terminal Point?

a) $t=\frac{5\pi}{6}$

$P\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

$\cos\frac{\pi}{6}$ $\quad\sin\frac{\pi}{6}$

b) $\frac{7\pi}{4}$



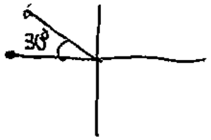
$P\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$

c) $-\frac{2\pi}{3}$

$P\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$

ex7

$t=\frac{29\pi}{6}=\frac{30\pi-\pi}{6}=5\pi-\frac{\pi}{6}$



$P\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

5.2 Trig functions

ex1

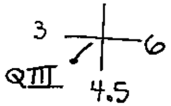
a	$\frac{\pi}{3}=60°$	$\frac{\pi}{2}=90°$
sin	$\frac{\sqrt{3}}{2}$	1
cos	$\frac{1}{2}$	0
tan	$\sqrt{3}$	dne
csc	$+\frac{2}{\sqrt{3}}$	1
sec	2	dne
cot	$\frac{1}{\sqrt{3}}$	0

ex2 Sign?

$\cos\frac{\pi}{3}$ (+)

QI

tan 4 (+)

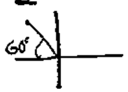


Q II

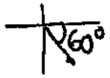
cos t (−)

sin t (+)

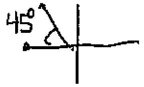
ex3 a) $\cos\frac{2\pi}{3}=-\frac{1}{2}$



b) $\tan\left(-\frac{\pi}{3}\right)=-\sqrt{3}$



c) $\sin\frac{19\pi}{4}=\frac{\sqrt{2}}{2}$



$\frac{20\pi-\pi}{4}=5\pi-\frac{\pi}{4}$

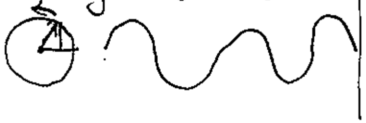
5.3~5.4 See SAT Pg 24 & Mem Library & $\underset{①}{A}f(\underset{①'}{B}(x+\underset{②'}{\frac{c}{B}}))+\underset{③}{D}$

$\underset{②''}{B}x+\underset{①''}{c}$

• 5.3

ex 1, 2, 3, 4, 5, 6, 7
6th 1 3 4 5 6
8 9 10

① Show youtube



② FT FFT STFT
(DFT
DTFT

-Voice frequencies
freq-time diagram
-Timbre

Amplitude modulation
8, 9, 10

-Interference & beats

③ AM/FM
youtube video

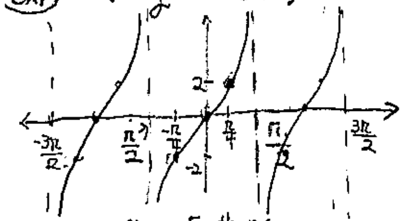
Barron's examples

Project/Bonus

Simple Harmonic Motion
Sound wave addition/timbre
Beats

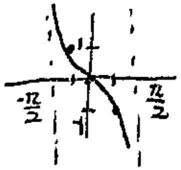
5.4

ex1 a) $y = 2\tan y$



Show 5 things

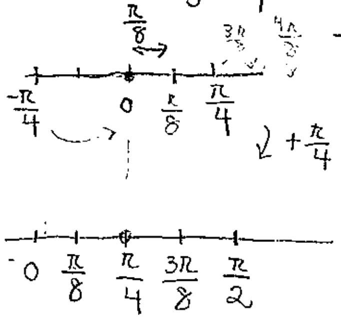
b) $y = -\tan x$

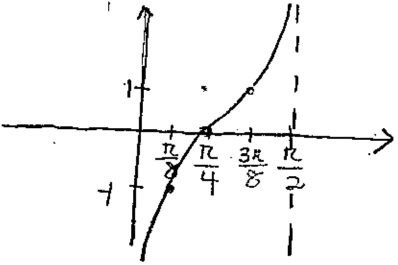


ex2 b) $y = \tan[2x - \frac{\pi}{2}] = \tan[2(x - \frac{\pi}{4})]$

$T = \frac{\pi}{2}$

Right $\frac{\pi}{4}$

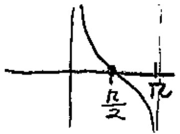


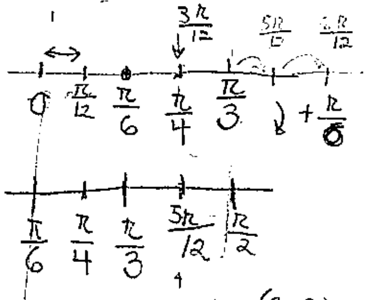


ex3 $y = 2\cot(3x - \frac{\pi}{2})$

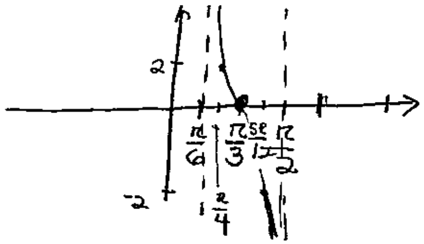
$= 2\cot[3(x - \frac{\pi}{6})]$

$T = \frac{\pi}{3}$



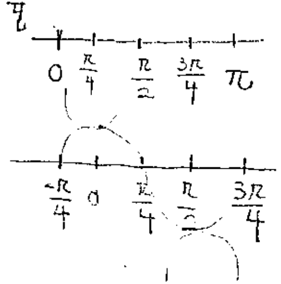


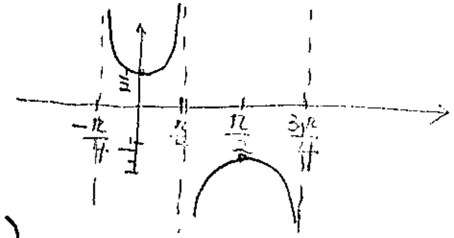
$\frac{\pi}{4} + \frac{\pi}{6} = \frac{(3+2)}{12}$



ex4 b) $y = \frac{1}{2}\csc(2x + \frac{\pi}{2}) = \frac{1}{2}\csc(2(x + \frac{\pi}{4}))$

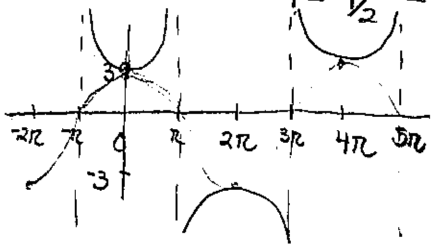
$T = \frac{2\pi}{2} = \pi$

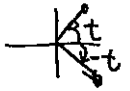




ex5 $y = 3\sec(\frac{1}{2}x)$

$T = \frac{2\pi}{1/2} = 4\pi$



★ $\sin t = \text{odd}$  $\csc t = \text{odd}$

$\cos t = \text{even}$ $\quad \sec t = \text{even}$

$\tan t = \text{odd} = \frac{\text{odd}}{\text{even}}$ $\quad \cot t = \text{odd}$

& 2π periodic

$\boxed{\text{ex5}}$ a) $\sin(-\frac{\pi}{6}) = -\sin\frac{\pi}{6} = -\frac{1}{2}$

30°

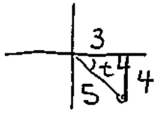
b) $\cos(-\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

c) $\sin\theta + \sin(-\theta) = 0$

d) $\cos\theta - \cos(-\theta) = 0$

e) $\sin(2\pi - \theta) = -\sin\theta$

$\boxed{\text{ex6}}$ $\cos t = \frac{3}{5}$ $\quad t \in Q\text{IV}$, Find others



$\sin t = \frac{-4}{5}$ $\quad \csc t = \frac{-5}{4}$

$\sec t = +\frac{3}{5}$

$\tan t = \frac{-4}{3}$ $\quad \cot t = -\frac{3}{4}$

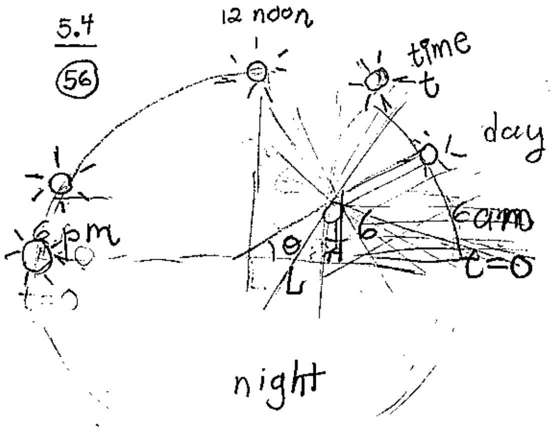
$t \in Q\text{III}$

$\boxed{\text{ex7}}$ Write $\tan t$ in terms of $\cos t$

- Graph sin/cos/… $\quad$ domain
- IDs $\quad$ range

period

$\frac{5.4}{56}$



$\theta = \frac{t \text{ hours}}{24} \times 2\pi$

θ

$$\boxed{L(t) = 6\left|\cot\left(\frac{\pi}{12}t\right)\right|}$$

Shadow Length

how?

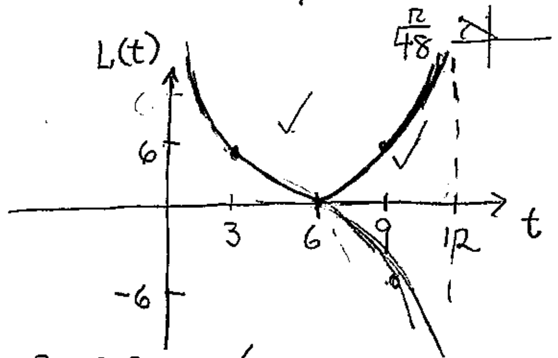
a)

Time	t	$L(t)$
8am	2	$6\cot\frac{\pi}{6} = 6\cdot\sqrt{3} \approx 10.39$ ✓ ✓
noon	6	$6\cot\frac{\pi}{2} = 0$ ✓
2pm	8	$6\left\|\cot\frac{2\pi}{3}\right\| = \frac{6}{\sqrt{3}} = 2\sqrt{3} \approx 3.46$ ✓
5:45pm	11.75 $11\frac{3}{4} = \frac{47}{4}$	$6\cot\left(\frac{\pi\times 47}{48}\right) \approx 6\cot\left(\frac{\pi}{48}\right) \approx 91.54$

(+8)

60°

b)



$T = \frac{\pi}{\pi/12} = 12$

c) $t = 3 \rightarrow 9$ am ✓

9 $\quad$ 3pm

d) infinitely long ✓

Bonus

$\sin' x = \cos x$

$\lim\limits_{h\to 0} \frac{\sin x}{x} = 1$ $\lim\limits_{h\to 0} \frac{\cos h - 1}{h} = 0$

Method 1

$$\lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \cos x \underbrace{\lim_{h\to 0} \frac{\sin h}{h}}_{1} + \sin x \underbrace{\left(\lim_{h\to 0} \frac{\cosh - 1}{h}\right)}_{0} = \cos x$$

Graphing CALC

OR

$$\lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0} \frac{2\cos(x+\frac{h}{2})\sin\frac{h}{2}}{h}$$

$$= \lim_{h\to 0} \frac{\sin\frac{h}{2}}{h/2} \cdot \lim_{h\to 0} \cos(x+\frac{h}{2})$$

$$= 1 \times \cos x$$

$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$\cos x - \cos y = -2\sin(\frac{x+y}{2})\sin(\frac{x-y}{2})$

$\frac{x+h+x}{2}$ $\frac{x+h-x}{2}$

$$D_x \cos x = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h\to 0} \frac{-2\sin(x+\frac{h}{2})\sin(\frac{h}{2})}{h}$$

$$= \underbrace{\lim_{h\to 0} \frac{\sin(x+\frac{h}{2})}{h}}_{=\ \sin x} \cdot \underbrace{\left(-\lim_{h\to 0} \frac{\sin(\frac{h}{2})}{h/2}\right)}_{-1}$$

5.5

Simple Harmonic Motion

- displacement is a sinusoid

$\boxed{y = a \sin \omega t, \quad y = a \cos \omega t}$

$\boxed{\text{Amplitude}}$

(1 cycle)

Period $\boxed{T = \frac{2\pi}{\omega}}$ time for 1cycle

Frequency $\boxed{f = \frac{\omega}{2\pi}}$ (Hz: cycles/sec)

$\boxed{y = a \overset{}{\underset{\cos}{\sin}}(\underline{2\pi f}\, t)}$

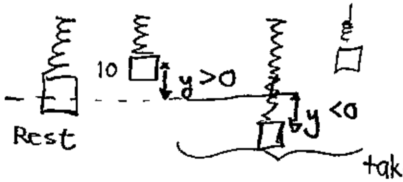
$2\pi 3\boxed{t}$

$\Rightarrow \Delta t = 1 \Rightarrow$ go around 3 times ⊕

3 cycles/second

$B = \frac{2\pi}{0.5} \times \frac{2\pi}{T} = 2\pi f$

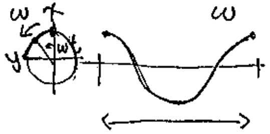
$\boxed{\text{ex1}}$ Vibrating Spring



takes 0.5 second

$y = 10\sin \underbrace{4\pi}_{\omega} t$

$10\cos(\underbrace{4\pi}_{\omega} t)$



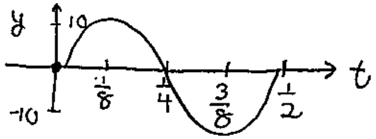
$T = \frac{1}{2}s, \ f = 2$

a) $A = 10$ cm

$T = \frac{2\pi}{4\pi} = \frac{1}{2}$ s ← OR $y = 10\sin(2\pi \times 2t)$

$f = 2$ Hz

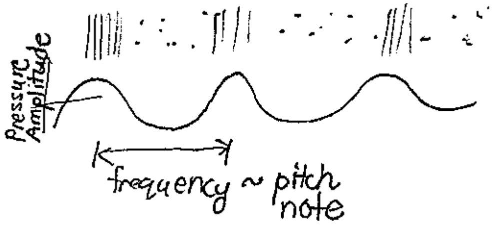
b)



ex2 Tuba note E

$V(t) = 0.2\sin(80\pi t)$

Pressure lb/in² $2\pi \times 40$

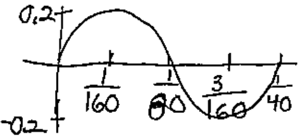


a) $A = 0.2$ lb/in²

$T = \frac{1}{40}$ s

$f = 40$ Hz

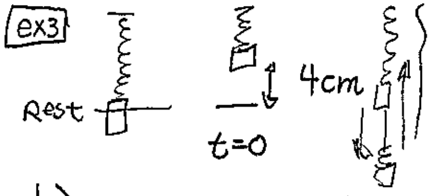
b)



c) Louder ⇒ A bigger > 0.2 $\underline{\underline{0.3}}\sin(2\pi \cdot \underline{\underline{35}}\, t)$

d) flat note $f < 40$ Hz

ex3



No friction
Return to 4cm after $\frac{1}{3}$s

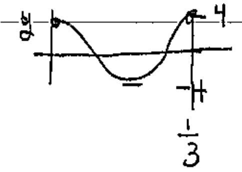
Model

a) $y = \boxed{4\cos(2\pi 3t)}$

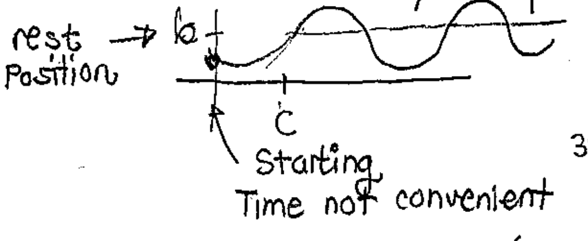
$T = \frac{1}{3}$ OR: $\frac{1}{3} = \frac{2\pi}{B}$

$f = 3$ $B = 6\pi$

b)



General: $\boxed{y = a\sin(\omega(t-c)) + b}$, $y = a\cos(\omega(t-c)) + b$



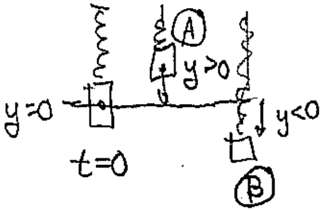
$y = 3\sin(\omega(t-2)) + 5$

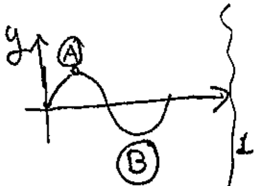
3cm oscillation

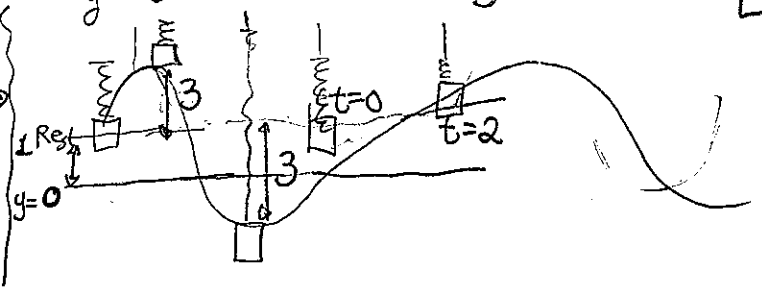
time for cycle is $\frac{1}{2}$s

4π

5cm above ground







ex4 star brightness

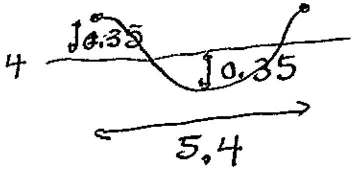
Delta Cephei

max brightness every 5.4 days
avg brightness 4.0 magnitude
Brightness varies ±0.35

a) function

$B(t) = \boxed{0.35\cos\left(\frac{2\pi}{5.4}t\right) + 4}$

b) Sketch



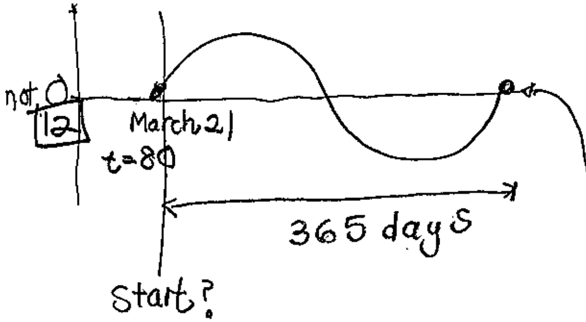
ex5 Number hours of daylight

$y(t)$

days from January 1

- Longest day is 14h 50min $14\frac{5}{6} \rightarrow$ ~~Figure 9 occurs on June 21 t=~~

Shortest: 9h 10min = $9\frac{1}{6} \rightarrow$ December 21



$\boxed{a} = \frac{14\frac{5}{6} - 9\frac{1}{6}}{2} = \boxed{2.83\text{h}}$

$\frac{5\frac{2}{3}}{2} = \boxed{\frac{17}{6}}$

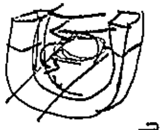
$14\frac{5}{6} - \frac{17}{6}$

$= 12$

$y = \boxed{\frac{17}{6}\sin\left(\frac{2\pi}{365}(t-80)\right) + 12}$

ex6 Demo

(motor) Generator



Changing $\vec{B} \Rightarrow$ current (Faraday)

Rotating. $E(t) = E_0 \cos \omega t$

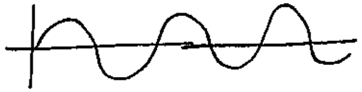
"110V"

a c varies ±155V

$f = 60 Hz$

• Eqn: $V(t) = 155 \sin(120\pi t)$

★why "110V"?



Root-mean-square

avg = 0 ; $\sqrt{avg[V^2]} \Rightarrow \frac{1}{\sqrt{2}} V_{max} = \frac{155}{\sqrt{2}} \approx 110$

Damped Harmonic Motion (friction stops  oscillation)

$y = \boxed{ke^{-ct}} \sin\omega t$ ← envelope
cos

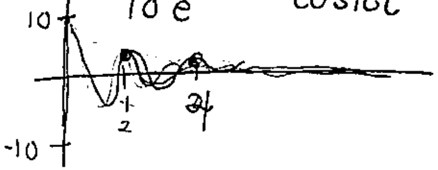
initial amplitude

c = damping constant

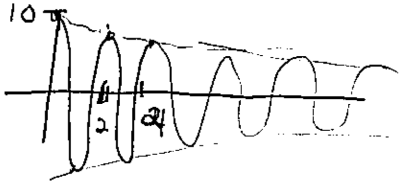
$T = \frac{2\pi}{\omega}$ "Period"

$T = 2$

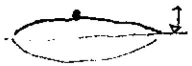
$C = -0.5$ $10e^{-0.5t}\cos\pi t$



$10e^{-0.1t}\cos\pi t$



ex8 G ($f = 200Hz$)

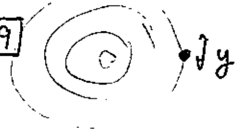


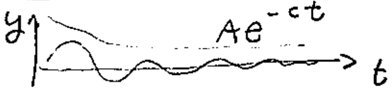
damping constant $c = 1.4$

$y(t) = Ae^{-ct}\cos\omega t$

$= \boxed{0.5e^{-1.4t}\cos(400\pi t)}$

ex9





Amplitude drops to $\frac{1}{10}$ of initial after 20s.

$c = ?$

$h(t+T) = \frac{1}{10} h(t)$

$\cancel{Ae^{-ct}} e^{-cT} = \cancel{\frac{1}{10}Ae^{-ct}}$

$e^{-cT} = \frac{1}{10}$

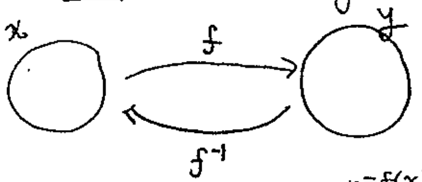
$-cT = -\ln 10$

$c = \frac{\ln 10}{T} = \frac{\ln 10}{20} \approx \boxed{0.12}$

$\frac{1}{10}\cancel{A} = \cancel{A}e^{-ct}$

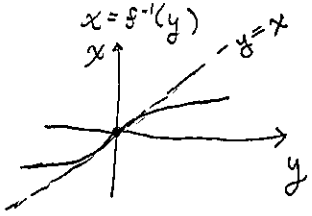
As the tides change, the water level in a bay varies sinusoidally. At high tide today at 8 A.M., the water level was 15 ft.
6 hrs later at 2 PM it was 3ft.
Model water level as fn of time using a sinusoid.

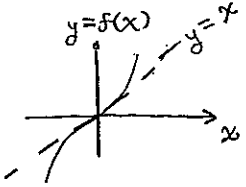
$\boxed{7.4}$ Inverse Trig Functions



Power? Angle?

$\sin^{-1}\frac{1}{2}$ e^2 $\log 100$ $\cos 60°$

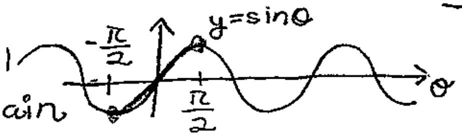




invertible | — Rule

- 1-1 (monotonic)

- Is sine invertible? No, not 1-1 ⇒ Restrict domain



calculator too

- $y = \sin\Theta$, $\Theta = \underbrace{\sin^{-1} y}$

  ★★ angle whose sine is y
  arcsine

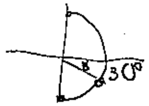
Ⓠ Solve $\sin\Theta = \frac{1}{2}$, $\Theta \in [0, 2\pi]$

$\Theta = \frac{\pi}{6}$, $\frac{5\pi}{6}$

30° , 150°
390° , 510°

Ⓠ $\sin^{-1}(\frac{1}{2}) = \underline{30°}$ not 150° Pick the angle between $(-\frac{\pi}{2}, \frac{\pi}{2})$

Ⓠ $\sin^{-1}(-\frac{1}{2}) = -\frac{\pi}{6}$



Ⓠ $\sin^{-1}(-1) = -\frac{\pi}{2}$

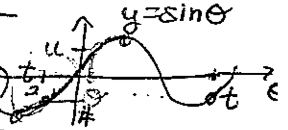
$\boxed{ex1}$ $\sin^{-1}(\frac{3}{2}) = dne$

$\boxed{ex2}$ CALC : $\sin^{-1}(0.82) \overset{RAD}{=} 0.96141$
$\overset{deg}{=} 55°$

$\sin^{-1}\frac{1}{3} = 19.5°$
$= 0.33984$ Rad

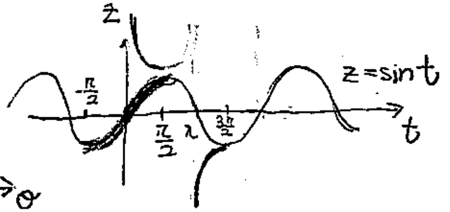
$\sin(\sin^{-1} u) \overset{?}{=} u$ Always (if u in domain)
$\sin(\Theta)$

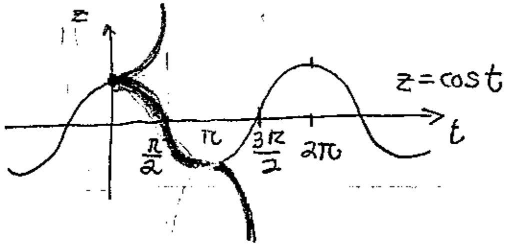
$\sin^{-1}(\sin t) \overset{?}{=} t$ only if $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$

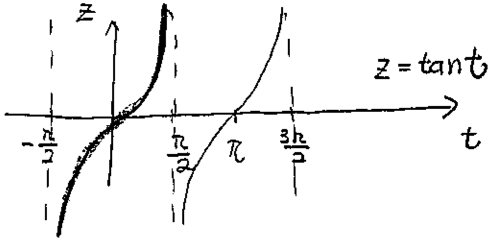


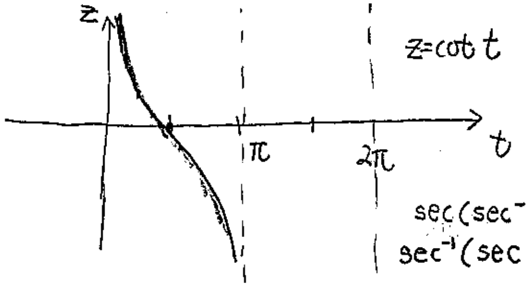
$\sin^{-1}(\sin t) = \sin^{-1}(\#) = t_2 \neq t$

Trigonometric Inverses









domain range

$\sin^{-1} z : [-1, 1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$

$z \longmapsto t$ angle

$\cos^{-1} z : [-1, 1] \longrightarrow [0, \pi]$

$\tan^{-1} z : \mathbb{R} \longrightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$

$\csc^{-1} z : (-\infty, -1] \cup [1, \infty) \longrightarrow (0, \frac{\pi}{2}] \cup (\pi, \frac{3\pi}{2}]$

$\sec^{-1} z : (-\infty, -1] \cup [1, \infty) \longrightarrow [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$

$\cot^{-1} z : \mathbb{R} \to (0, \pi)$

$\sec(\sec^{-1} 2) = 2$ ✓
$\sec^{-1}(\sec \frac{\pi}{4}) = \frac{\pi}{4}$ ✓

Ⓠ T/F
✓ $\sin(\sin^{-1} 0.1) = 0.1$
✓ $\cos(\cos^{-1} 0.5) = 0.5$
× $\sin(\sin^{-1} 3) = 3$
✓ $\tan(\tan^{-1} 50) = 50$
× $\cos^{-1}(\cos 2\pi) = \overset{(0)}{\cancel{2\pi}}$
× $\cos^{-1}(\cos \frac{3\pi}{2}) = \cancel{\frac{3\pi}{2}} \to \frac{\pi}{2}$
✓ $\sin^{-1}(\sin \frac{\pi}{2}) = \frac{\pi}{2}$
× $\sin^{-1}(\sin \pi) = \cancel{\pi} \to 0$
✓ $\cos^{-1}(\cos \frac{\pi}{2}) = \frac{\pi}{2}$

7.1 TRIG IDs

Fundamental IDs (Pg 528) $\forall x \in \mathbb{R}$

- Reciprocal: (by definition on unit circle)

$\csc x = \frac{1}{\sin x}$ $\tan x = \frac{\sin x}{\cos x}$ $\cot = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$

$\sec x = \frac{1}{\cos x}$

- Pythagórean (from unit circle)

$\sin^2 x + \cos^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$1 + \cot^2 x = \csc^2 x$

Even-odd

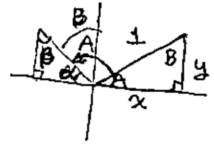
$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$
sec	csc
	tan
	cot

★ Cofunction

$\sin(\frac{\pi}{2} - u) = \cos u$	
cot	tan
sec	csc
cos	sin

★ $\boxed{\sin A = \cos B \text{ true when } A+B=90^\circ}$

$\beta = 90 - A$



assuming acute

QI ✓

$\sin A = y$

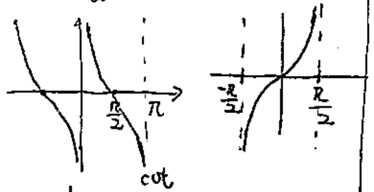
$\cos B = x$

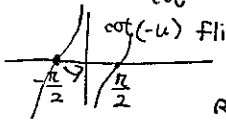
Just show triangle QII ✓ $\sin A > 0$

$\cos B = \cos\beta > 0$

when not acute?

$\cot(\frac{\pi}{2} - u) = \tan u$

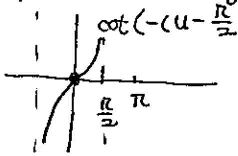




cot(-u) flip y-axis

Right

$\cot(-(u - \frac{\pi}{2}))$



$\sec(u - 90^\circ) = \csc u$? ✓

$\sec(90^\circ - u)$

$\sin(u - 90^\circ) = -\cos u$ ✓

T/F

$\sec(u + \frac{\pi}{2}) = \csc(-u)$ ✓

$\sec(u - \frac{\pi}{2}) = \csc u$ ✓ even

$\cot(-u + 90^\circ) = \tan u$ ✓

$\sin(90^\circ - u) = \cos u$ ✓

$\tan(u + 90^\circ) = \cot(u)$ ×

correction $-\cot(u)$

$\sec(90^\circ - u) = \csc(-u)$ ×

correction $\csc u$

$\csc(90^\circ - u) = \sec(-u)$ OK even

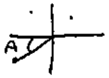
② $\tan A = \cot(A + 60^\circ)$. $A = ?$

a) $2A + 60^\circ = 90^\circ$

A acute $A = 15^\circ$

b) A is in QIII

⇒ 15° is reference angle. $A = \boxed{195^\circ}$ why?



$\tan A = \cot(90^\circ - A)$

$= \cot(A + 60^\circ)$

⇓

$90 - A = A + 60$

$30 = 2A$

$15 = A$

True for any $A \in \mathbb{R}$

tan is π periodic

$15^\circ + k \cdot 180^\circ$

are all solutions

FALSE: 1 counterexample

TRUE: einstein (a single exp can prove him wrong)

Simplify / Prove a Trig ID

- Start from complicated side (OR meet in the middle)
- factor / use definition IDs / $\frac{1}{A-B}$, $\frac{A+B}{A+B}$

ex1 $\cos t + \tan t \sin t$

$= \cos t + \frac{\sin^2 t}{\cos t} = \frac{\cos^2 t + \sin^2 t}{\cos t} = \frac{1}{\cos t} = \sec t$

ex2 simplify.

$\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{1+\sin\theta}$

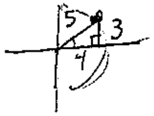
$= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{1+\sin\theta} \cdot \frac{1-\sin\theta}{1-\sin\theta} = \frac{\sin\theta}{\cos\theta} + \frac{\cancel{\cos\theta}(1-\sin\theta)}{\underbrace{1-\sin^2\theta}_{\cos^2\theta}}$

$= \frac{\sin\theta + (1-\sin\theta)}{\cos\theta} = \frac{1}{\cos\theta} = \sec\theta$

OR $\frac{\sin\theta(1+\sin\theta) + \cos^2\theta}{\cos\theta(1+\sin\theta)} = \frac{\cancel{\sin\theta + 1}}{\cos\theta(\cancel{1+\sin\theta})} = \sec\theta$

$\boxed{\text{ex3}}$ $\cos(\sin^{-1}\frac{3}{5}) = ?$

Method 2 : $\cos\theta = +\frac{4}{5}$



$(\sin\theta = \frac{3}{5})$

$\tan(\sin^{-1}\frac{3}{5}) = \frac{3}{4}$

$\sec(\sin^{-1}\frac{3}{5}) = \frac{5}{4}$

$\csc(\sin^{-1}\frac{3}{5}) = \frac{5}{3}$

Method 1 : $\cos(\sin^{-1}\frac{3}{5})$

$= \cos\theta = \oplus\sqrt{1-\sin^2\theta}$

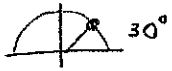
$-\frac{\pi}{2}$ $\frac{\pi}{2}$

$= \sqrt{1-[\sin(\sin^{-1}\frac{3}{5})]^2}$

$= \sqrt{1-(\frac{3}{5})^2} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

$\boxed{\text{ex4}}$

a) $\cos^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{6}$

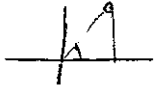


b) $\cos^{-1}0 = \frac{\pi}{2}$

90°

c) $\cos^{-1}\frac{5}{7} = \theta$

$= 0.77519$



$\cos\theta = \frac{5}{7}$

$\boxed{\text{ex5}}$ Write as algebraic expressions in x for $-1 \le x \le 1$

a) $\sin(\underset{\theta}{\underbrace{\cos^{-1}x}})$

change to cos so $\cos(\cos^{-1}x) = x$

$= \pm\sqrt{1-\cos^2\theta} = \pm\sqrt{1-[\cos(\cos^{-1}x)]^2} = \sqrt{1-x^2}$

$\sin\theta \ge 0$ ← $\cos^{-1}$ angle restriction

$\sin^{-1}(\frac{1}{2}) = 30°$

$\sec^{-1}(2) = 60°$

$\csc^{-1}(-\frac{2}{\sqrt{3}}) = 240°$

60°

$\tan^{-1}(-1) = -45°$

$\tan^{-1}(\sqrt{3}) = 60°$

$\sin(\sin^{-1}u) = u \quad (u \in [-1,1])$

$\sin^{-1}(\sin 7\pi) = 0$

$\cos^{-1}(\cos\frac{3\pi}{2}) = \frac{\pi}{2}$

$\sin^{-1}(\sin 150°) = 30°$

$\sin^{-1}(\sin 240°) = -60°$

$\cos^{-1}(\cos 240°) = 120°$

60°

$\sec^{-1}(\sec 120°) = 240°$

$\csc^{-1}(\csc 120°) = 60°$

$\tan^{-1}(\tan 240°) = 60°$

$\tan^{-1}(\tan 120°) = -60°$

$\sin^{-1}(\cos 60°) = 30°$

$\sin^{-1}(\cos 210°) = -60°$

$-\frac{\sqrt{3}}{2}$ 30°

$\cos^{-1}(\tan 225°) = 180°$

$\csc^{-1}(\cot 135°) = -45°$

-1

b) $\tan(\underset{\theta}{\underbrace{\cos^{-1}x}}) = \frac{\sin\theta}{\cos\theta} = \frac{\pm\sqrt{1-\cos^2\theta}}{\cos\theta} = \boxed{\frac{\sqrt{1-x^2}}{x}}$

θ

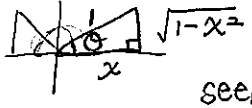
Method 2 :

$\sin(\underset{\theta}{\underbrace{\cos^{-1}x}}) = \oplus\sqrt{1-x^2}$

$\tan\theta = \frac{\sqrt{1-x^2}}{x}$

$\checkmark \left[\begin{matrix} x<0 \\ \sqrt{\ } > 0 \end{matrix}\right.$ − + $\begin{matrix} \sqrt{\ } > 0 \\ x > 0 \end{matrix}$ >0 ✓

$\cos\theta = x$



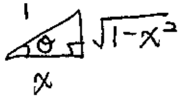
seems θ acute but works for $x, \theta \in \mathbb{R}$ & defined

★ TRIANGLE Method always works so that $S(T^{-1}(x))$ correct (Restrictions chosen well)

★ Bonus show/prove

$\boxed{\text{ex6}}$ $\sin(2\underset{\theta}{\underbrace{\cos^{-1}x}})$ algebraic. $-1 \le x \le 1$

$\cos\theta = x$



$= 2\sin\theta\cos\theta$

$= \boxed{2\sqrt{1-x^2}\ x}$

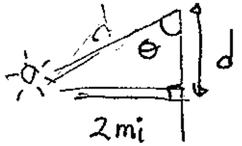
$\boxed{\text{ex7}}$ a) $\tan^{-1}1 = \frac{\pi}{4}$

b) $\tan^{-1}\sqrt{3} = 60° = \frac{\pi}{3}$

c) $\tan^{-1}(-20) = -1.52084$ radians

$\tan^{-1}(-\frac{\sqrt{3}}{3}) = -30° = -\frac{\pi}{6}$

$\boxed{\text{ex8}}$



$\tan\theta = \frac{2}{d}$

$\theta(d) = \tan^{-1}(\frac{2}{d})$

- Show $\sin x + \cos x = 1$ is False

A single counterexample

$$\sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2} \neq 1 \quad \#$$

- Warning:

$$A = B$$
$$\Downarrow \; \not\Uparrow$$
$$A^2 = B^2$$

NOT same eqn unless reversible operation's performed

$\boxed{ex3}$ $\cos\theta(\sec\theta - \cos\theta) \overset{?}{=} \sin^2\theta$

rewrite in sine

show 1 ID at a time

$$LHS = \cos\theta\left(\frac{1}{\cos\theta} - \cos\theta\right)$$
$$= 1 - \cos^2\theta$$
$$= \sin^2\theta = RHS$$

$\boxed{ex4}$ $2\tan x \sec x \overset{?}{=} \frac{1}{1-\sin x} - \frac{1}{1+\sin x}$

$$RHS = \frac{1+\sin x - (1-\sin x)}{1-\sin^2 x}$$
$$= \frac{2\sin x}{\cos^2 x} = 2\tan x \sec x = LHS$$

$\boxed{ex5}$ $\frac{\cos u}{1-\sin u} = \sec u + \tan u$

$$LHS = \frac{\cos u}{1-\sin u} \cdot \frac{1+\sin u}{1+\sin u} = \frac{\cancel{\cos u}(1+\sin u)}{\underbrace{1-\sin^2 u}_{\cos^2 u}}$$
$$= \sec u + \tan u$$

$\boxed{ex6}$ Meet in the middle

$$\frac{1+\cos\theta}{\cos\theta} = \frac{\tan^2\theta}{\sec\theta - 1}$$

$$LHS = \frac{1+\cos\theta}{\cos\theta} = \sec\theta + 1$$

$$RHS = \frac{\tan^2\theta}{\sec\theta - 1} \cdot \frac{\sec\theta+1}{\sec\theta+1} = \frac{\cancel{\tan^2\theta}(\sec\theta+1)}{\cancel{\tan^2\theta}}$$

$\sec^2\theta - 1$

$\boxed{ex7}$ Trig Substitution (in calculus)

$$\int_0^1 \sqrt{1-x^2}\,dx \qquad \text{Let } x = \sin\theta \qquad x: 0 \sim 1$$
$$dx = \cos\theta\,d\theta \qquad \theta: 0 \sim \frac{\pi}{2}$$

restriction

$$\int_0^{\pi/2} \sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta$$

☆

$$\sqrt{\cos^2\theta}$$
$$|\cos\theta| = \cos\theta$$

$$= \int_0^{\pi/2} \cos^2\theta\,d\theta = \int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\,d\theta = \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

☆

Formulas

use formulas

5th → 7.1 (Pg. 533) #2, 6, 8, 10, 14, 18, 23, 27, 30, 33, 36, 38, 43, 46, 50, 53, 57, 61,
6th → 7.1 (P.498) 4 7 10 11 16 19 26 32 46 36 37 40 45 47 51 56 59 64

62, 72, 77, 78, 91, 93, 94. Bonus ~~100~~.
63 74 79 80 94 96 95

± formula

5th → 7.2 (Pg. 539) # 4, 7, 8, 9, 11, 14, ~~15~~, 17, ~~19~~, 27, (29,) 30, 36, 37, ~~43~~, 44, 46, 47,
6th → 7.2 (P.505) 5 9 10 12 14 16 ~~18~~ 20 ~~21~~ 29 (32) 31 38 40 ~~58~~ 57 59 62
class practice too

(48,) 49, 50, 54, 55. Bonus 56, 57, 48 $\cos' x = -\sin x$
63 64 68 69 Bonus: 70 71 61
in class too
~~44~~, 45, 47, 49, ~~51~~, 52

47, 49, 51, 52, 50

----

more formula

7.3 (Pg548) ~~#2~~, 3, 5, 7, 11, 12, 23, 24, ~~25~~, 27, 29, ~~30~~, 31, 32, 36, 38, 41, 44,
6th → 7.3 (P.514) ~~4~~ 6 7 10 14 13 25 26 ~~28~~ 29 32 ~~31~~ 33 34 38 39 55 57

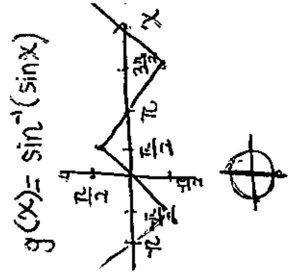
45, 46, 47, 50, 51, 55, 56, 58, 59, 62, 65, 73, 74, 77, 80, 81, 91, 93.
60 59 61 63 65 69 70 71 74 76 79 88 87 92 94 96 106 107

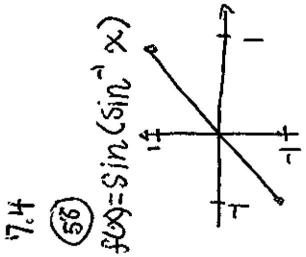
Bonus # 88.
102

7.4 || eqns || 7.5

7.5 (Pg. 568) # 4, 6, 8, (~~15~~,) 17, 19, 24, (27,) (54,) 67.
CR 7.4 (P523) 28 31 33 ~~43~~ 53 41 60 3 7 43

7.5 (Pg. 568) #56, 58, 61, 64, 66, 68, 72, 75, 79, 80, 81, 82.
(528) ~~49~~ 36 40 41 44 45 56 59 64 63 57 58
43 30
27 28





Your next quiz will be based on these problems (al of chapter 7)

- Worksheet MTC (You need to know how to show work) Correct all errors.

6th (P. 530)

5th →
- Ch. 7 Review (Pg. 571)
  - (Check # 7, 8, 9)
  - Exercises # 2, 5, 6, 7, 9, 15, 20, 23, 26, 31, 32, 35, 37, 39, 41, 42, 47,
    1 6 5 7 10 15 20 24 25 34 33 38 39 41 43 44 50

    (51, 54~57,) 59, 61, 64, 72, 74, 75, 76.
    ~~53~~ 61 63 66 67 69 --------------
    51 ~60 Standing Waves activity

(536) 1 3 4 ~~5~~ 6 7

6th Traveling Waves (Pg. 578) # 1, 3, 4, 6, 7.

For # 1 and 4, just sketch the waves for the first 2 times.

6th Pg. 519 # 57~62; Pg. 521 # 11~14 (don't solve. Just say Law of sines or cosines
P. 486 #67~72 P. 488 #13~16

3 soln none 1

6.4 Law of Sines (P.506) # 6, 10, 13, 21, 22, 23, 24 25, (26,) 27, 30, 39, 43.
6.5 (473) P 7 ~~11~~ ~~13~~ 16 23 24 ↑ ↑ 28 30 31 42 45
26 25

(Pg. 523) # 1~2 map (Great Trigonometric Survey of India (our classroom))
490 1~2 (Theodolite, Sir George Everest CW

P472 note

(480) 6 ~~10~~ 9 ~~17~~ ~~18~~ ~~20~~------~~19~~ ~~22~~ ~~21~~ 23 29 36 37

- 6.6 ~~6.5~~ Law of Cosines (P. 513) # 3, ~~7~~, 8, 15, (Sines) 16, 17, 18, 19, (20) 21, 27, ~~30~~, 33, 35, 37,
  44 42, 45 43, 46 44, 50 47, 51 49.
  2 soln sine, no soln sine, cos, 1 soln sine
- 8.1 (Pg. 586) # 7, 9~23 odd, 27, 31, 35, 39, 41, 42, 43, 46, 50, 51, 52, 53, 57, 59,
  546 9 ~ 25 odd 29 33 38 42 44 43 45 47 56 66 65 62 54 67
  61. 11, 24, 49, 51, 52
- Bonus: Prove Heron's formula
- Bonus: Is more than one solution possible in the Law of Sines? Why? Is more than one solution possible in the Law of Cosines? Why?

Bonus: beat frequency. See http://en.wikipedia.org/wiki/Beat_frequency Answer these questions

7.2 ± formulas

$\boxed{7.2}$

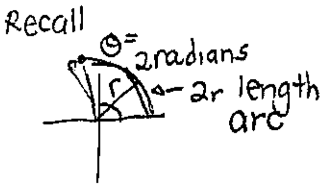
$\sin(a\pm b) = \sin a\cos b \pm \cos a\sin b$

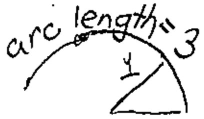
$\cos(a\pm b) = \cos a\cos b \mp \sin a\sin b$

$\tan(a\pm b) = \dfrac{\tan a\pm \tan b}{1\mp \tan a\tan b}$

Proof of cos(a+b). . cos(a−b) by even/odd

- #56~57 for other proofs

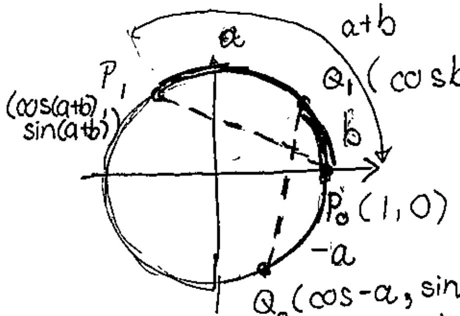




Θ = 3

unit circle

\# radian for Θ = #along arc



$Q_0(\cos -a, \sin -a) = Q_0(\cos a, -\sin a)$

arc a+b for both chords

$d(P_0, P_1) = d(Q_0, Q_1)$

$\sqrt{[\cos(a+b)-1]^2 + \sin^2(a+b)} = \sqrt{(\cos a - \cos b)^2 + (-\sin a - \sin b)^2}$

$\underline{\cos^2(a+b)} - 2\cos(a+b) + 1 + \underline{\sin^2(a+b)} = \underline{\cos^2 a} - 2\cos a\cos b + \underline{\cos^2 b}$
$+ \underline{\sin^2 a} + 2\sin a\sin b + \underline{\sin^2 b}$

$\cancel{2} - \cancel{2}\cos(a+b) = \cancel{2} - \cancel{2}\cos a\cos b + \cancel{2}\sin a\sin b$

$\cos(a+b) = \cos a\cos b - \sin a\sin b$

$\cos(a-b) = \cos(a+-b) = \cos a\cos(-b) - \sin a\sin(-b)$
$= \cos a\cos b + \sin a\sin b$

$\boxed{ex1}$ a) $\cos 75°$ $\quad 75 = 45 + 30°$

$= \cos(45° + 30°) = \cos 45°\cos 30° - \sin 45°\sin 30°$

$= \frac{\sqrt2}{2}\frac{\sqrt3}{2} - \frac{\sqrt2}{2}\frac{1}{2} = \frac{\sqrt6 - \sqrt2}{4}$

b) $\cos\frac{\pi}{12}$ $\quad \frac{\pi}{6} \div 2 = 30° \div 2 = 15°$
$= 45° - 30° = 60° - 45°$

$\cos(\frac{\pi}{4} - \frac{\pi}{6}) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6}$

$= \frac{\sqrt2}{2}\frac{\sqrt3}{2} + \frac{\sqrt2}{2}\frac{1}{2} = \frac{\sqrt6 + \sqrt2}{4}$

OR $\cos(\frac{\pi}{3} - \frac{\pi}{4}) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}$
$= \frac{1}{2}\frac{\sqrt2}{2} + \frac{\sqrt3}{2}\frac{\sqrt2}{2} = \frac{\sqrt2 + \sqrt6}{4}$

$\boxed{ex2}$ $\sin 20°\cos 40° + \cos 20°\sin 40°$
$= \sin(20° + 40°) = \sin 60° = \frac{\sqrt3}{2}$

$\boxed{ex3}$ $\cos(\frac{\pi}{2} - u) \overset{?}{=} \sin u$

$\cancel{\cos\frac{\pi}{2}}\cos u + \sin\frac{\pi}{2}\sin u = \sin u$

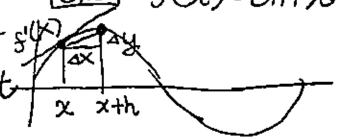
0

$\boxed{ex4}$ verify $\dfrac{1+\tan x}{1-\tan x} = \tan(\frac{\pi}{4} + x)$

(use a+b → one angle)

RHS $= \dfrac{\tan\frac{\pi}{4} + \tan x}{1 - \tan\frac{\pi}{4}\tan x} = \dfrac{1+\tan x}{1-\tan x} = $ LHS

$\boxed{ex5}$ $f(x) = \sin x$ was bonus $\quad$ new: ex6 } inverses
★ ex7 }



★Bonus on next Quiz

Difference Quotient

$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h} = \frac{\sin(x+h) - \sin x}{h} = \frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$

secant Line slope

$= \sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)$

graph: 0

sync function

$\frac{dy}{dx} = f'(x) = \sin x \cdot \lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) + \cos x \lim_{h\to 0}\frac{\sin h}{h} = \cos x$

Expressions of Form $A\sin x + B\cos x$

ⓐ $\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = \sin(60° + x)$

write as one sinusoid

$\frac{1}{2} \downarrow \cos\theta$ $\qquad \frac{\sqrt{3}}{2} \downarrow \sin\theta$

$\theta = 60°$

Ⓠ $A\sin x + B\cos x = \sin(\phi + x)$ Phi

$\sqrt{A^2+B^2}$ $\qquad \cos\phi \qquad \sqrt{A^2+B^2} \qquad \sin\phi \qquad \sqrt{A^2+B^2}$

(1#)

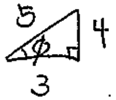
$\Downarrow$

THM

$$\boxed{\begin{array}{l} A\sin x + B\cos x = k\sin(\phi + x) \\ k = \sqrt{A^2+B^2} \qquad \phi \text{ not arcsin's restriction.} \\ \cos\phi = \frac{A}{\sqrt{A^2+B^2}} \qquad \sin\phi = \frac{B}{\sqrt{A^2+B^2}} \end{array}}$$

$\boxed{\text{ex6}}$ (ex8 new) $3\sin x + 4\cos x = k\sin(x+\phi) = 5\sin(x+53.1°)$

3·4·5 $\qquad k = 5$



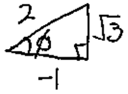
$\phi = \sin^{-1}\left(\frac{4}{5}\right) = 53.1°$

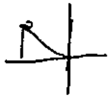
$(\boxed{\text{ex7}})$ ex9 $f(x) = -\sin 2x + \sqrt{3}\cos 2x$

$= k\sin(2x+\phi) = \boxed{2\sin\left(2x + \frac{2\pi}{3}\right)}$

$2\sin 2\left(x + \frac{\pi}{3}\right)$

$k = 2$

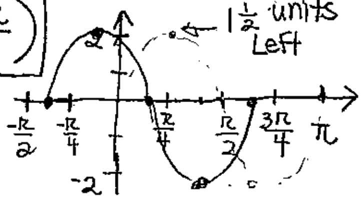




$\alpha = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = 60°$

$\phi = 120°$

*sketch



$T = \pi$

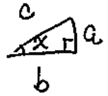
7.3 Double-Angle, Half-Angle, Product-Sum

$$\boxed{\begin{array}{l} \sin 2x = 2\sin x\cos x \\ \cos 2x = \cos^2 x - \sin^2 x \\ \qquad = 2\cos^2 x - 1 \\ \qquad = 1 - 2\sin^2 x \\ \tan 2x = \frac{2\tan x}{1-\tan^2 x} \end{array}}$$

By addition formula

$\because \sin^2 x + \cos^2 x = 1$

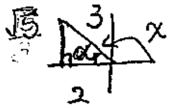
for



$\sin 2x = ?$

$\boxed{\text{ex1}}$ $\cos x = -\frac{2}{3}$ in QII

$\begin{cases} \cos 2x = ?\ 2\cos^2 x - 1 = 2\left(\frac{4}{9}\right) - 1 = \frac{8-9}{9} = -\frac{1}{9} \\ \sin 2x = ?\ 2\sin x\cos x = 2\left(+\frac{\sqrt{5}}{3}\right)\left(\frac{-2}{3}\right) = \frac{-4\sqrt{5}}{9} \end{cases}$



Recall why $\cos x = -\frac{2}{3}$ $\qquad$ α is reference angle

$\sin x = +\frac{\sqrt{5}}{3}$ $\qquad$ same magnitude. Check ± by Quadrant

$\boxed{\text{ex2}}$ $\cos(3x)$ in terms of $\cos \underline{\underline{x}}$

$\cos 3x = \cos(2x + x)$

$= \cos(2x)\cos x - \sin(2x)\sin x$

$= (2\cos^2 x - 1)\cos x - 2\sin^2 x\cos x$

$\qquad (1 - \cos^2 x)$

$= 4\cos^3 x - 3\cos x$

half-angle ex 5, 6 , sum-Product 8, 9

$\boxed{\text{ex3}}$ Prove $\dfrac{\sin \overset{(2x+x)}{3x}}{\sin x \cos x} = 4\cos x - \sec x$

LHS = (fraction&3x) $\dfrac{\sin 2x \cos x + \cos 2x \sin x}{\sin x \cos x}$

$= \dfrac{2\cancel{\sin x}\cos^{\cancel{2}} x + (\cos^{\cancel{2}} x - \sin^2 x)\cancel{\sin x}}{\cancel{\sin x}\ \cancel{\cos x}}$ Cancel sine: all terms

$= 2\cos x + \cos x - \dfrac{\sin^2 x}{\cos x}$

$= 3\cos x - \dfrac{(1-\cos^2 x)}{\cos x}$

$= 3\cos x + \cos x - \sec x = 4\cos x - \sec x$

faster: use $\cos 2x = 2\cos^2 x - 1$ since $\overset{\text{RHS}}{\text{all}}$ in cosines

$\dfrac{2\cancel{\sin x}\cos^2 x + (2\cos^2 x - 1)\cancel{\sin x}}{\cancel{\sin x}\cos x} = 2\cos x + 2\cos x - \sec x$

$= \text{RHS}$

Formula to <u>Lower Powers</u> from Double-Angle ID

*Quiz where from IDs BONUS

Dx → $\sin^{\cancel{2}} x = \cos x$

$\int \cos x\,dx = \sin x$

$\int \cos^2 x\,dx = ??$

$$\boxed{\begin{aligned} \sin^2 x &= \frac{1-\cos 2x}{2} \\ \cos^2 x &= \frac{1+\cos 2x}{2} \\ \tan^2 x &= \frac{1-\cos 2x}{1+\cos 2x} \end{aligned}}$$

∵ $\frac{\sin}{\cos}$

$\boxed{\text{ex4}}$ Lower to first power of cosine

$\sin^2 x \cos^2 x$ Lowered 1st Time

$$= \frac{1-\cos 2x}{2} \; \frac{1+\cos 2x}{2}$$

$$= \frac{1-\cos^2 2x}{4} = \frac{1}{4} - \frac{1+\cos 4x}{4\cdot 2}$$

$\frac{2}{2\cdot 4} - \frac{1}{4\cdot 2} = \frac{1}{8}$

$$= \frac{1}{8}(1-\cos 4x)$$

OR:

$$(\sin x \cos x)^2 = \left(\frac{1}{2}\sin 2x\right)^2 = \frac{1}{4}\sin^2 2x$$

$$= \frac{1}{4}\;\frac{1-\cos 4x}{2}$$

Product to Sum Formulas ← from ± IDs

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

$$\sin a \cos b = \frac{\sin(a+b) + \sin(a-b)}{2}$$

cancels

$$\cos a \sin b = \frac{\sin(a+b) - \sin(a-b)}{2}$$

cancels +

$$\cos a \cos b = \frac{\cos(a+b) + \cos(a-\textcircled{b})}{2}$$

cancels

$$\sin a \sin b = \frac{\cos(a\overset{+}{-}\textcircled{b}) - \cos(a+\textcircled{b})}{2}$$

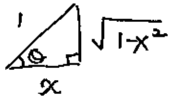
cancel

($\boxed{\text{ex7}}$) ex9 $\sin 3x \sin 5x$ as a sum

$$= \frac{\cos(a-b) - \cos(a+b)}{2}$$

$$= \frac{1}{2}\left[\cos(-2x) - \cos(8x)\right] = \frac{1}{2}(\cos 2x - \cos 8x)$$

$\boxed{\text{ex7}}$ new $\sin(2\underbrace{\cos^{-1} x}_{\theta})$ as algebraic expr in x only $-1 \le x \le 1$

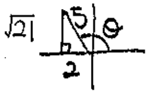


$2\sin\theta\cos\theta = 2x\sqrt{1-x^2}$

$\boxed{\text{ex8}}$ $\sin 2\theta = ?$

$\cos\theta = -\frac{2}{5}$

θ in QII



$2\sin\theta\cos\theta$

$2\,\frac{\sqrt{21}}{5}\left(-\frac{2}{5}\right) = \boxed{-\frac{4\sqrt{21}}{25}}$

Sum-to-Product

~~$\sin a \cos b \rightarrow \sin \quad + \sin$~~

~~$\cos a \sin b \leftarrow \sin^{(A+B)} - \sin^{(A-B)}$~~

~~$\cos a \cos b \leftarrow \cos^{(A+B)} + \cos(A-B)$~~

~~$\sin a \sin b \leftarrow -(\cos \quad - \cos \quad)$~~

2 terms added of $\sin(A \pm B)$

$x = \underset{A}{\frac{\text{sum}}{2}} + \underset{B}{\frac{\text{diff}}{2}}$, $y = \underset{A}{\frac{\text{sum}}{2}} - \underset{B}{\frac{\text{diff}}{2}}$

$\sin(A+B) + \sin(A-B)$ 1st term

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

2nd

② $$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$\cos(A+B) \rightarrow$ 1st term ②

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

→ 2nd term

④ $$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

Proof

② $$\cos\overset{A}{\left(\frac{x+y}{2}\right)}\sin\overset{B}{\left(\frac{x-y}{2}\right)}$$

$$= \frac{1}{2}\left[\sin\overset{A+B}{\left(+\frac{y}{2}\cdot 2\right)} + \sin\overset{A+B}{(x)}\right]$$

$$= \frac{\sin y + \sin x}{2}$$

④ $$\sin\overset{A}{\left(\frac{x+y}{2}\right)}\sin\overset{B}{\left(\frac{x-y}{2}\right)}$$

$$= \frac{\cos(A-B) - \cos(A+B)}{2}$$

$$= \frac{1}{2}[\cos y - \cos x]$$

($\boxed{ex8}$) 10

1st term $\sin(A+B)$ →

$$\sin 7x + \sin 3x = 2\sin\left(\frac{10x}{2}\right)\cos\left(\frac{4x}{2}\right)$$

($\boxed{ex9}$) 11 Prove $\frac{\sin 3x - \sin x}{\cos 3x + \cos x} = \tan x$

→ $\sin(\quad)$ 2nd term

→ 1st term $\cos(\ \cdot\ )$

$$\text{LHS} = \frac{\cancel{2\cos\left(\frac{4x}{2}\right)}\sin\left(\frac{2x}{x}\right)}{\cancel{2\cos\left(\frac{4x}{2}\right)}\cos\left(\frac{2x}{2}\right)} = \tan x = \text{RHS}$$

OR $\sin(2x+x) - \sin x \ldots$

Half-Angle Formulas

$\cos 2x = \cos^2 x - \sin^2 x$
$= 2\cos^2 x - 1$
$= 1 - 2\sin^2 x$

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}$$

by Quadrant

$\leftarrow \sin^2 x = \frac{1-\cos 2x}{2}$

same reason

same thing!

$$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$$

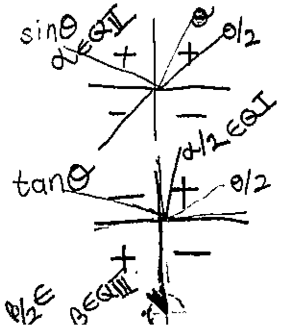
$\tan^2 x = \frac{1-\cos 2x}{1+\cos 2x}$

$$\tan\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{1+\cos u}\cdot\left(\frac{1+\cos u}{1+\cos u}\right)}$$

$$= \pm\sqrt{\frac{\sin^2 u}{(1+\cos u)^2}} = \pm\left|\frac{\sin u}{1+\cos u}\right|$$

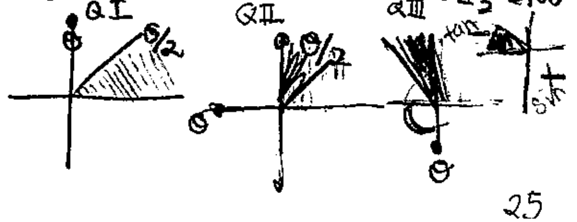
always > 0

$$= \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\cos u}$$



★ $\tan\frac{u}{2}$ & $\sin u$
always same sign!

QIV
270° ≤ θ ≤ 360°
$135° \le \frac{\theta}{2} \le 180°$

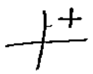


OR (BOOK)

$$\tan\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{1+\cos u}\cdot\left(\frac{1-\cos u}{1-\cos u}\right)}$$

85~46

$\boxed{ex5}$ $$\sin 22.5^\circ = \sin\left(\frac{45^\circ}{2}\right) = \pm\sqrt{\frac{1-\cos 45^\circ}{2}}$$



$$= +\sqrt{\frac{1-\left(\frac{\sqrt{2}}{2}\right)}{2}} = \sqrt{\frac{2-\sqrt{2}}{4}} > 0$$

$$= \frac{1}{2}\sqrt{2-\sqrt{2}}$$

$\boxed{ex6}$ $\tan\left(\frac{u}{2}\right)$ $\sin u = \frac{2}{5}$ $u \in Q\text{II}$

$$\cos 2u = 1-2\sin^2 u = 1-2\left(\frac{4}{25}\right) = \frac{17}{25}$$

$$\tan 2u = \frac{2\tan u}{1-\tan^2 u} = \frac{2\left(\frac{-2}{\sqrt{21}}\right)}{1-\frac{4}{21}}$$

tan u < 0

$$= \frac{4 \times \frac{\cancel{21}}{\sqrt{21}}\cdot\frac{\sqrt{21}}{\sqrt{21}}}{17} = \frac{-4\sqrt{21}}{17}$$

OR

$\cos u = \ominus\sqrt{1-\sin^2 u}$

QII

$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{1-\left(-\sqrt{21}/5\right)}{2/5} = \frac{5+\sqrt{21}}{2}$$

$$\csc\frac{u}{2} = \frac{1}{\sin\frac{u}{2}} = \pm\sqrt{\frac{2}{1-\cos u}}$$

QI

$$= \textcircled{\pm}\sqrt{\frac{2}{1+\frac{\sqrt{21}}{5}}} = \sqrt{\frac{10}{5+\sqrt{21}}}$$

7.4 Eqns

7.5 Trig Eqns . What's the difference/relationship?

Equation   formula   expression   function   identity

solution

curve

region

$\sec^{-1}x \overset{?}{=} \frac{1}{\cos^{-1}x}$ eqn . Solve?

Expression x^2+2

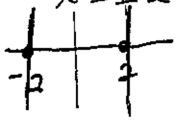
Identity / formula $\sin^2 x+\cos^2 x=1$ always true

$\tan x=\frac{\sin x}{\cos x}$ $\cos^2 x-\sin^2 x=1$

Equation

$x^2+2=6$

$x=\pm 2$



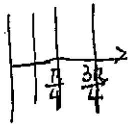
Solution.

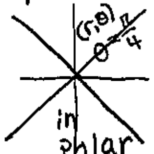
function?

$\sin^2 x-\cos^2 x=0$

$|\sin x|=|\cos x|$

$x=\frac{\pi}{4}+k\frac{\pi}{2}$





×

$x^2+y^2\leq 4$

$x^2+y^2\leq 4$ distance to O

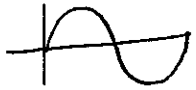
soln. curve

$r=2$, θ anything

implicitly

$y=\pm\sqrt{1-x^2}$

$y=\sin x$

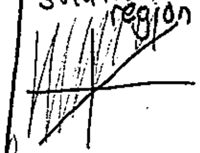


✓

$y\geq x$

inequality

solution set



$\sin^2 x+\cos^2 x=1$

ID / formula

all x

$\tan x=\frac{\sin x}{\cos x}$

all x (defined)

$\sec^{-1}x=\cos^{-1}(\frac{1}{x}$

ID all x

$$\frac{dy}{dx}=ky$$

$y=Ce^{kx}$ solution

family of functions

(ex1) ex5 a $2\sin x - 1 = 0$

6th ed 7.4 ~ 7.5 skip a few ex

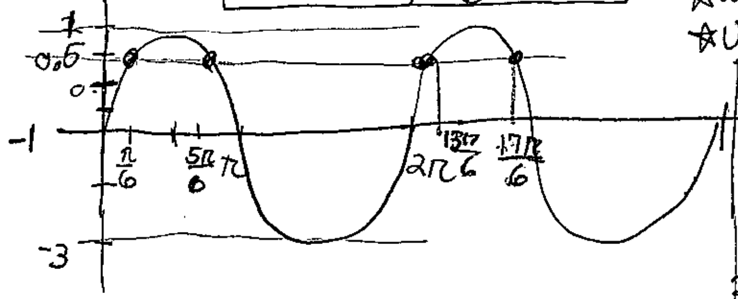
$\sin x = \frac{1}{2}$

$x = 30^\circ + k360^\circ, 150^\circ + k360^\circ$

$\boxed{\frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi}$ ← HW

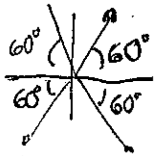
★write like this

★Use radians



(ex2) ex 5b $\tan^2 x - 3 = 0$

$\tan x = \pm\sqrt{3}$



$x = \frac{\pi}{3} + k\pi, -\frac{\pi}{3} + k\pi$

(ex3) $f(x) = \sin x$

$g(x) = \cos x$

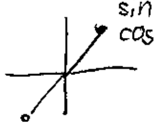
at what x intersect?

$\sin x = \cos x$

$\tan x = 1$

$x = 45^\circ + k360^\circ$

~~$= \frac{\pi}{4} + 2\pi k$ OR $\frac{5\pi}{4} + 2\pi k$~~



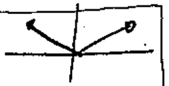
simpler

$\boxed{\frac{\pi}{4} + k\pi}$

$45^\circ + k180^\circ$

7.4 ① 6th ex2 Solve $\cos\theta = -\frac{\sqrt{2}}{2}$ in $[0, 2\pi)$ $k \in \mathbb{Z}$

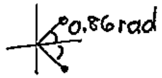
$\theta = \underbrace{\frac{3\pi}{4}, \frac{5\pi}{4}}_{k=0} + 2k\pi$



$\underbrace{\frac{11\pi}{4}, \frac{13\pi}{4}}_{k=1}, \underbrace{-\frac{5\pi}{4}, -\frac{3\pi}{4}}_{k=-1}$

7.4 ex3 Solve

$\cos\theta = 0.65$



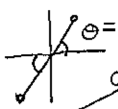
$2\pi - 0.86 = 5.42$

$0.65 + 2k\pi$

$5.42 + 2k\pi$

7.4 see p30

ex4 Solve $\tan\theta = 2$



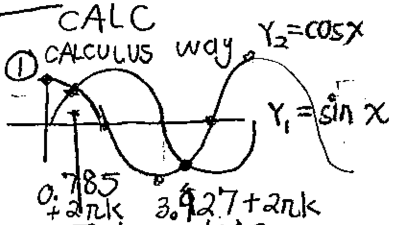
$\theta = \boxed{1.12 + k\pi}$

OR

~~$\pi + 1.12$ 4.26~~

CALC

① CALCULUS way



② Solve(sin x = cos x, x)

slow

x = 0.785398 (4.@n3-3)

??

(ex4) By factoring

ex6

$2\cos^2 x - 7\cos x + 3 = 0$

$2u^2 - 7u + 3 = 0$

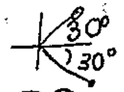
1 -3

2 -1

$(2u - 1)(u - 3) = 0$

$|\cos x| \le 1$

$u = \cos x = \frac{1}{2}$ OR ~~$\cos x = 3$~~



$\boxed{x = \frac{\pi}{3} + 2k\pi \text{ OR } \frac{5\pi}{3} + 2k\pi}$

Using an I.D

(ex5) $1 + \sin x = 2\cos^2 x$

Get as sin

$1 + \sin x = 2(1 - \sin^2 x)$

$2\sin^2 x + \sin x - 1 = 0$

2 -1

1 1

$(2u - 1)(u + 1) = 0$

$\sin x = \frac{1}{2}$ or $\sin x = -1$

$x = \frac{\pi}{6} + 2k\pi,$ (30°)

$\frac{5\pi}{6} + 2k\pi$

$x = \frac{3\pi}{2} + 2k\pi$

28

$\boxed{\text{ex6}}$ Using a Trig ID & Factoring

Like 7.4 $\boxed{\text{ex 7}}$ 6th

$$\sin \underline{\underline{2x}} - \cos x = 0$$

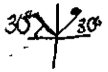
$$\rightarrow 2\sin x \cos x - \cos x = 0$$

$$\cos x\,(2\sin x - 1) = 0$$

$$\downarrow \qquad\qquad \downarrow$$

$$0 \quad \text{or} \quad 0$$

$$\boxed{\frac{\pi}{2} + k\pi}$$



$$\boxed{\begin{matrix}\frac{\pi}{6} + 2k\pi \\ \frac{5\pi}{6} + 2k\pi\end{matrix}}$$

$k \in \mathbb{Z}$

Techniques
① u-sub
② change to sin
③ $AB = 0$
$\swarrow$ $\searrow$
0 or 0
$\boxed{\text{ex7}}$ sqr sides → check

$\boxed{\text{ex7}}$ squaring & ID

$$\cos x + 1 = \sin x \qquad [0, 2\pi)$$

(can't 2x or $\cos^2 x = 1 - \sin^2 x$) yet

$$(\cos x + 1)^2 = \sin^2 x$$

$$\cos^2 x + 2\cos x + 1 = \sin^2 x$$

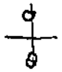
$$\underbrace{\cos^2 x - \sin^2 x}_{\cancel{\cos 2x} \quad \cancel{+2}} + 2\cos x + 1 = 0$$

All $\underline{\underline{\cos}}$

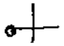
$$\cos^2 x + 2\cos x + \cos^2 x = 0$$

$$2\cos x\,(1 + \cos x) = 0$$

$$\downarrow \qquad\qquad \downarrow$$



$$\frac{\pi}{2} + k\pi$$



$$\pi + 2k\pi$$

$$x = \boxed{\frac{\pi}{2}, \cancel{\frac{3\pi}{2}}, \pi}$$

$0+1=1$ ✓ $\quad \cancel{0+1=-1} \quad$ $0=0$ ✓

$a^2 = b^2$

$\Downarrow$!!

$a = b$

Check Answer

- $\frac{f(x)}{h(x)} = \frac{g(x)}{h(x)}$ ($\times$ $\div$)
  Loses soln $h(x) = 0$
- $f^2(x) = g^2(x)$
  adds soln
  $f = \pm g$

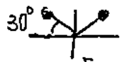
★ Before trying HW review each example's purpose (Q&A in class at each lesson)

When you have multiple Angles

$\boxed{\text{ex8}}$ $2\sin 3x - 1 = 0$

a) All solns:

$$\sin 3x = \frac{1}{2}$$



$$3x = \begin{cases}\frac{\pi}{6} + 2k\pi \\ \frac{5\pi}{6} + 2k\pi\end{cases}$$

$$\left[\begin{matrix} \frac{\pi}{6} & \overset{2\times 6}{\frac{13\pi}{6}}, & \overset{4\times 6}{\frac{25\pi}{6}} & \overset{6\times 6}{\frac{37\pi}{6}} \\ \frac{5\pi}{6}, & \overset{3\times 6}{\frac{17\pi}{6}} & \overset{5\times 6}{\frac{29\pi}{6}} & \overset{7\times 6}{\frac{41\pi}{6}} \end{matrix}\right]$$

$$x = \boxed{\begin{cases}\frac{\pi}{18} + \frac{2k\pi}{3} \\ \frac{5\pi}{18} + \frac{2k\pi}{3}\end{cases}}$$

b) Solns in $[0, 2\pi)$

b) Solns in $[0, 2\pi)$

$$\boxed{\begin{matrix} \overset{k=0}{\frac{\pi}{18}}, & \overset{k=1}{\frac{13\pi}{18}}, & \overset{k=2}{\frac{25\pi}{18}}, \\ \frac{5\pi}{18}, & \frac{17\pi}{18}, & \frac{29\pi}{18} \end{matrix}} \cancel{\frac{37\pi}{18}}$$

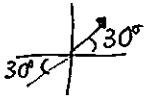
Faster:

$\frac{2k\pi}{3}$: $k = 3 \Rightarrow 2\pi$
one revolution

$$2\pi + \boxed{\frac{\pi}{18}} \notin [0, 2\pi)$$

$\boxed{\text{ex9}}$ $\sqrt{3}\tan\frac{x}{2} - 1 = 0$

$$\tan\frac{x}{2} = \frac{1}{\sqrt{3}}$$



$$\frac{x}{2} = \frac{\pi}{6} + k\pi$$

a) all solns: $\boxed{x = \frac{\pi}{3} + 2k\pi}$

b) in $[0, 4\pi)$

$k = 2 \Rightarrow 4\pi$

$$\boxed{\underset{k=0}{\frac{\pi}{3}}, \underset{k=1}{\frac{7\pi}{3}}}$$

$\cancel{k=3 \quad \frac{\pi}{3} + 6\pi}$

$k=2$ $\quad 4\pi + \frac{\pi}{3} \notin [0, 4\pi)$

(ex10) Using Inverse Trig

Like 7.4 ex4

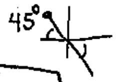
HW - Special Angle, finish by memorized chart

- not special angle: Leave as inverse fn.

$\tan^2 x - \tan x - 2 = 0.$

$(u-2)(u+1)=0$

$\tan x = 2 \qquad \tan x = -1$



$$\left(\boxed{x=\tan^{-1}2 + k\pi \qquad x=\frac{3\pi}{4}+k\pi}\right)$$

OR to use the restricted domain

$$\boxed{x=\tan^{-1}2 + k\pi \quad , \quad -\frac{\pi}{4}+k\pi}$$

tan is π-periodic

Like 7.4 ex3

(ex11) Using Inverse Trig Fn.

a) Solve $3\sin\theta - 2 = 0$

$\sin\theta = \frac{2}{3}$

$$\boxed{\theta=\begin{cases}\sin^{-1}\frac{2}{3} + 2k\pi \\ \pi - \sin^{-1}\frac{2}{3} + 2k\pi\end{cases}}$$

b) Calculator $[0, 2\pi)$ $\quad \theta = \begin{cases} 0.72973 \\ 2.41186 \end{cases}$

7.5

(66)

$\tan\frac{x}{2} - \sin x = 0$ or

$\frac{1-\cos x}{\sin x} - \frac{\sin^2 x}{\sin x} = 0$

$1-\cos x = \underbrace{\sin^2 x}_{1-\cos^2 x}$

$\cos x = \cos^2 x$

$\cos^{\cancel{2}} x\,(1-\cos x)=0.$

$\downarrow \qquad \downarrow$

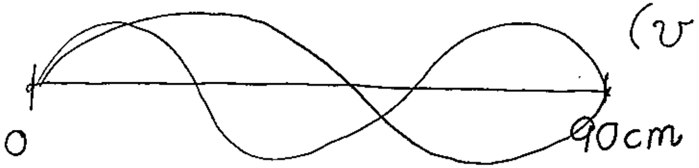
$\cos x = 0 \quad$ OR $\quad \cos x = 1$

$$\boxed{x=\frac{\pi}{2}+k\pi \qquad x=2k\pi}$$

$$\boxed{x=0, \frac{\pi}{2}, \frac{3\pi}{2} \text{ in } [0, 2\pi)}$$

Standing Waves Activity

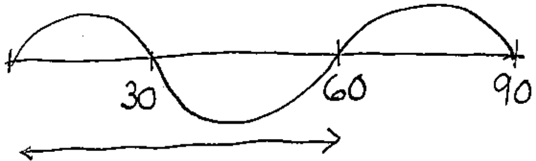
- Standing Wave Generator Demo



$(v = 3720\,cm/s)$

Recall $\underset{\substack{\uparrow\\ \text{by medium}}}{v} = \lambda \underset{\text{by oscillator}}{f}$

① Estimate the speed of the wave



$\lambda = 60cm$

$f = 62Hz$

$v = 3720\,cm/s$

② incident / reflected waves

$$y_{\substack{1\\2}}(x,t) = A\sin\frac{2\pi}{60}(x \mp 3720t)$$

③ Standing wave (superposition)

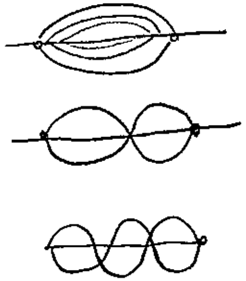
$$y(x,t) = 2A\sin\frac{2\pi}{60}x\cos\left(\frac{\pi}{30}\times 3720\,t\right)$$

④ Use eqn in ③ to verify location of nodes

$$\frac{\pi}{30}x = n\pi \Rightarrow x = 30n$$
$$= 0, 30, 60, 90, \dots$$

⑤ Say you were given $v = 3720\,cm/s$ & setup is $L = 90\,cm$ long

Show how to calculate f_n for n loops in terms of v, L, n



since nodes must be at ends

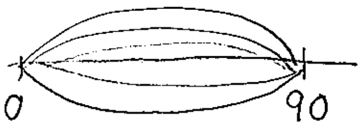
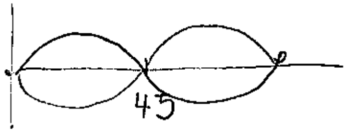
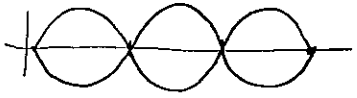
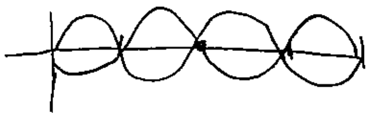
$\frac{n\lambda}{2} = L$ & $v = \lambda f$

$\Rightarrow \lambda = \frac{2L}{n}$

$\Rightarrow f = \frac{v}{\lambda} = \boxed{\frac{n v}{2L}}$

⑥

$(n\frac{\lambda}{2} = L)$

n	Sketch	f predicted	f experiment	λ_{exp}	v
n=1		$f_1 = \frac{62}{3} = 20.\bar{6}$ Hz	very close!	180cm	3720cm
2		$f_2 = 2f_1 \approx 41.\bar{3}$		45	"
3		$f_3 = 62$ Hz		60	"
4		$f_4 = 83$ Hz		45	"
5		$f_5 = 103$ Hz		36	"

Traveling Waves

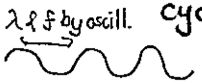
Pg 575 Ch 7
(6th ed P533)

- Simple harmonic motion (recall Ch 6)

$$y(t) = A\cos \underset{\uparrow}{\omega}(t - t_0) + y_0$$

time for 1 cycle is T

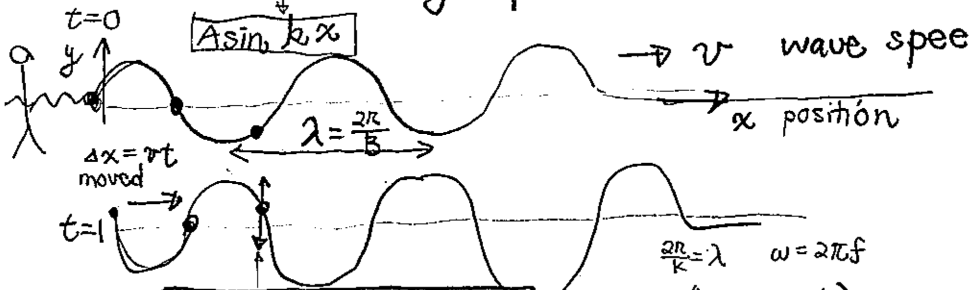
$$\frac{2\pi}{T} = 2\pi f$$

- Waves  → v medium $= \lambda f$

In Lab:

- Wave on string
  - energy moves right
  - pass by makes part of rope go up/down in SHM

$$\boxed{A\sin \overset{\frac{2\pi}{\lambda}}{k}x}$$



$\frac{2\pi}{k} = \lambda$ $\omega = 2\pi f$

$$\boxed{y(x,t) = A\sin k(x - vt)} = A\sin(kx - \omega t)$$

Fix x.. point oscillates up & down

$$y(x,t) = A \underset{\cos}{\sin} \frac{2\pi}{\lambda}(x - \underset{\uparrow}{v}t)$$

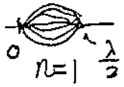
$$v = \lambda f = \frac{\lambda}{T}$$

---

nodes. At all times, places at rest

$\sin kx = 0$

 $kx = n\pi$



$$x = \frac{n\pi}{k} : 0, \frac{\pi}{k}, \frac{2\pi}{k}, \frac{3\pi}{k}, \frac{4\pi}{k}, \ldots$$

$$= \frac{n\pi}{2\pi}\cdot\lambda = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, \ldots$$

→

$\boxed{\text{ex1}}$ $y(x,t) = 3\sin\left(2x - \frac{\pi}{2}t\right)$

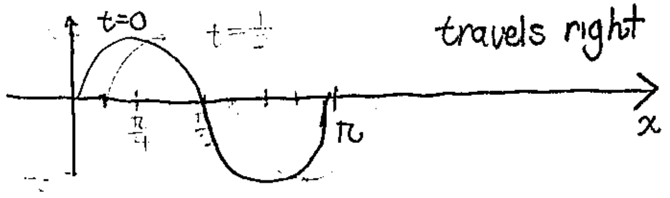
a) $x = \frac{\pi}{6}$

$$\Rightarrow y(t) = 3\sin\left(\frac{\pi}{3} - \frac{\pi}{2}t\right)$$

b) Shape (sketch) of wave at $t = 0, 0.5, 1, 1.5, 2$ → see pg 576

$y(x) = 3\sin \underset{\nwarrow}{2}x$ $y(x, \ldots) = \ldots \sin 2\left(x - \frac{\pi}{2}\right)$

$T = \pi$



travels right

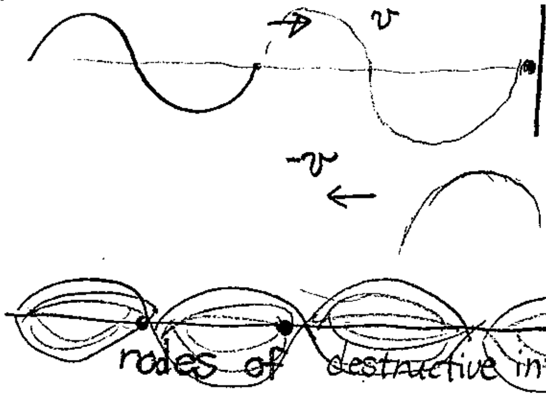
c) wave speed $y = A\sin k(x - vt)$

$$3\sin 2\left(x - \frac{\pi}{4}t\right)$$

$$v = \boxed{\frac{\pi}{4}}$$

Standing Waves

– show pdf CISE 11.1



wall

$y_1(x,t) = A\sin k(x - vt)$

$y_2(x,t) = A\sin k(x + vt)$

(freq must be so that $\frac{\lambda}{2}k = L$)

$$y(x,t) = y_1 + y_2$$

$$= A\sin \overset{A}{k(x - vt)} + A\sin \overset{B}{k(x + vt)}$$

$$= 2A\sin\left(\underset{\frac{A+B}{2}}{kx}\right)\cos\left(\underset{\frac{A-B}{2}}{kvt}\right)$$ sum-to-product formula

OR just $A\sin kx\cos(\omega t) - A\cos kx\sin\omega t) + A\sin k\omega\cos\omega t + A\cos \cancel{kx}\sin\omega t$ 21

$\boxed{\text{ex2}}$ $y_{\substack{1\\2}} = 1.5\sin\left(\frac{\pi}{5}x \mp 3t\right)$ feet

a) standing wave

Total

$y(x,t) = 3\sin\left(\frac{\pi}{5}x\right)\cos 3t$

nodes $\frac{\cancel{\pi}}{5}x = n\cancel{\pi}$

$x = 5n$

$= 0,\ 5ft,\ 10ft,\ 15ft, \ldots$

b) Sketch... see pg 577

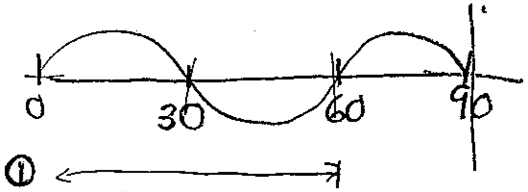
$v = \sqrt{\frac{T}{\mu}}$ ← Newtons ← kg/m

For [illegible]

1 loop $\boxed{f_1 = \frac{1}{2L}\sqrt{\frac{T}{\mu}}}$

2 Loops: $2f_1, \ldots$

Demo Standing Wave Generator.



$\cancel{\frac{2\pi}{\alpha}} = \frac{90}{3}$

Amplify [illegible] 4~8.

① $\lambda = 60$ cm

$f = 62$ HZ

$v = 3720$ cm/s

① Sketch/Label in cm

② $v=$ $\lambda=$ $f=$ $\omega=$

③ incident reflected waves Superposition

④ Verify nodes

② incident reflected

$y_{\substack{1\\2}}(x,t) = A\sin\frac{2\pi}{60}(x \mp 3720t)$

Superposition

③ $y(x,t) = 2A\sin\overset{\alpha=}{\left(\frac{2\pi}{60}\right)}x\cos\left(\frac{\pi}{30}\times 3720t\right)$

Graphing calculator
- hard to do 2 variables

④ Nodes verify

$\frac{\pi}{30}x = n\pi$

$x = 30n$ ✓

Of course $\frac{2\pi}{\lambda}x = n\pi$

$x = n\left(\frac{\lambda}{2}\right)$

HW pg 578

#1, 3, 4, 6, 7

↑ Just sketch first two times

⑦ Vibrating String : Sound from combination of standing waves like...

$y = A\sin\alpha x$

a) $\frac{2\pi}{\alpha} = 2\pi$ $\quad \alpha = 1$

$\frac{2\pi}{\alpha} = \pi$ $\quad \alpha = 2$

$\frac{2\pi}{\alpha} = \frac{2\pi}{3}$ $\quad \alpha = 3$

$\frac{2\pi}{\alpha} = \frac{\pi}{4}$ $\quad \alpha = 4$

b) $\alpha = 5$ $\quad \alpha = 6$

c) $f = 440$ HZ

fix t

$\frac{2\pi}{\beta} = \frac{1}{440}$

$y = A\cos\beta t$

$\beta = 880\pi$

d) $y(x,t) = A\sin\alpha x\cos\beta t$ standing Wave

$= 1\sin x\cos 880\pi t$

$\boxed{\text{wikipedia}}$ - see picture

fix x

$\sin 2\pi f_1 t + \sin 2\pi f_2 t = 2\cos\left(2\pi\underbrace{\frac{f_1 - f_2}{2}}_{\text{slow}}t\right)\sin\left(2\pi\underbrace{\frac{f_1 + f_2}{2}}_{f_2 \approx f_1}t\right)$

★ BONUS : beat frequencies

Beat Frequency : Show wikipedia
Firefox for sounds

- Fix position x
- 2 sounds of similar frequency interfere to hear beats (loud/soft ...)

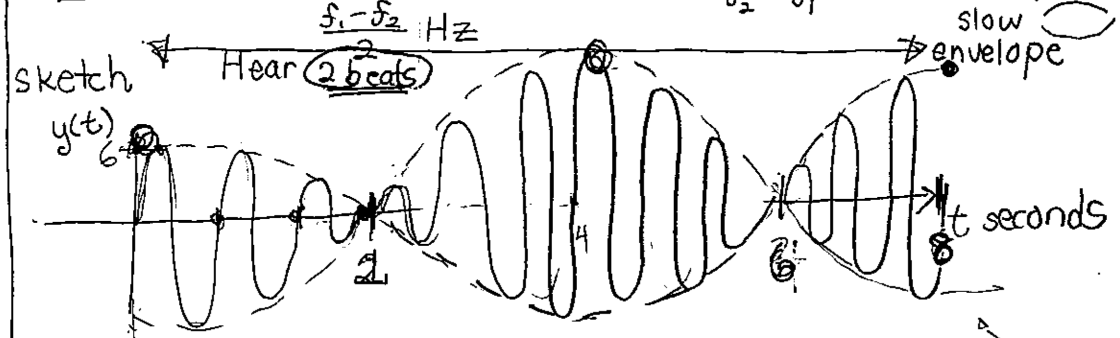
$f_1 \approx f_2$

$$\boxed{\underset{(A+B)}{\sin 2\pi f_1 t} + \underset{(A-B)}{\sin 2\pi f_2 t} = 2\sin 2\pi\underset{f_2 \approx f_1}{\left(\frac{f_1+f_2}{2}\right)} t \; \cos 2\pi \underbrace{\left(\frac{f_1-f_2}{2}\right)}_{\approx 0} t}$$

almost like standing waves but envelope now loud/soft sounds

slow envelope

sketch



$2\sin 2\pi(1.5t)\cos(2\pi 0.2t)$

$A=3$ ✓

$T = 5 \overset{\checkmark}{=} \frac{2}{f_1-f_2}$

-Beat frequency $\approx \frac{2}{8} = 0.25$ Hz ✓

$f_1 \approx f_2 = \cancel{\frac{3}{2}}$

- $\frac{f_1+f_2}{2} = 1.5$ ✓

$f_1+f_2 = 3$

$\underline{-f_1 + f_2 = 0.25}$ ✓

$2f_2 = 3.25$

$\boxed{f_2 = 1.625}$ ✓

$\boxed{f_1 = 1.375}$ ✓

$T_{sine} = \frac{2}{3}$ ✓

$\frac{f_2 - f_1}{2} = \frac{1}{8}$ ✓

Beat frequency $= \boxed{|f_1 - f_2|}$

serway & Beichner Ch. 18

(50) note C 523 Hz

$f_1 = 523$ Hz reference

piano tuner tries to get the C note but hears 2 beats/second between piano string & oscillator

a) Possible frequencies of piano?

$|f_2 - f_1| = 2$ Hz

$\boxed{f_2 = 525 \text{ Hz or } 521 \text{ Hz}}$

b) 3 beats/s $|f_2 - f_1| = $ Hz

$f_2 = \boxed{526 \text{ Hz or } 520 \text{ Hz}}$

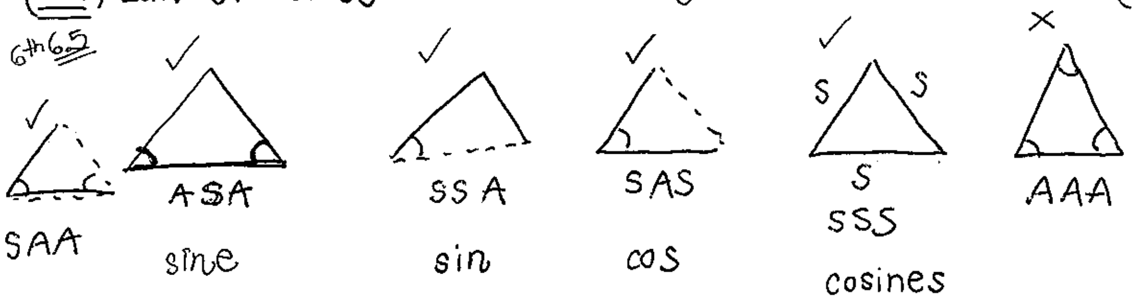
c) Is more beats better or worse?

Worse. Envelope $\frac{f_1-f_2}{2} \uparrow \Rightarrow f_1 \neq f_2$

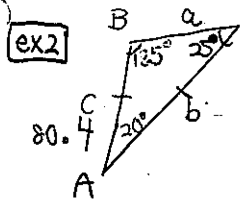
(6.4) Law of Sines

6th 6.5

which can you solve?



(skip) ex2



Solve

$$\frac{a}{\sin 20°} = \frac{80.4}{\sin 25°} = \frac{b}{\sin 135°}$$

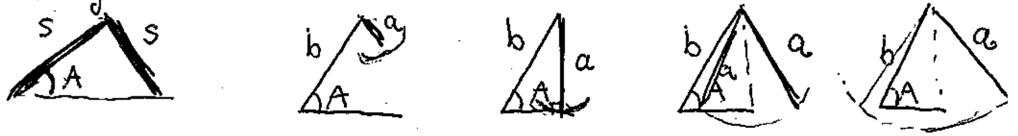
$$\boxed{a = 65.1 \qquad b = 134.5}$$

- To solve a triangle: 3 parts are given
  (1 must be a side)

(L7 P13~14)

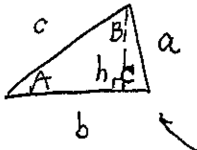
- see SAT : when to use Law of Sines or Cosines

Ambiguous Case Given SSA



$a < b\sin A$	$a = b\sin A$	$b\sin A < a < b$	$a > b$
no soln.	1 soln.	2 solns.	one soln

★ Careful for more than one answer for Law of Sines!



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Proof

$$\text{area} = \frac{1}{2}ab\sin C$$

$$\frac{1}{2}\text{base} \times \underset{\text{height}}{(a\sin C)}$$

$$= \frac{1}{2}ac\sin B$$

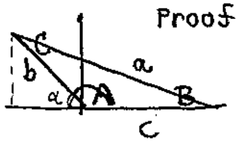
$$= \frac{1}{2}bc\sin A$$

$$\frac{\underline{ab\sin C = ac\sin B = bc\sin A}}{abc}$$

$$\frac{\sin C}{c} = \frac{\sin B}{b} = \frac{\sin A}{a}$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Proof

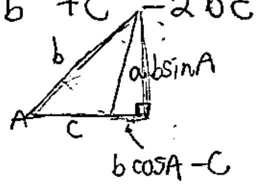


distance formula

$$a^2 = (b\sin A)^2 + (-b\underset{\uparrow \text{negative}}{\cos} A + C)^2$$

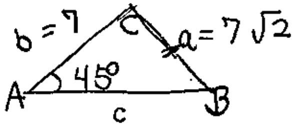
$$= \underbrace{b^2\sin^2 A + b^2\cos^2 A}_{} - 2bc\cos A + c^2$$

$$= b^2 + c^2 - 2bc\cos A$$



ex3

(S SAT L7 Note pg4 CW)



$$\frac{\sin B}{\cancel{7}} = \frac{\sin 45°}{\cancel{7}\sqrt{2}}$$

$$\sin B - \frac{1}{2}$$

$$\boxed{B = 30°} \text{ OR } 150°$$

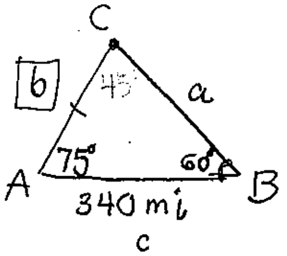
+45°

$$\boxed{C = 105°} \qquad \cancel{195°} > 180°$$

$$\frac{c =}{\sin C} \quad \frac{7\sqrt{2}}{\sin 45°}$$

$$c = 7 \times 2 \times \sin 105° \approx \boxed{13.5}$$

ex1

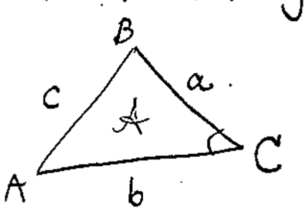


opp ∠ & side

$$\frac{b}{\sin 60°} = \frac{340}{\sin 45°}$$

$$b = 340\sqrt{2} \cdot \frac{\sqrt{3}}{2} \approx \boxed{416} \text{ miles}$$

Area of Triangle



Heron's Formula

$$\boxed{\begin{array}{l} A = \sqrt{s(s-a)(s-b)(s-c)} \\ s = \frac{1}{2}(a+b+c) \\ \text{semiperimeter} \end{array}}$$

Proof (BONUS)

$$A = \frac{1}{2} ab\sin C$$

$$A^2 = \frac{1}{4} a^2 b^2 (\overset{\sin^2 C}{1-\cos^2 C})$$

$$= \frac{1}{4} a^2 b^2 (1-\cos C)(1+\cos C)$$

$$c^2 = a^2+b^2-2ab\cos C$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

$$= \frac{1}{4} a^2 b^2 \left(1-\frac{a^2+b^2-c^2}{2ab}\right)\left(1+\frac{a^2+b^2-c^2}{2ab}\right)$$

$$= \frac{1}{4} a^2 b^2 \left(\underbrace{2ab-a^2-b^2}+c^2\right)\left(\underbrace{2ab+a^2+b^2}-c^2\right)$$

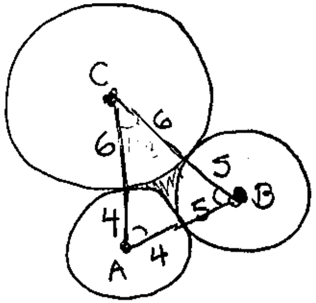
$$\frac{(-(a-b)^2+c^2)}{2ab}\frac{((a+b)^2-c^2)}{2ab}$$

$$= \frac{(c-(a-b))}{2}\frac{(c+(a-b))}{2}\frac{(a+b-c)}{2}\frac{(a+b+c)}{2}$$

$$\underbrace{\frac{b+c+a}{2}-a}_{s-a} \quad \underbrace{\frac{a+c+b}{2}-b}_{s-b} \quad \underbrace{\frac{a+b+c}{2}-c}_{s-c} \quad s$$

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

(35)
(6th 37)



$$s = \frac{1}{2}(10+11+9)$$

$$= 15$$

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{15(15-10)(15-11)(15-9)}$$

42.4264

$$A_{triangle} - S_C - S_A - S_B$$

$$S_A = \frac{1}{2}r^2\theta_{A\,RAD} = \pi r^2 \times \frac{\theta_{A\,deg}}{360^\circ}$$

- $11^2 = 10^2+9^2-2(10)(9)\cos A$

$$A = \boxed{70.53^\circ}$$

$$\frac{\sin B}{10} = \frac{\sin 70.53^\circ}{11} \Rightarrow B = \boxed{58.99^\circ}$$

$$\frac{\sin C}{9} = \frac{\sin 70.53^\circ}{11} \Rightarrow C = 50.48^\circ$$

- $A_{triangle} = \frac{1}{2}\, 10 \times 9 \sin \overset{70.53^\circ}{A} = \boxed{42.426}$

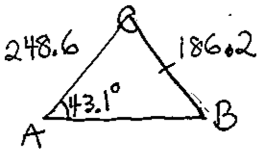
✓

$$\boxed{3.85\text{cm}^2}$$

- $A_{triangle} - S_A - S_B - S_C$

$$= 42.426 - \frac{\pi}{360^\circ}(16\times 70.53^\circ + 25\times 58.99 + 36\times 50.48)$$

ex4 Solve $\angle A = 43.1^\circ$
$a = 186.2$
$b = 248.6$

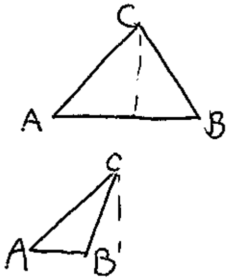


$$\frac{\sin B}{248.6} = \frac{\sin 43.1^\circ}{186.2} \qquad \frac{\sin C}{c} = \frac{\sin 43.1^\circ}{186.2}$$

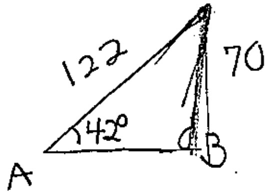
$B = \sin^{-1}(0.912) \longrightarrow C = \longrightarrow c = 257.79$
$= 65.8188^\circ \qquad 71.08^\circ$
OR $114.18^\circ \longrightarrow 22.72^\circ \qquad 105.25$

★ Always consider $\boxed{\theta \in [0, 180^\circ)}$

Sketch



ex5 Solve $\angle A = 42^\circ$
$a = 70$
$b = 122$

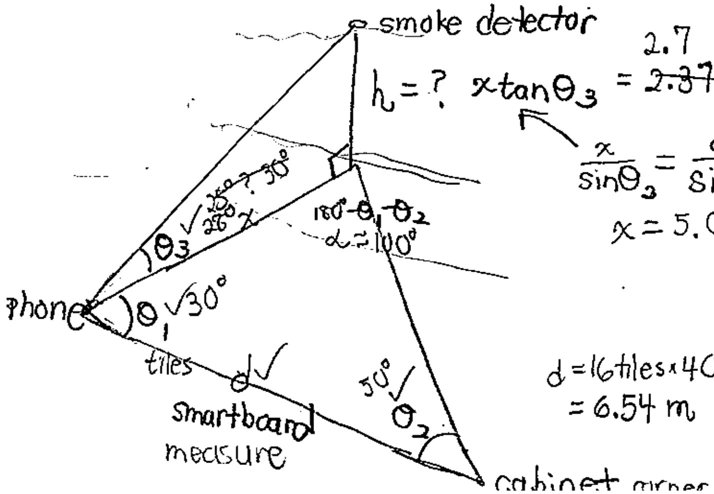


$$\frac{\sin B}{122} = \frac{\sin 42^\circ}{70}$$

$$\sin B = \frac{122}{70} \sin 42^\circ \approx 1.17$$

no soln.

CW: Find smoke detector height, baseline: along board
(Hint: #4 P 489 6th) use protractor



$h = ?\ x \tan\theta_3 = \overset{2.7}{\cancel{2.37}}\,m \quad 2.9$ (Real height 2.61 m)

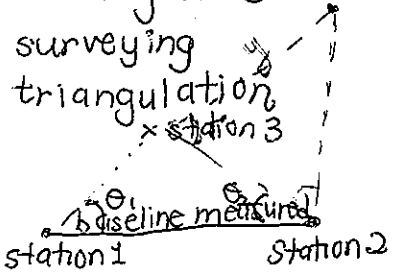
$$\frac{x}{\sin\theta_3} = \frac{d}{\sin\alpha}$$

$x = 5.09$ cm

$d = 16$ tiles × 40cm + 14cm
$= 6.54$ m

6th (P. 489) Interesting note:

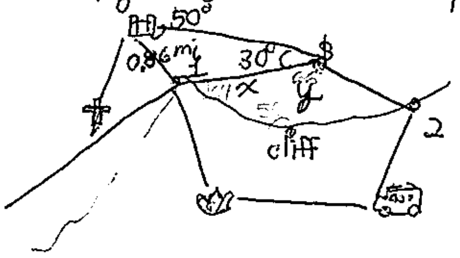
surveying
triangulation



- ★ Only measure 1st baseline
- ★ Then just angles Law of Sines

Mt.Everest

pg. 523 #1~2 Map



$$\frac{x}{\sin 50^\circ} = \frac{0.86}{\sin 30^\circ} \Rightarrow x = 1.32\,mi$$

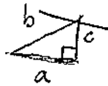
$$\frac{y}{\sin 64} = \frac{1.32}{\sin 50^\circ} \Rightarrow y = 1.55\,mi$$

6th 6.6

(6.5) Law of Cosines ← OK since cos θ ~~$\cos^{-1} y = \theta$ only QI soln.~~

see SAT examples

cos θ ~~θ ∈ [0, 180°]~~ only QI

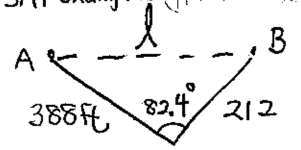


~~$b^2 = a^2 + c^2 - 2ac\cos\theta$~~

~~$|a+c| > |b| > |a-c|$~~ > 0

~~$b^2 > a^2 - 2ac + c^2$~~

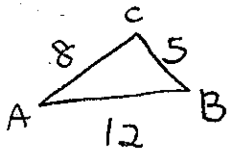
ex1



$\ell^2 = 388^2 + 212^2 - 2(388) \times 212 \cos 82.4°$

$\ell \approx 416.8$

ex2



solve

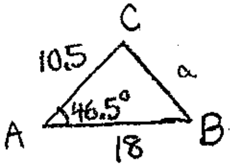
$5^2 = 8^2 + 12^2 - 2(8)(12)\cos A$

$\cos A = 0.953125 \rightarrow \boxed{\angle A = 18°}$

$\cos B = 0.875 \rightarrow \angle B = \boxed{29°}$

$\angle C = 180° - 18 - 29 = 133°$

ex3



$a^2 = 10.5^2 + 18^2 - 2(10.5)(18)\cos 46.5°$

$a = 13.2$

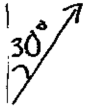
$\frac{\sin B}{10.5} = \frac{\cos 46.5°}{13.2} \rightarrow \boxed{\begin{array}{l} B = 33.2197° \\ C = 100.2803° \end{array}}$ ~~$146.8° + 46.5$ > 180~~

Roundoff error?

Navigation

N 30° E "30° to the E of due N"

N60°W S70°W

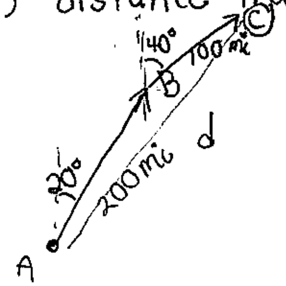


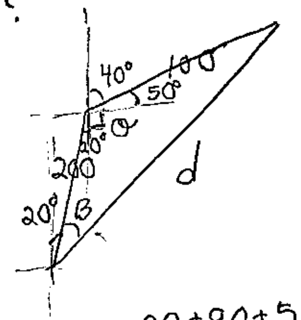
ex4 Ⓐ N20°E fly 200mi/h

Ⓑ After 1hr: N40°E

Ⓒ 30min: Land

a) distance traveled?





$\theta = 20 + 90 + 50 = 160°$

$d^2 = 100^2 + 200^2 - 2(100)(200)\cos 160°$

$d = \boxed{295.95 \text{ mi}}$

b) $\frac{\sin\beta}{100} = \frac{\sin 160°}{295.95}$

$\beta = 6.636°$

$\boxed{N\ 26.636°\ E}$

$c^2 = a^2 + b^2 - 2ab\cos\theta$

θ only one answer because θ ∈ [0, 180°]

cos θ < 0 ⇒ must be QII

cos θ > 0 ⇒ must be QI

not both

$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

sin θ must be > 0

θ ∈ [0, 180°]

QI or QII